1. (**Just for fun**) (4p)

   (a) In the lectures, when deriving and using equations of motion in different contexts, we have taken the pressure as a product of density and temperature, $p = nT$. Now, when discussing magnetic confinement of plasma, exactly the same product is used for energy density! Contemplate if this is OK – or has Taina once again made a typo somewhere. (2p)

   (b) The Sun produces energy at the rate of about 3.8×10^{26} W. This is generated by converting mass into energy. How much mass is converted into energy every second? Where does this energy go to? (2p)

   Solution.

   (a) In the magnetic confinement context, the temperature T is expressed in units of electronvolts, so it represents average particle energy E_{av} (or $E_{av} = \frac{3}{2}T$, depending on the convention). Thus, nT is the number density of particles multiplied by their energy, giving us the energy density.

   For the last lecture, n and T are related by the ideal gas law, $pV = Nk_BT$, where p is the pressure, V the volume of the system, N the total number of particles in the system, and k_BT is the average thermodynamic energy of a particle in the system (or proportional to it). Dividing by V, we have that $p = nk_BT$, where n is now the number density of particles. Thus, in our system, energy density is equal to (or proportional to) the pressure, and there's no problem with our reasoning.

   (b) We use the formula $E = mc^2$. Dividing by c^2 and taking a time derivative gives us that

   $$\frac{\mathrm{d}m}{\mathrm{d}t} = \frac{1}{c^2}\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{P}{c^2} = \frac{3.8 \times 10^{26}\,\mathrm{W}}{(3.0 \times 10^8\,\mathrm{W})^2} \approx 4.22 \times 10^9\,\mathrm{kg\,s^{-1}}$$

   The energy is radiated into space. Its intensity decreases quickly as a function of the distance from the Sun (inverse-square law far from the Sun, except that space is not completely empty).

   □

2. (**Collapse of a neutron star**) (6p)

   The energy source of a star is nuclear fusion, starting with fusion of hydrogen to helium, and then moving up in the periodic table until reaching the most stable elements around iron. If the star is sufficiently massive, that is. Take such a star at the end of its fusion career, so that it has ran out of exothermal reactions and is cooling down. Then at some point there is not enough kinetic pressure to counteract the gravitational pull and the star will collapse into a so-called *neutron star*. It is called so because, in the process of collapsing, weak interaction turns protons into neutrons. At the end, the only remaining 'force', preventing the whole system from collapsing into a singularity, is the Fermi pressure (i.e., Pauli's exclusion principle). Let's assume that we can and do measure the magnetic field on the surface of the star before and after the collapse, yielding the not-so-unrealistic values of 10 µT for the first measurement and 100 kT for the latter. Assuming that the plasma in and around this massive star behaves according to *ideal MHD*, what is the radius of the neutron star if the original star had the radius of $R = 1\,000\,000$ km?

   Solution.

   We assume that, outside the star, the magnetic field can be approximated as a dipole field. If the magnetic moment is aligned with the z-axis in cylindrical coordinates, then we can in spherical coordinates express the field as

   $$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0 M}{4\pi r^3}[2\cos\theta\hat{\boldsymbol{r}} + \sin\theta\hat{\boldsymbol{\theta}}],$$

   where the constant in front is not important here. Take the star to be a sphere of radius R and set (B_1, R_1) and (B_2, R_2) as the magnetic field strength and radius at the first and second measurements, respectively. The magnetic field at the equatorial plane (with respect to the magnetic moment) outside the star before and after ($\boldsymbol{B}'$) the collapse, in spherical coordinates, can be written

   $$\boldsymbol{B}([r, \varphi, \theta = \pi/2]) = \frac{B_1 R_1^3}{r^3}\hat{\boldsymbol{\theta}}$$
   $$\boldsymbol{B}'([r, \varphi, \theta = \pi/2]) = \frac{B_2 R_2^3}{r^3}\hat{\boldsymbol{\theta}}$$

According to the frozen-in condition, the flux of $\boldsymbol{B}$ through the annulus (disk with hole in the middle) defined by R_1 and xR_1 is the same as the flux of $\boldsymbol{B}'$ through an annulus defined by R_2 and xR_2, where x is an arbitrary constant. This can be expressed mathematically as

$$\begin{aligned}
\oiint_{S_1} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} &= \oiint_{S_2} \boldsymbol{B}' \cdot \mathrm{d}\boldsymbol{S} \\
\int_{R_1}^{xR_1} Br \,\mathrm{d}r &= \int_{R_2}^{xR_2} B'r \,\mathrm{d}r \\
B_1 R_1^3 \int_{R_1}^{xR_1} \frac{1}{r^2} \,\mathrm{d}r &= B_2 R_2^3 \int_{R_2}^{xR_2} \frac{1}{r^2} \,\mathrm{d}r \\
B_1 R_1^3 \Big[-\frac{1}{r}\Big]\Big|_{R_1}^{xR_1} &= B_2 R_2^3 \Big[-\frac{1}{r}\Big]\Big|_{R_2}^{xR_2} \\
B_1 R_1^2 [1 - 1/x] &= B_2 R_2^2 [1 - 1/x] \\
R_2 &= \sqrt{\frac{B_1}{B_2}} R_1 = \sqrt{\frac{10\,\mu\mathrm{T}}{100\,\mathrm{kT}}} \times 1\,000\,000\,\mathrm{km} = 10\,\mathrm{km}
\end{aligned}$$

□

3. (**Justifying the existence of magnetosphere**) (6p)

The solar wind introduced in the lectures comes with a pressure exerted on the Earth's magnetic field $\boldsymbol{B}$. This kinetic pressure can be expressed as ρV^2, where ρ is the mass density of the solar wind and V its speed. For Earth to maintain its magnetosphere, the magnetic pressure has to be able to stand up against this external kinetic pressure. Justify the existence of a magnetosphere around Earth by assigning the magnetosphere an effective radius R_M. Evaluate R_M (in terms of R_E) using typical solar wind parameters $\rho = 5\,\mathrm{u\,cm^{-3}}$ and $V = 400\,\mathrm{km\,s^{-1}}$. Naturally, R_M has to be larger than the Earth radius R_E for the magnetosphere to exist.

Solution.

For the magnetosphere to exist in an equilibrium, Earth's magnetic pressure together with the magnetosphere's kinetic plasma pressure needs to compensate the solar wind's kinetic pressure:

$$p_{kin,wind} = \rho V^2 = p_{Earth} = \frac{B^2}{2\mu_0} + p_{kin,plasma} = \frac{B^2}{2\mu_0}(1+\beta),$$

where the plasma beta, $\beta = p_{kin,plasma}/p_{magnetic}$, is familiar from lecture 10. Neglecting the magnetization of the magnetosphere, the magnetic field is that of a dipole field: $B \propto r^{-3}$. Therefore, the magnetic field at the magnetosphere boundary can be written as

$$B_M = B_0 \left(\frac{R_E}{R_M}\right)^3,$$

where $B_0 \approx 30\,\mu\mathrm{T}$ is the magnetic field at Earth's surface. Combining the previous expressions gives

$$\begin{aligned}
\rho V^2 &= \frac{B_M^2}{2\mu_0}(1+\beta) \\
&= \frac{B_0^2}{2\mu_0}\left(\frac{R_E}{R_M}\right)^6 (1+\beta) \\
\frac{R_M}{R_E} &= \left(\frac{B_0^2/2\mu_0}{\rho V^2}(1+\beta)\right)^{1/6}
\end{aligned}$$

Recalling from the lectures that the β parameter is typically much smaller than one, we see that the size of the magnetosphere is dictated by the strength of Earth's own magnetic field. We approximate

$$\frac{R_M}{R_E} \approx \left(\frac{B_0^2/2\mu_0}{\rho V^2}\right)^{1/6}$$

For the given parameters, we have

$$R_M \approx 8 \times R_E.$$

The effective radius of the magnetosphere is larger than the radius of Earth, thus justifying the existence of the magnetosphere around Earth. □

4. (**Magnetic dipole field**) (8p)

Recall from basic electromagnetism: far enough from the magnetic dipole with magnetization M, the field can be expressed as $\boldsymbol{B} = -\nabla\Psi$, where $\Psi = -(\mu_0/(4\pi))\boldsymbol{M}\cdot\nabla(1/r)$.

(a) Using spherical coordinates and setting the z-axis along the magnetic dipole, show that the magnetic field is

$$\boldsymbol{B} = \frac{\mu_0}{4\pi}\frac{(3\boldsymbol{M}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}} - \boldsymbol{M}}{r^3}.$$

(2p)

(b) Calculate the components of the magnetic field (still in spherical coordinates) as well as the magnitude of the magnetic field. (2p)

(c) In space and geophysics, instead of the conventional azimuthal angle θ, ranging from 0 (north pole) to π (south pole), one typically uses another angle λ, which measures the azimuthal angle not from the z-axis but, rather, from the (x,y) plane. So, λ ranges from $-\pi/2$ (north pole) to $+\pi/2$ (south pole). Express the magnetic field components in terms of λ. (2p)

(d) You can follow the field lines by making sure you always keep the same direction as the field line, i.e., $r\frac{d\lambda}{dr} = B_\lambda/B_r$. Using this, show that the distance of a field line from Earth can be expressed as $r = r_0\cos^2\lambda$, where r_0 is the distance of the field line at the equator. (2p)

Solution.

(a) The magnetic field is obtained by combining the given expressions:

$$\begin{aligned}
\boldsymbol{B} &= -\nabla\left[-\frac{\mu_0}{4\pi}\boldsymbol{M}\cdot\nabla\left(\frac{1}{r}\right)\right] \\
&= \frac{\mu_0}{4\pi}\left\{\boldsymbol{M}\times\underbrace{\left[\nabla\times\nabla\left(\frac{1}{r}\right)\right]}_{=0} + \nabla\left(\frac{1}{r}\right)\times\underbrace{(\nabla\times\boldsymbol{M})}_{=0} + (\boldsymbol{M}\cdot\nabla)\nabla\left(\frac{1}{r}\right) + \nabla\left(\frac{1}{r}\right)\cdot\underbrace{\nabla\boldsymbol{M}}_{=0}\right\} \\
&= \frac{\mu_0}{4\pi}(\boldsymbol{M}\cdot\nabla)\nabla\left(\frac{1}{r}\right),
\end{aligned}$$

where we used the fact that the magnetization, $\boldsymbol{M} = M\hat{\boldsymbol{z}}$, is constant in space and $\nabla\times\nabla f = 0$ for any scalar f. Noting that $\nabla(1/r) = -\boldsymbol{r}/r^3$ and $\nabla\boldsymbol{r} = \mathbf{I}$, where $\mathbf{I}$ is the unit dyadic tensor, we have that

$$\begin{aligned}
\boldsymbol{B} &= \frac{\mu_0}{4\pi}(\boldsymbol{M}\cdot\nabla)\left(-\frac{\boldsymbol{r}}{r^3}\right) \\
&= -\frac{\mu_0}{4\pi}\boldsymbol{M}\cdot\left[\frac{\nabla\boldsymbol{r}}{r^3} + \boldsymbol{r}\nabla\left(\frac{1}{r^3}\right)\right] \\
&= -\frac{\mu_0}{4\pi}\boldsymbol{M}\cdot\left[\frac{\mathbf{I}}{r^3} - 3\boldsymbol{r}\frac{\hat{\boldsymbol{r}}}{r^4}\right] \\
&= \frac{\mu_0}{4\pi}\boldsymbol{M}\cdot\left[3\frac{\hat{\boldsymbol{r}}\hat{\boldsymbol{r}}}{r^3} - \frac{\mathbf{I}}{r^3}\right] \\
&= \frac{\mu_0}{4\pi}\frac{(3\boldsymbol{M}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}} - \boldsymbol{M}}{r^3}
\end{aligned}$$

(b) We can evaluate the components by noting that $\boldsymbol{M} = M\hat{\boldsymbol{z}} = M(\cos\theta\,\hat{\boldsymbol{r}} - \sin\theta\,\hat{\boldsymbol{\theta}})$. Thus we have that:

$$\begin{cases}
B_r &= \dfrac{\mu_0 M}{2\pi r^3}\cos\theta \\
B_\theta &= \dfrac{\mu_0 M}{4\pi r^3}\sin\theta \\
B_\varphi &= 0
\end{cases}$$

The magnitude of the magnetic field is given by

$$\begin{aligned} B &= \sqrt{B_r^2 + B_\theta^2} \\ &= \sqrt{\left(\frac{\mu_0 M}{2\pi r^3}\cos\theta\right)^2 + \left(-\frac{\mu_0 M}{4\pi r^3}\sin\theta\right)^2} \\ &= \frac{\mu_0 M}{4\pi r^3}\sqrt{4\cos^2\theta + \sin^2\theta} \\ &= \frac{\mu_0 M}{4\pi r^3}\sqrt{3\cos^2\theta + 1} \end{aligned}$$

(c) The new azimuthal angle λ is related to the spherical angle θ (where $\theta = 0$ at the north pole and $\theta = \pi$ at the south pole) by

$$\theta = \pi/2 + \lambda \quad \Rightarrow \quad \cos\theta = \cos(\pi/2 + \lambda) = -\sin\lambda, \quad \sin\theta = \sin(\pi/2 + \lambda) = \cos\lambda$$

The magnetic field components are thus

$$\begin{cases} B_r &= -\dfrac{\mu_0 M}{2\pi r^3}\sin\lambda \\ B_\lambda &= \dfrac{\mu_0 M}{4\pi r^3}\cos\lambda \\ B_\varphi &= 0 \end{cases}$$

(d) Following the field line, we have

$$\begin{aligned} r\frac{\mathrm{d}\lambda}{\mathrm{d}r} &= \frac{B_\lambda}{B_r} \\ &= \left[\frac{\mu_0 M}{4\pi r^3}\cos\lambda\right]\left[-\frac{\mu_0 M}{2\pi r^3}\sin\lambda\right]^{-1} \\ &= -\frac{1}{2}\frac{\cos\lambda}{\sin\lambda} \\ -2\int_0^\lambda \frac{\sin x}{\cos x}\,\mathrm{d}x &= \int_{r_0}^r \frac{1}{y}\,\mathrm{d}y \end{aligned}$$

Let $u = \cos x \Rightarrow \mathrm{d}u = -\sin x\,\mathrm{d}x$, then

$$\begin{aligned} 2\int_1^{\cos\lambda}\frac{1}{u}\,\mathrm{d}u &= \int_{r_0}^r \frac{1}{y}\,\mathrm{d}y \\ 2\big(\ln(\cos\lambda) - \ln 1\big) &= \ln r - \ln r_0 \\ \ln(\cos^2\lambda) &= \ln(r/r_0) \\ \cos^2\lambda &= \frac{r}{r_0} \\ r &= r_0\cos^2\lambda \end{aligned}$$

□

5. (**Food for thought**)

During the lectures we learned many things about our very own 'magnetic cage', the dipole field that protects us from being directly hit by the potentially deadly components of the solar wind. What we did not learn is that this field is not permanent but has changed polarity in the course of Earth's history – on a geological time scale. Try to find information about this reversal of the field and contemplate its possible consequences. Return your short write-up in MyCourses.