1 INTRODUCTION

LEARNING OUTCOMES

Students get an overall picture about prerequisites of the course, the roles of engineering models in structure modelling, and finite element method in displacement analysis of structures. The topics of week 2 are

- Structure modelling
- Structure analysis
- Mathematica language and the finite element solver of MEC-E8001

EXPERIMENT VS. MODELLING

In design of a simple pendulum of a tall-case clock, the required information is the dependency of period T on mass m, initial angle ϕ_0 from the stable equilibrium position, acceleration by gravity g, and length L. The main options are

Straightforward experiment: Measurement of T on various physical structures (characterized by m and L), with various initial angles ϕ_0, and on various places on earth (characterized by g).

Dimension analysis: Application of generic principles of physics to get $T = \sqrt{L / g} f(\phi_0)$ and measurement of $T\sqrt{g / L}$ as the function of ϕ_0.

Mathematical modelling: Application of simplifying assumptions, the basic laws of mechanics, and rules of mathematics to get $T = 2\pi\sqrt{L / g}$.

ERROR COMPONENTS

Measuring error $\|\bar{u}-u\|$: measured ($\bar{u}$) and exact (u)

Modeling error $\|u-\hat{u}\|$: exact (u) and exact to model ($\hat{u}$)

Numerical error $\|\hat{u}-\tilde{u}\|$: exact to model ($\hat{u}$) and numerical ($\tilde{u}$)

Error $\|\bar{u}-\tilde{u}\| = \|\bar{u}-u+u-\hat{u}+\hat{u}-\tilde{u}\| \leq \|\bar{u}-u\|+\|u-\hat{u}\|+\|\hat{u}-\tilde{u}\|$

An engineering model and, thereby, predictions by the model contain always error due to the simplifications made. Therefore, in practice, numerical error of the same order in finding the predictions (solution to the model) can be considered as acceptable.

1.1 STRUCTURE MODELLING

MODELLING STEPS

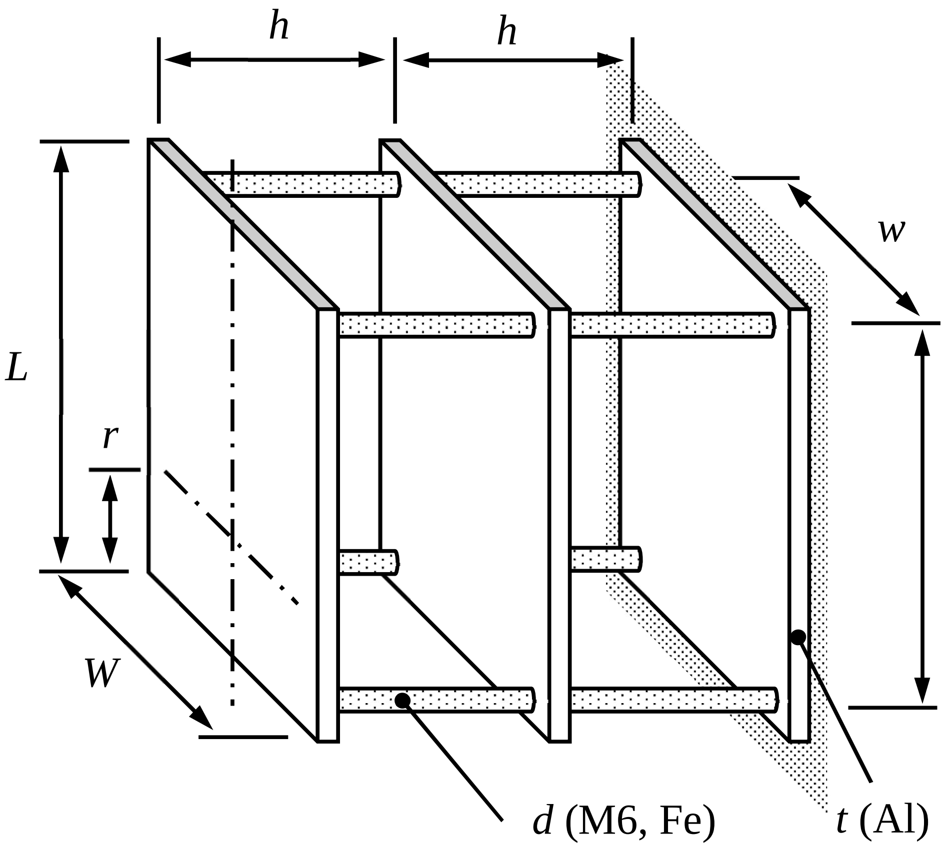
- **Crop**: Decide the boundary of structure. Interaction with surroundings need to be described in terms of known forces, moments, displacements, and rotations. All uncertainties with this respect bring uncertainty to the model too.

- **Idealize**: Simplify the geometry and decide the model. Ignoring the details, not likely to affect the outcome, may simplify analysis a lot.

- **Parameterize**: Assign symbols to geometric and material parameter of the idealized structure. Measure or find the values needed in numerical calculations.

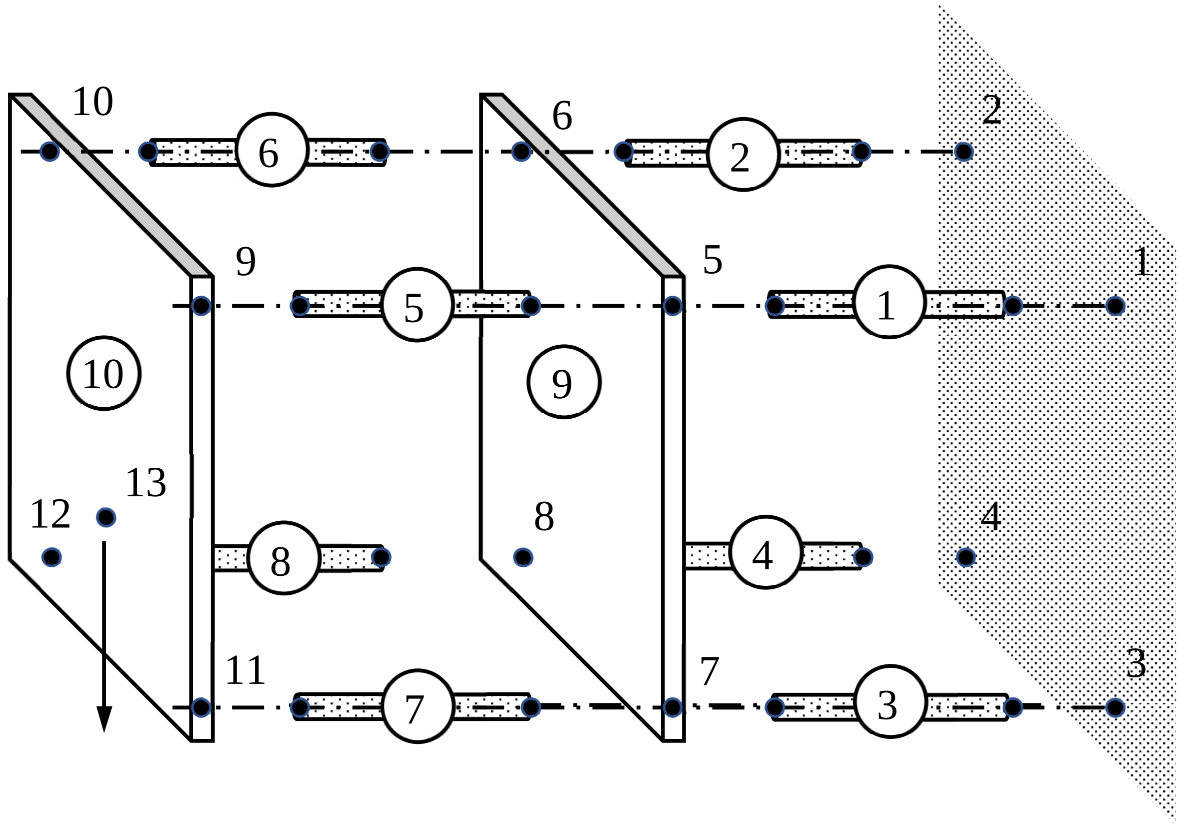
- **Divide-and-rule**: Represent a complex structure as a set of structural parts interacting with each other through connection points and surroundings with interaction models.

CROP-IDEALIZE-PARAMETERIZE



d	4.8 mm
h	0.156 m
l	0.4 m
w	0.243 m
L	0.44 m
W	0.295 m
t	1.5 cm
r	6.5 cm

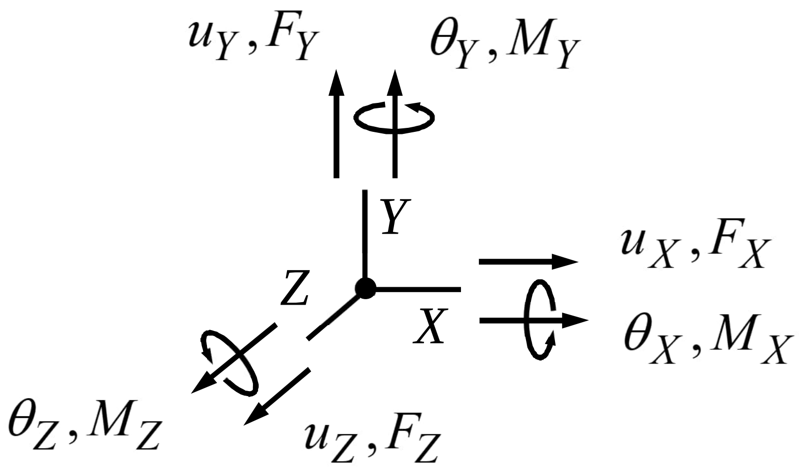
DIVIDE-AND-RULE



The book-keeping with unique identifiers for the structural parts and connection points allow a consistent naming of the kinematic and kinetic quantities of analysis.

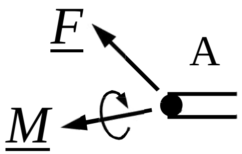
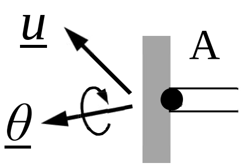
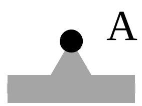
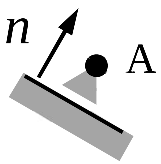
KINEMATIC AND KINETIC QUANTITIES

The primary quantities of analysis are displacements, rotations, forces and moments at the connection points of the structural parts. The components of the vector quantities (magnitude and direction) are taken to be positive in the directions of the coordinate axes.

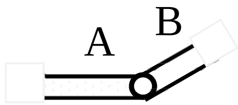


Vector quantities are invariants in the sense $\vec{a} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k} = a_X\vec{I} + a_Y\vec{J} + a_Z\vec{K}$, and can be transformed from one coordinate system to another using the property.

INTERACTION MODELS

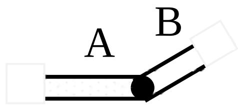
name	symbol	equations
force		$\vec{F}_{\mathrm{A}} = \underline{\vec{F}},\ \vec{M}_{\mathrm{A}} = \underline{\vec{M}}$
fixed		$\vec{u}_{\mathrm{A}} = \underline{\vec{u}},\ \vec{\theta}_{\mathrm{A}} = \underline{\vec{\theta}}$
joint		$\vec{u}_{\mathrm{A}} = 0,\ \vec{M}_{\mathrm{A}} = 0$
slider		$\vec{n} \cdot \vec{u}_{\mathrm{A}} = 0,\ \vec{F}_{\mathrm{A}} - (\vec{F}_{\mathrm{A}} \cdot \vec{n})\vec{n} = 0,\ \vec{M}_{\mathrm{A}} = 0$

joint



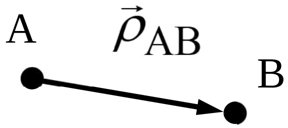
$$\vec{u}_\mathrm{B} = \vec{u}_\mathrm{A},\ \vec{M}_\mathrm{A} = 0,\ \vec{M}_\mathrm{B} = 0$$

fixed



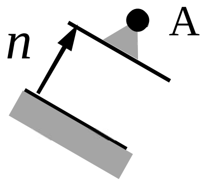
$$\vec{u}_\mathrm{B} = \vec{u}_\mathrm{A},\ \vec{\theta}_\mathrm{B} = \vec{\theta}_\mathrm{A}$$

rigid



$$\vec{u}_\mathrm{B} = \vec{u}_\mathrm{A} + \vec{\theta}_\mathrm{A} \times \vec{\rho}_\mathrm{AB},\ \vec{\theta}_\mathrm{B} = \vec{\theta}_\mathrm{A}$$

contact



$$\vec{n} \cdot \vec{u}_\mathrm{A} \geq 0,\ \vec{n} \cdot \vec{F}_\mathrm{A} \geq 0,\ (\vec{n} \cdot \vec{u}_\mathrm{A})(\vec{n} \cdot \vec{F}_\mathrm{A}) = 0$$

Interaction models define a kinematic quantity (displacements and rotations) or its work conjugate (forces and moments)!

NEWTON's LAWS OF MOTION

I In an inertial frame of reference, an object either remains at rest or continues to move at a constant velocity, unless acted upon by a force.

II The vector sum of the forces on an object is equal to the mass of that object multiplied by the acceleration of the object (assuming that the mass is constant).

III When one body exerts a force on a second body, the second body simultaneously exerts a force equal in magnitude and opposite in direction on the first body.

Newton's laws in their original forms apply to particles only. The formulation for rigid bodies and deformable bodies require slight modifications.

MODELS FOR STRUCTURAL PARTS

Structural part model is a relationship between the displacements, rotations, forces, moments at the connection points and external given forces (like weight) acting on the structural parts. The relationship is. One may consider the relationship as the generalization of the simple spring model affected by the assumptions used (beam, plate, solid model), material model, the number of connection and additional points, and the shape of the structural parts. Assuming a linear and stationary case

nodal displacements distributed forces

$$\{F\} = [K]\{u\} - \{f\}$$

nodal forces stiffness

1.2 STRUCTURE ANALYSIS

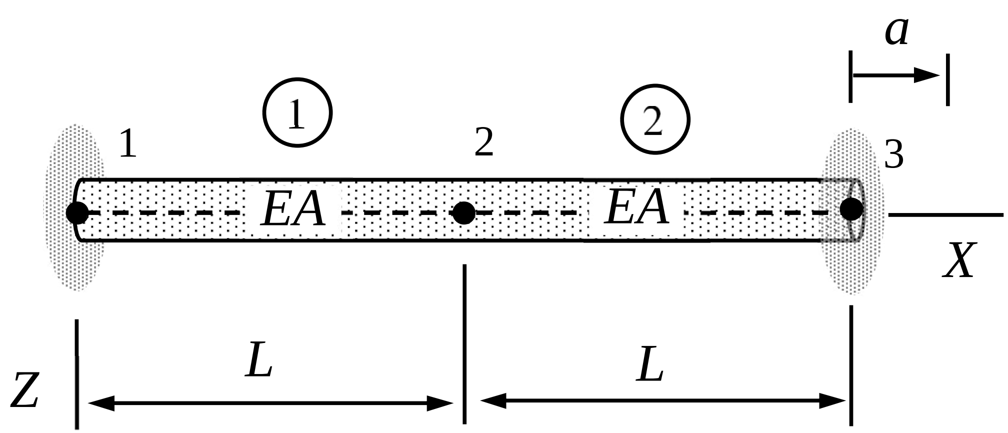
- Idealize a complex structure as a set of structural parts, whose behavior can be approximated by using the usual engineering models (bar, beam, plate, rigid body etc.).

- Write down the equilibrium equations at the connections (Newton **III**), the force-displacement relationships of the structural parts, and constraints concerning the nodal displacements (displacements and rotations should match).

- Solve the nodal displacements and rotations and the forces and moments acting on the structural parts from the equation system.

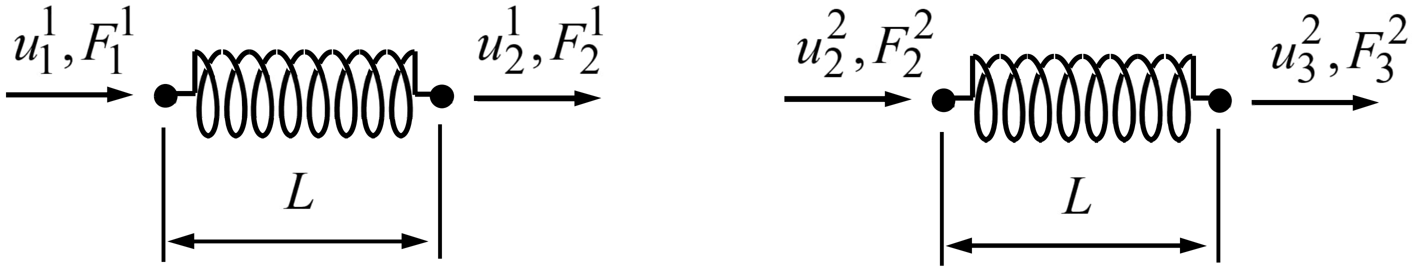
- Determine the stress in the structural parts one-by-one according to the engineering model used (optional step).

EXAMPLE 1.1 A connector bar is welded at its ends to rigid walls. If the right end wall displacement is a, determine the displacements of connection points 1, 2, and 3 and the forces acting on structural parts 1 and 2. Cross sectional area A and Young's modulus of the material E are constants and the displacement force relationship of a bar is the same as that of a spring with coefficient $k = EA / L$.



Answer $u_1^1 = 0$, $u_2^1 = u_2^2 = \frac{1}{2}a$, $u_3^2 = a$, $F_1^1 = F_2^2 = -\frac{1}{2}ka$, $F_2^1 = F_3^2 = \frac{1}{2}ka$.

- Let us omit the index for the component (*x*,*y*,*z* as unnecessary in this simplistic case) and use the two-index notation u_i^e , F_i^e for the displacements and forces acting on the structure. Superscript denotes the structural element and subscript the connection point. The exploded structure with the displacements and forces is given by



- Interaction model between the bars and with the surroundings is of type “fixed” so $u_1^1 = 0$, $u_2^1 = u_2^2$, and $u_3^2 = a$ (left edge welding, integrity of structure at the connection, and displacement of the right end wall). The force constraints are due to Newton **III** which requires that F_2^1 and F_2^2 are equal in magnitude and opposite in signs i.e. $F_2^1 + F_2^2 = 0$.

- As the structural parts can be considered as springs of coefficient $k = EA/L$, $F_1^1 = k(u_1^1 - u_2^1), F_1^2 = k(u_2^1 - u_1^1)$, $F_2^2 = k(u_2^2 - u_3^2)$, and $F_3^2 = k(u_3^2 - u_2^2)$.

- Altogether, the 8 equations determining the 4 displacement components u_1^1, u_2^1, u_2^2, u_3^2 and the 4 force components F_1^1, F_2^1, F_2^2, F_3^2 are given by

$$F_1^1 = k(u_1^1 - u_2^1),\ F_1^2 = k(u_2^1 - u_1^1),\ F_2^2 = k(u_2^2 - u_3^2),\ F_3^2 = k(u_3^2 - u_2^2).$$

$$u_1^1 = 0,\ u_2^1 = u_2^2,\ u_3^2 = a,$$

$$F_2^1 + F_2^2 = 0.$$

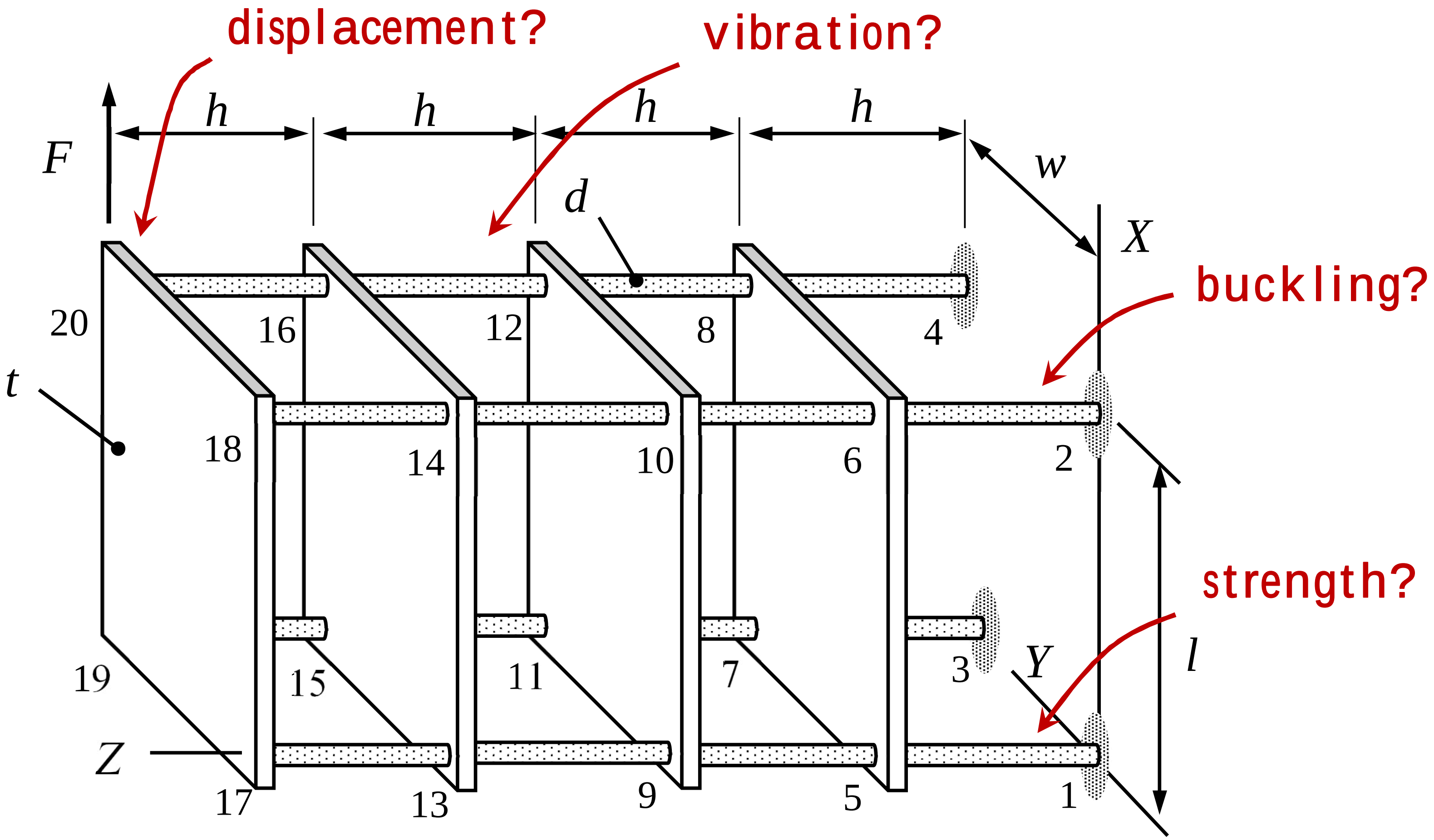
- The linear equation system can be solved, e.g., by considering the equations in a proper order (to be discussed later in more detail), by Gauss elimination, by Mathematica, …

$$u_1^1 = 0,\ \ u_2^1 = \frac{1}{2}a,\ u_2^2 = \frac{1}{2}a,\ \ u_3^2 = a,$$

$$F_1^1 = -\frac{1}{2}ka,\ \ F_2^1 = \frac{1}{2}ka,\ \ F_2^2 = -\frac{1}{2}ka,\ F_3^2 = \frac{1}{2}ka.$$

The example and exercise problems of MEC-8001 can be solved either by hand calculation (above) or by representing the problem in the two-table form used by the FE-code of the course.

1.3 FINITE ELEMENT ANALYSIS



Displacement and stress analysis according to the linear elasticity theory may not entirely explain the behavior of a structure!

WHY FINITE ELEMENTS ?

Design of machines and structures: Solution to stress or displacement by analytical method is often impossible due to complex geometry, heterogeneous material etc. Lack of the "exact solution" to an "approximate problem" is not an issue in engineering work.

Finite element method is the standard of solid mechanics: Commercial codes in common use are based on the finite element method. A graphical user interface may make living easier, but a user should always understand what the problem is and in what sense it is solved!

Finite element method has a strong theory: Although approximate solution is acceptable, knowing nothing about the numerical error is not acceptable.

NUMERICAL ERROR

Numerical method replaces the original problem (solution $\hat{u}$**)** by a *numerically convenient problem* (solution $\tilde{u}$). In FEM the error $e = \|\hat{u} - \tilde{u}\|$ can be made as small as wanted by increasing the numerical work.

Error is usually of the form $e = \|\hat{u} - \tilde{u}\| \le C_e n^{-\alpha}$, in which n characterizes the size of *numerically convenient problem* (typically the number of linear equations) and C_e and α are positive constants.

Numerical work (number of arithmetic operations needed) depends on the details of the recipe, but it grows typically at a polynomial rate $w \le C_w n^{\beta}$ (hence $e \sim 1/w^{(\alpha/\beta)}$)

DISPLACEMENT FEA

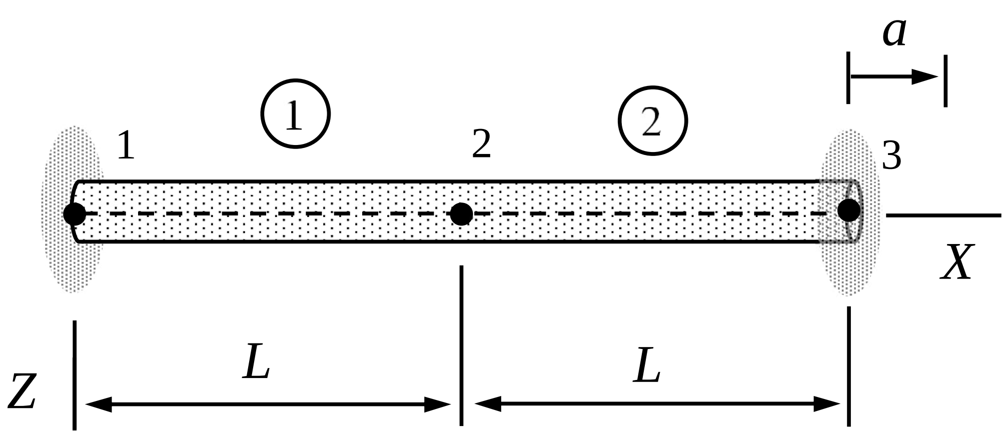
- Model the structure as a collection of elements (solid, plate, beam). Derive the element contributions $\delta W^e = \delta W^{\text{int}} + \delta W^{\text{ext}}$ in terms of the nodal displacement and rotation components of the structural coordinate system.

- Sum the element contributions to end up with the virtual work expression of the structure $\delta W = \sum_{e \in E} \delta W^e$. Re-arrange to get the "standard" form $\delta W = -\delta \mathbf{a}^{\mathrm{T}} (\mathbf{K}\mathbf{a} - \mathbf{F}) = 0$.

- Use the principle of virtual work $\delta W = 0 \ \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus for $\delta \mathbf{a} \in \mathbb{R}^n$ to deduce the linear equations $\mathbf{K}\mathbf{a} - \mathbf{F} = 0$.

- Solve the equations for displacements and rotations $\mathbf{a}$.

EXAMPLE 1.2 A connector bar is welded at its ends to rigid walls. Assuming linearly elastic behavior and the right end wall displacement a, determine the displacements of nodes 1, 2 and 3. Model the structure as a collection of two bar elements of cross-sectional area A and Young's modulus E.



Answer $u_{X2} = \frac{1}{2}a$

In finite element analysis, the conditions related with the integrity of the structure are (usually) satisfied ‘a priori’ to eliminate the internal forces automatically. Structure needs to modelled as a set of elements but exploded structure, with all kinematic and kinetic quantities in it, is not needed. Let us follow the recipe:

- First, virtual work expressions of the elements in terms of the displacement components in the structural coordinate system

$$\delta W^1 = -\begin{Bmatrix} 0 \\ \delta u_{X2} \end{Bmatrix}^{\mathrm{T}} \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ u_{X2} \end{Bmatrix} = -\delta u_{X2} \frac{EA}{L} u_{X2},$$

$$\delta W^2 = -\begin{Bmatrix} \delta u_{X2} \\ 0 \end{Bmatrix}^{\mathrm{T}} \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{X2} \\ a \end{Bmatrix} = -\delta u_{X2} \frac{EA}{L} (u_{X2} - a).$$

- Second, virtual work expression of the structure in its “standard” form

$$\delta W = \delta W^1 + \delta W^2 = -\delta u_{X2}\frac{EA}{L}u_{X2} - \delta u_{X2}\frac{EA}{L}(u_{X2}-a) = -\delta u_{X2}\frac{EA}{L}(2u_{X2}-a).$$

- Third, principle of virtual work $\delta W = 0 \ \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus for $\delta \mathbf{a} \in \mathbb{R}^n$ imply a linear equation to the unknown u_{X2} and, thereby, the solution

$$\frac{EA}{L}(2u_{X2}-a) = 0 \quad \Leftrightarrow \quad u_{X2} = \frac{1}{2}a. \quad \Leftarrow$$

The recipe works no matter the complexity of the structure, analysis type, and method of calculation (hand or FE-code) with slight modifications depending mainly on the analysis type.

VIBRATION FEA

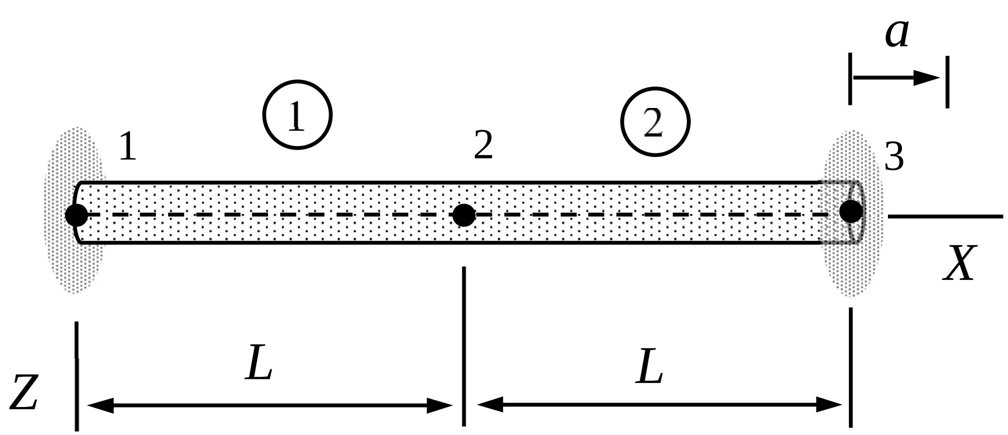
- Model the structure as a collection of elements (solid, plate, beam). Derive the element contributions $\delta W^e = \delta W^{\text{int}} + \delta W^{\text{ext}} + \delta W^{\text{ine}}$ in terms of the nodal displacement and rotation components of the structural coordinate system.

- Sum the element contributions to end up with the virtual work expression of the structure $\delta W = \sum_{e \in E} \delta W^e$. Re-arrange to get $\delta W = -\delta \mathbf{a}^{\text{T}} (\mathbf{M}\ddot{\mathbf{a}} + \mathbf{K}\mathbf{a} - \mathbf{F})$.

- Use the principle of virtual work $\delta W = 0 \; \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus for $\delta \mathbf{a} \in \mathbb{R}^n$ to deduce the ordinary differential equations $\mathbf{M}\ddot{\mathbf{a}} + \mathbf{K}\mathbf{a} - \mathbf{F} = 0$.

- Solve the equations for the natural angular speeds of vibrations and the corresponding modes $(\omega, \mathbf{a})_i$ or for displacements and rotations as the functions of time $\mathbf{a}(t)$.

EXAMPLE 1.3 A connector bar is welded at its ends to rigid walls. If the welding fails at (time) $t = 0$ when the right end wall displacement is a, determine the displacement of the midpoint 2 as the function of time. Model the structure as a collection of two bar elements of cross-sectional area A, Young's modulus E, and density ρ.



Answer $u_{X2}(t) = \frac{1}{4} a(1+\sqrt{2})\cos(t\sqrt{\frac{6}{7}(5-3\sqrt{2})\frac{E}{\rho L^2}}) - \frac{1}{4} a(\sqrt{2}-1)\cos(t\sqrt{\frac{6}{7}(5+3\sqrt{2})\frac{\mathrm{E}}{\rho L^2}})$

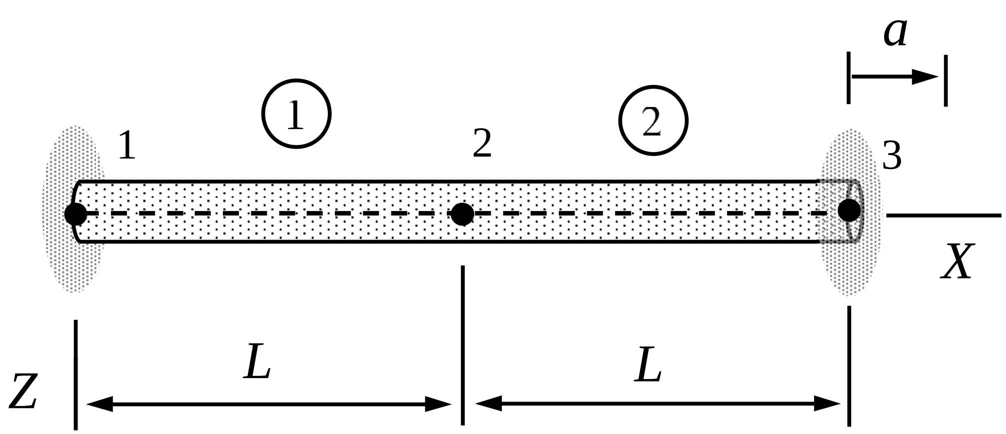
STABILITY FEA

- Model the structure as a collection of beam, plate, etc. elements. Derive the element contributions $\delta W^e = \delta W^{\text{int}} + \delta W^{\text{ext}} + \delta W^{\text{sta}}$ in terms of the nodal displacement and rotation components of the structural coordinate system.

- Sum the element contributions to end up with the virtual work expression of the structure $\delta W = \sum_{e \in E} \delta W^e$. Re-arrange to get $\delta W = -\delta \mathbf{a}^{\mathrm{T}} \mathbf{R}(\mathbf{a})$ and use the principle of virtual work $\delta W = 0 \ \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus for $\delta \mathbf{a} \in \mathbb{R}^n$ to deduce the (non-linear) equilibrium equations $\mathbf{R}(\mathbf{a}) = 0$.

- Find the values of the control parameter values and the corresponding modes $(p, \mathbf{a})_i$ for non-unique solutions of the equilibrium equations. The smallest of the control parameter values is the critical one.

EXAMPLE 1.4 A connector bar is welded at its ends to rigid walls. Determine the displacement a at which the buckling of structure occurs. Model the structure as a collection of two beams of cross-section moments A, I, Young's modulus E and shear modulus G.



Answer $a = -20\dfrac{I}{AL}$

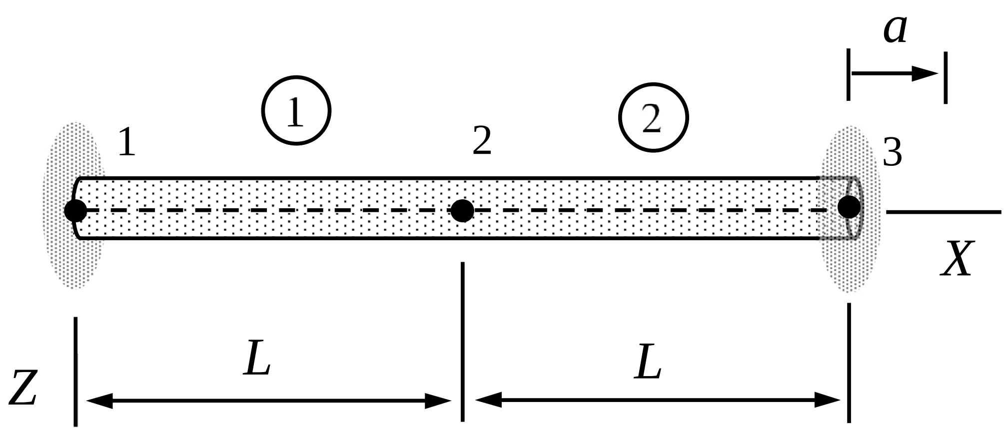
NON-LINEAR FEA

- Model the structure as a collection of beam, plate, etc. elements by considering the initial geometry. Derive the element contributions $\delta W^e = \delta W^{\text{int}} + \delta W^{\text{ext}}$ in terms of the nodal displacement and rotation components of the structural coordinate system.

- Sum the element contributions to end up with the virtual work expression of the structure $\delta W = \sum_{e \in E} \delta W^e$. Re-arrange to get $\delta W = -\delta \mathbf{a}^{\mathrm{T}} \mathbf{R}(\mathbf{a})$ and use the principle of virtual work $\delta W = 0 \;\; \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus for $\delta \mathbf{a} \in \mathbb{R}^n$ to deduce the equilibrium equations $\mathbf{R}(\mathbf{a}) = 0$.

- Find a physically meaningful solution $\mathbf{a}$ by using any of the standard numerical methods for non-linear algebraic equation systems.

EXAMPLE 1.5 A connector bar is welded at its ends to rigid walls. Determine the axial displacement of midpoint 2 according to the large displacement theory when the right end wall displacement is a. Model the structure as a collection of two bar elements of cross-section area A and Young's modulus E. Use the problem parameter values $L = 1\text{m}$, $A = 0.01\text{m}^2$, $E = 100\,\text{N/m}^2$, and $a = -L/10$.



Answer $u_{X2} = -0.05$ ($u_{X2} = -0.05 - 1.31i$, $u_{X2} = -0.05 + 1.31i$)

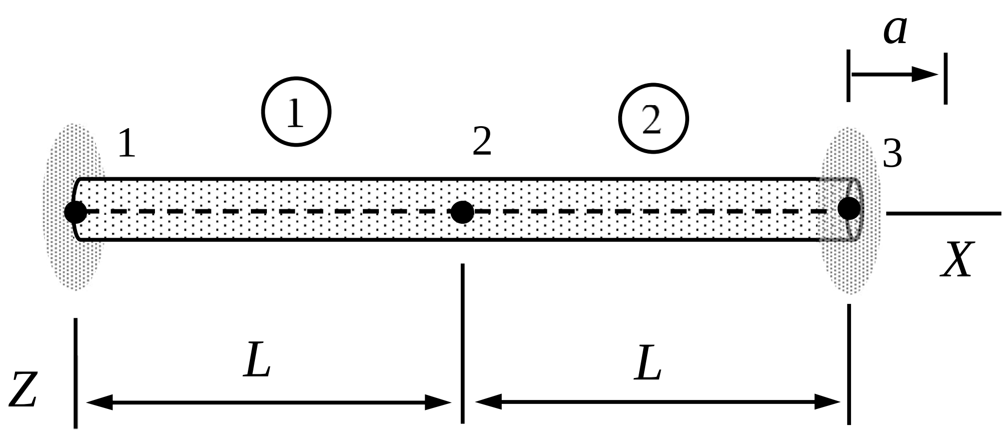
THERMO-MECHANICAL (MULTI-PHYSICS) FEA

- Model the structure as a collection of beam, plate, etc. elements. Derive the element contributions δW^e and δP^e in terms of nodal displacements/rotation components of the structural coordinate system and temperature.

- Sum the element contributions to end up with the variational expression for the structure. Re-arrange to get $\delta W + \tau\delta P = -\delta\mathbf{a}^{\mathrm{T}}\mathbf{R}(\mathbf{a},\mathbf{b}) - \tau\delta\mathbf{b}^{\mathrm{T}}\mathbf{R}(\mathbf{b})$ (τ is a dimensionally correct but otherwise arbitrary constant). Use the principle $\delta W + \tau\delta P = 0 \quad \forall\,\delta\mathbf{a},\delta\mathbf{b}$ and the fundamental lemma of variation calculus to deduce $\mathbf{R}(\mathbf{a},\mathbf{b}) = 0$ and $\mathbf{R}(\mathbf{b}) = 0$.

- Solve the linear algebraic equations for the nodal displacements, rotations, and temperatures (due to the one-sided coupling of the stationary problem, solving the temperature first is always possible).

EXAMPLE 1.6 A connector bar is welded at its ends to rigid walls. Determine the stationary displacement u_{X2} and temperature ϑ_2 at node 2, when the temperature of the right end is increased to $2\vartheta^\circ$ and the right end wall displacement is a, Model the structure as a collection of two bar elements of cross-section area A, Young's modulus E, thermal conductivity k, and thermal expansion coefficient α. Stress in the bar vanishes, when the temperature in the wall and bar is ϑ° and $a=0$.



Answer $u_{X2} = \frac{1}{4}(4a - L\alpha\vartheta^\circ)$, $\vartheta_2 = \frac{3}{2}\vartheta^\circ$

MATHEMATICAL REPRESENTATIONS

- **Small displacement analysis** $\mathbf{R}(\mathbf{a}) = \mathbf{Ka} - \mathbf{F} = 0$
- **Vibration analysis** $\mathbf{R}(\mathbf{a}) = \mathbf{M}\ddot{\mathbf{a}} + \mathbf{Ka} - \mathbf{F} = 0 \;\; t > t_0 \;,\quad \dot{\mathbf{a}} = \dot{\mathbf{a}}_0 \;\; t = t_0, \quad \mathbf{a} = \mathbf{a}_0 \;\; t = t_0$
- **Eigenfrequency analysis** $\mathbf{R}(\mathbf{a}, \omega) = (-\mathbf{M}\omega^2 + \mathbf{K})\mathbf{a} = 0$
- **Stability analysis** $\mathbf{R}(\mathbf{a}, p) = (-p\mathbf{F} + \mathbf{K})\mathbf{a} = 0$
- **Large displacement analysis** $\mathbf{R}(\mathbf{a}) = 0$
- **Thermo-mechanical analysis** $\mathbf{R}(\mathbf{a}, \mathbf{b}) = 0$

PREREQUISITE; MATRIX ALGEBRA I

Addition	$\mathbf{C} = \mathbf{A} + \mathbf{B}$	$C_{ij} = A_{ij} + B_{ij}$
Multiplication (scalar)	$\mathbf{C} = \alpha \mathbf{A}$	$C_{ij} = \alpha A_{ij}$
Multiplication (matrix)	$\mathbf{C} = \mathbf{A}\mathbf{B}$	$C_{ij} = \sum_{k \in \{1 \ldots n\}} A_{ik} B_{kj}$

Unit matrix	$\mathbf{I}$	$\delta_{ij} = 1 \quad i = j, \quad \delta_{ij} = 0 \quad i \neq j$
Symmetric matrix	$\mathbf{A} = \mathbf{A}^{\mathrm{T}}$	$A_{ij} = A_{ji}$
Skew symmetric matrix	$\mathbf{A} = -\mathbf{A}^{\mathrm{T}}$	$A_{ij} = -A_{ji}$
Positive definite matrix	$\mathbf{x}^{\mathrm{T}} \mathbf{A} \mathbf{x} > 0 \quad \forall \mathbf{x} \neq 0$	

PREREQUISITE; MATRIX ALGEBRA II

Transpose	$\mathbf{A}^{\mathrm{T}}$	$A_{ij}^{\mathrm{T}} = A_{ji}$
Inverse	$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$	$\sum_{k\in\{1\ldots n\}} A_{ik} A_{kj}^{-1} = \delta_{ij}$
Derivative	$\dot{\mathbf{x}}$	$\dot{x}_i = dx_i / dt$

Linear equation system	Find $\mathbf{x}$ such that $\mathbf{A}\mathbf{x} = \mathbf{b}$
Eigenvalue problem	Find all $(\lambda, \mathbf{x})$ such that $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0$
Eigenvalue composition	$\mathbf{A} = \mathbf{X}\boldsymbol{\lambda}\mathbf{X}^{-1}$, where $\mathbf{X} = [\mathbf{x}_1 \ldots \mathbf{x}_n]$ and $\boldsymbol{\lambda} = diag[\lambda_1 \ldots \lambda_n]$
Matrix function	If $\mathbf{A} = \mathbf{X}\boldsymbol{\lambda}\mathbf{X}^{-1}$, then $f(\mathbf{A}) = \mathbf{X} f(\boldsymbol{\lambda})\mathbf{X}^{-1}$

EXAMPLE 1.7 Determine the eigenvalues λ_1, λ_2 and the corresponding eigenvectors $\mathbf{x}_1$, $\mathbf{x}_2$ of the 2×2 matrix $\mathbf{A}$. Write down also the eigenvalue decomposition $\mathbf{A} = \mathbf{X\lambda X}^{-1}$ when

$$\mathbf{A} = \begin{bmatrix} 3 & 0 \\ -2 & 1 \end{bmatrix}.$$

Answer $\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ -2 & 1 \end{bmatrix}$

- In an eigenvalue problem of matrix $\mathbf{A}$, the goal is to find all pairs $(\lambda, \mathbf{x})$ such that $(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=0$. As the equation is homogeneous, a non-zero solution to $\mathbf{x}$ requires that the matrix is singular, i.e., $\det(\mathbf{A}-\lambda\mathbf{I})=0$. Hence

$$\det\begin{bmatrix} 3-\lambda & 0 \\ -2 & 1-\lambda \end{bmatrix} = (3-\lambda)(1-\lambda)=0 \;\Rightarrow\; \lambda_1 = 3 \quad \text{or} \quad \lambda_2 = 1.$$

- After finding the possible values of λ, the corresponding vectors (eigenvectors) are given by the original equation $(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=0$. Solution to $\mathbf{x}$ is not unique and any *non-zero* solution suffices (in practice one may choose one or more components of $\mathbf{x}$ and solve the equation for the remaining)

$$\lambda_1: \;\; \begin{bmatrix} 3-3 & 0 \\ -2 & 1-3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 0 \;\Rightarrow\; \mathbf{x}_1 = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} -1 \\ 1 \end{Bmatrix},$$

$$\lambda_2 : \begin{bmatrix} 3-1 & 0 \\ -2 & 1-1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 0 \quad \Rightarrow \quad \mathbf{x}_2 = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}.$$

- Matrix of eigenvalues $\boldsymbol{\lambda}$, matrix of eigenvectors $\mathbf{X}$ and its inverse $\mathbf{X}^{-1}$ are now

$$\boldsymbol{\lambda} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{X} = [\mathbf{x}_1 \; \mathbf{x}_2] = \left[\begin{array}{c:c} -1 & 0 \\ 1 & 1 \end{array}\right] \text{ and } \mathbf{X}^{-1} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}.$$

- Eigenvalue decomposition $\mathbf{A} = \mathbf{X}\boldsymbol{\lambda}\mathbf{X}^{-1}$ is a very useful representation of the original matrix (for example $f(\mathbf{A}) = \mathbf{X} diag[f(\lambda_1) \ldots f(\lambda_n)]\mathbf{X}^{-1}$)

$$\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ -2 & 1 \end{bmatrix}. \quad \Leftarrow$$

1.4 FE-CODE OF MEC-E8001

"Structure is a collection of elements connected by nodes. Geometry, displacement, temperature etc. of the structure are defined by the nodal values of coordinates, translation, rotation, temperature etc. of which some are known and some unknown."

DATA STRUCTURE

$prb = \{ele, fun\}$ where

$ele = \{prt_1, prt_2, \ldots\}$ .... elements

$fun = \{val_1, val_2, \ldots\}$ .... nodes

Element

$prt = \{typ, pro, geo\}$ where

$typ = \mathrm{BAR} \mid \mathrm{TORSION} \mid \mathrm{BEAM} \mid \mathrm{RIGID} \mid \ldots \mid$ .... model

$pro = \{p_1, p_2, \ldots, p_n\}$ ....properties

$geo = \mathrm{Point}[\{n_1\}] \mid \mathrm{Line}[\{n_1, n_2\}] \mid \mathrm{Polygon}[\{n_1, n_2, n_3\}] \mid \ldots \mid$.............................. geometry

Nodes

$val = \{crd, trn, rot\} \mid \{crd, trn, rot, tmp\}$ where

$crd = \{X, Y, Z\}$ .... structural coordinates

$trn = \{u_X, u_Y, u_Z\}$ .... translation components

$rot = \{\theta_X, \theta_Y, \theta_Z\}$ ....rotation components

$tmp = \vartheta$ .... temperature

DISPLACEMENT ANALYSIS

Constraint

$\{\text{JOINT},\{\} \mid \{\{\underline{u}_X, \underline{u}_Y, \underline{u}_Z\}\}, \text{Point}[\{n_1\}]\}$ ....displacement constraint

$\{\text{JOINT},\{\}, \text{Line}[\{n_1, n_2\}]\}$ ....displacement constraint

$\{\text{RIGID},\{\} \mid \{\{\underline{u}_X, \underline{u}_Y, \underline{u}_Z\},\{\underline{\theta}_X, \underline{\theta}_Y, \underline{\theta}_Z\}\}, \text{Point}[\{n_1\}]\}$ ... displacement/rotation constraint

$\{\text{RIGID},\{\}, \text{Line}[\{n_1, n_2\}]\}$ .... rigid constraint

$\{\text{SLIDER},\{n_X, n_Y, n_Z\}, \text{Point}[\{n_1\}]\}$ ....slider constraint

Force

$\{\text{FORCE},\{F_X, F_Y, F_Z\}, \text{Point}[\{n_1\}]\}$ .... point force

$\{\text{FORCE},\{F_X, F_Y, F_Z, M_X, M_Y, M_Z\}, \text{Point}[\{n_1\}]\}$ .... point load

$\{\text{FORCE},\{f_X, f_Y, f_Z\}, \text{Line}[\{n_1, n_2\}]\}$ .... distributed force

$\{\text{FORCE},\{f_X, f_Y, f_Z\}, \text{Polygon}[\{n_1, n_2, n_3\}]\}$ .... distributed force

Beam model

$\{\text{BAR},\{\{E\},\{A\},\{f_X, f_Y, f_Z\}\},\text{Line}[\{n_1,n_2\}]\}$ ....bar mode

$\{\text{TORSION},\{\{G\},\{J\},\{\{m_X, m_Y, m_Z\}\}\},\text{Line}[\{n_1,n_2\}]\}$ .... torsion mode

$\{\text{BEAM},\{\{E,G\},\{A,I_{yy},I_{zz}\},\{f_X, f_Y, f_Z\}\},\text{Line}[\{n_1,n_2\}]\}$ ....beam

$\{\text{BEAM},\{\{E,G\},\{A,I_{yy},I_{zz},\{j_X, j_Y, j_Z\}\},\{f_X, f_Y, f_Z\}\},\text{Line}[\{n_1,n_2\}]\}$ ....beam

Plate model

$\{\text{PLANE},\{\{E,\nu\},\{t\},\{f_X, f_Y, f_Z\}\},\text{Polygon}[\{n_1,n_2,n_3\}]\}$ ....thin slab mode

$\{\text{PLANE},\{\{E,\nu\},\{t\},\{f_X, f_Y, f_Z\}\},\text{Polygon}[\{n_1,n_2,n_3,n_4\}]\}$..................thin slab mode

$\{\text{PLATE},\{\{E,\nu\},\{t\},\{f_X, f_Y, f_Z\}\},\text{Polygon}[\{n_1,n_2,n_3\}]\}$ ....bending mode

$\{\text{SHELL},\{\{E,\nu\},\{t\},\{f_X, f_Y, f_Z\}\},\text{Polygon}[\{n_1,n_2,n_3\}]\}$ ....plate

Solid model

$\{\text{SOLID},\{\{E,\nu\},\{f_X, f_Y, f_Z\}\},\text{Tetrahedron}[\{n_1,n_2,n_3,n_4\}]\}$ ....solid

$\{\text{SOLID},\{\{E,\nu\},\{f_X, f_Y, f_Z\}\},\text{Hexahedron}[\{n_1,n_2,n_3,n_4,n_5,n_6,n_7,n_8\}]\}$ ....solid

$\{\text{SOLID},\{\{E,\nu\},\{f_X, f_Y, f_Z, m_X, m_Y, m_Z\}\},\text{Tetrahedron}[\{n_1,n_2,n_3,n_4\}]\}$ ....solid

Operations

$prb = \text{REFINE}[prb]$ ....refine structure representation

$\text{Out} = \text{FORMATTED}[prb]$ .... display problem definition

$\text{Out} = \text{STANDARDFORM}[prb]$ .... display virtual work expression

$\text{sol} = \text{SOLVE}[\{\text{DISP}\}, prb] \mid \text{SOLVE}[prb]$ ....solve the unknowns

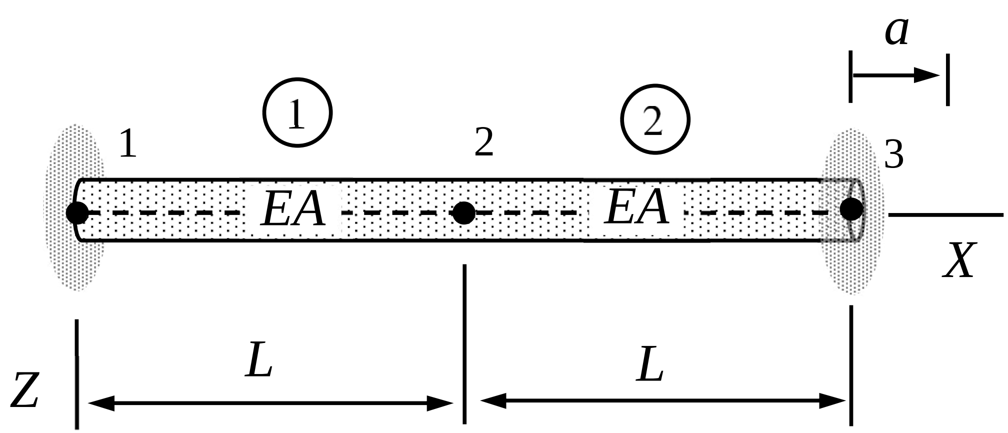
VIBRATION ANALYSIS…

STABILITY ANALYSIS…

NON-LINEAR ANALYSIS…

THERMOMECHANICAL ANALYSIS…

EXAMPLE 1.1 A connector bar is welded at its ends to rigid walls. If the right end wall displacement is a, determine the displacements of connection points 1, 2, and 3 and the forces acting on structural parts 1 and 2. Cross sectional area A and Young's modulus of the material E are constants and the displacement force relationship of a bar is the same as that of a spring with coefficient $k = EA/L$.



Answer $u_1^1 = 0,\ u_2^1 = u_2^2 = \frac{1}{2}a,\ u_3^2 = a,\ F_1^1 = F_2^2 = -\frac{1}{2}ka,\ F_2^1 = F_3^2 = \frac{1}{2}ka.$

- Problem description for the FE-solver uses duplicate node at the center point. Solution to the problem uses the replacement rule concept of Mathematica.

	model	properties	geometry
1	BAR	{{E}, {A}}	Line[{1, 2}]
2	BAR	{{E}, {A}}	Line[{3, 4}]
3	JOINT	{}	Point[{1}]
4	JOINT	{}	Line[{2, 3}]
5	JOINT	{a, 0, 0}	Point[{4}]

	$\{X, Y, Z\}$	$\{u_X, u_Y, u_Z\}$	$\{\theta_X, \theta_Y, \theta_Z\}$
1	{0, 0, 0}	{uX[1], 0, 0}	{0, 0, 0}
2	{L, 0, 0}	{uX[2], 0, 0}	{0, 0, 0}
3	{L, 0, 0}	{uX[3], 0, 0}	{0, 0, 0}
4	{2 L, 0, 0}	{uX[4], 0, 0}	{0, 0, 0}

$$\left\{FX[1] \to -\frac{a\,A\,E}{2\,L},\ FX[2] \to \frac{a\,A\,E}{2\,L},\ FX[4] \to \frac{a\,A\,E}{2\,L},\ uX[1] \to 0,\ uX[2] \to \frac{a}{2},\ uX[3] \to \frac{a}{2},\ uX[4] \to a\right\}$$