MEC-E8001 Finite Element Analysis, week 4/2022

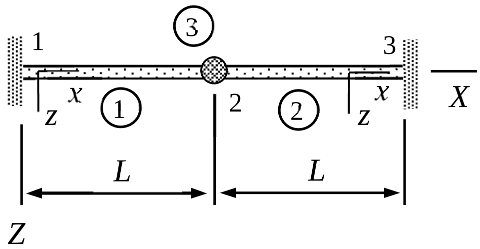
1. Determine the eigenvalue decomposition $\mathbf{A} = \mathbf{X}\boldsymbol{\lambda}\mathbf{X}^{-1}$ and $\sqrt{\mathbf{A}}$ when $\mathbf{A} = \begin{bmatrix} 4 & 0 \\ -1 & 1 \end{bmatrix}$.

Answer $\mathbf{A} = \mathbf{X}\boldsymbol{\lambda}\mathbf{X}^{-1} = \begin{bmatrix} -3 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1/3 & 0 \\ 1/3 & 1 \end{bmatrix}$ and $\sqrt{\mathbf{A}} = \pm\begin{bmatrix} 2 & 0 \\ -1/3 & 1 \end{bmatrix}$

2. Derive the consistent mass matrix $\mathbf{M}$ of a two-node beam element (bending in *xz*-plane). Assume that density is constant, cross section is constant, and the beam element is thin in the sense $t/h \ll 1$, so that $\delta w_{\Omega}^{\text{ine}} = -\delta w \rho A \ddot{w}$.

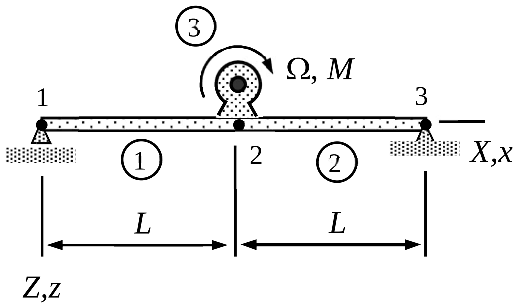
Answer $\mathbf{M} = \dfrac{\rho A h}{420}\left[\begin{array}{cc|cc} 156 & -22h & 54 & 13h \\ -22h & 4h^2 & -13h & -3h^2 \\ \hline 54 & -13h & 156 & 22h \\ 13h & -3h^2 & 22h & 4h^2 \end{array}\right]$

3. The *XZ*-plane structure shown consists of two *massless* beams and a homogeneous disk considered as a rigid body. Derive the equations of motion in terms of displacements u_{Z2} and θ_{Y2}. Young's modulus of the beam material and the second moment of area are E and I, and the mass and moment of inertia of the disk are m and J, respectively.



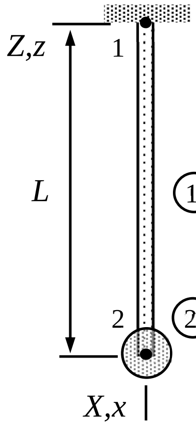
Answer $\dfrac{EI}{L^3}\begin{bmatrix} 24 & 0 \\ 0 & 8L^2 \end{bmatrix}\begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \end{Bmatrix} + \begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix}\begin{Bmatrix} \ddot{u}_{Z2} \\ \ddot{\theta}_{Y2} \end{Bmatrix} = 0$

4. The rotor of the machine shown rotates with angular speed Ω. Determine the bending stiffness EI so that the angular speed (free vibrations) of the foundation-machine system coincides with Ω. The foundation is modeled as two *massless* beams and the machine as a particle of mass *M*. Assume that $\theta_{Y1} = -\theta_{Y3}$ and $\theta_{Y2} = 0$.



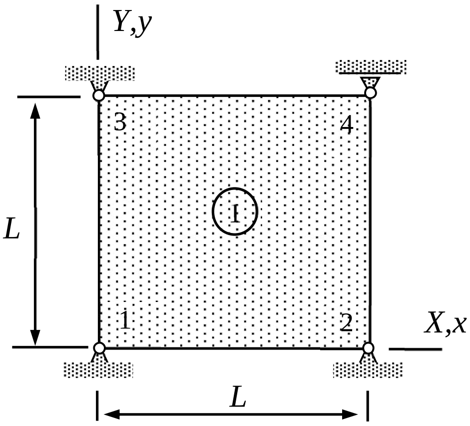
Answer $EI = \dfrac{1}{6} m L^3 \Omega^2$

5. *XZ*-plane structure shown consists of a beam and a homogeneous disk considered as a rigid body. Derive the equations of motion in terms of u_{Z2}, θ_{Y2} and determine the angular speeds of free vibrations. Assume that mass of the beam is negligible compared to that of the disk and that the beam is inextensible in the axial direction. Young's modulus E of the beam material and the second moment of area I are constants. Mass and moment of inertia of the disk are m and $\mathbf{J} = \frac{1}{5} mL^2 \mathbf{I}$, respectively.



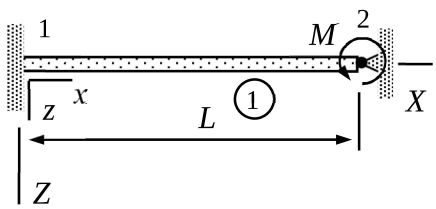
Answer $\omega_1 = \sqrt{\lambda_1} = \sqrt{2}\sqrt{\frac{EI}{mL^3}}$, $\omega_2 = \sqrt{\lambda_2} = \sqrt{30}\sqrt{\frac{EI}{mL^3}}$

6. Node 4 of a thin rectangular slab (assume plane stress conditions) is allowed to move horizontally and nodes 1, 2, and 3 are fixed. Derive the initial value problem giving as its solution the horizontal displacement $u_{X4}(t)$ of node 4 as function of time, if $u_{X4}(0) = U$ and $\dot{u}_{X4}(0) = 0$. Use just one bilinear element. Material parameters E, $\nu = 0$, ρ and thickness h of the slab are constants.



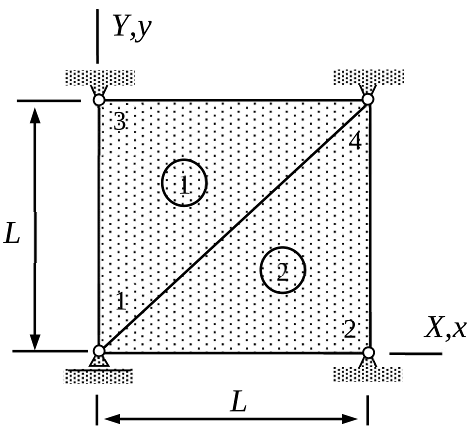
Answer $\ddot{u}_{X4} + \frac{9}{2}\frac{E}{L^2\rho} u_{X4} = 0 \quad t > 0$, $u_{X4}(0) = U$, $\dot{u}_{X4}(0) = 0$

7. The beam of the figure is subjected to moment M when $t < 0$. At $t = 0$, the moment is suddenly removed and the beam starts to vibrate. Derive the initial value problem giving $\theta_{Y2}(t)$ for $t > 0$. The beam is thin so that the rotational part of the inertia term is negligible. The geometrical quantities of the cross-section are *A*, *I* and the material constants *E* and ρ.



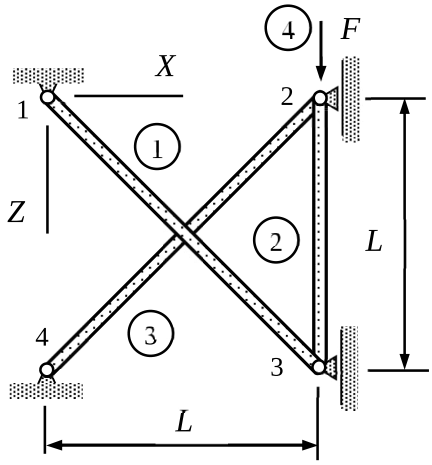
Answer $4\frac{EI}{L}\theta_{Y2} + \frac{\rho A L^3}{105}\ddot{\theta}_{Y2} = 0 \quad t > 0$, $\theta_{Y2}(0) = \frac{1}{4}\frac{ML}{EI}$, $\dot{\theta}_{Y2}(0) = 0$

8. Node 1 of a thin rectangular slab (assume plane stress conditions) is allowed to move horizontally at node 1 whereas nodes 2, 3 and 4 are fixed. Derive the expression of horizontal displacement $u_{X1}(t)$ of node 1 as function of time, if $u_{X1}(0) = U$ and $\dot{u}_{X1}(0) = 0$. Use two linear triangle elements. Material parameters E, ν, ρ, and thickness h of the slab are constants.



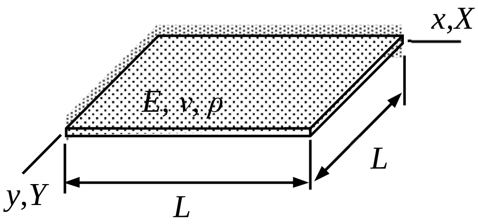
Answer $u_{X1}(t) = U\cos\left(t\sqrt{\frac{3}{2}\frac{3-\nu}{1-\nu^2}\frac{E}{\rho L^2}}\right) \quad t > 0$

9. Bars 1 and 3 of the structure shown are massless and bar 2 is rigid. Force F is acting on node 2. Determine the displacement $u_{Z2}(t)$ of node 2 for $t>0$, if the force is removed at $t=0$. Young's modulus of bars 1 and 3 is E and density of bar 2 is ρ. Cross-sectional area is constant A.



Answer $u_{Z2}(t)=F\dfrac{\sqrt{2}L}{EA}\cos(\dfrac{t}{L}\sqrt{\dfrac{E}{\sqrt{2}\rho}})\quad t>0$

10. A plate is simply supported on two edges and free on the other two edges as shown. Use the approximation $w(x,y,t)=a(t)xy/L^2$ to determine the transverse displacement as function of time $t>0$. Material properties E, ν, and ρ are constants and thickness of the plate is h. At $t=0$, initial conditions are $\dot{w}(x,y,0)=0$ and $w(x,y,0)=Uxy/L^2$. Assume that the plate is thin so that the rotation part of the inertia term is negligible.



Answer $w(x,y,t)=U\cos(\sqrt{3\dfrac{G}{\rho}}\dfrac{h}{L^2}t)\dfrac{xy}{L^2}\quad t>0$

Determine the eigenvalue decomposition $\mathbf{A} = \mathbf{X}\boldsymbol{\lambda}\mathbf{X}^{-1}$ and $\sqrt{\mathbf{A}}$ when $\mathbf{A} = \begin{bmatrix} 4 & 0 \\ -1 & 1 \end{bmatrix}$.

Solution

Let us solve for the eigenvalues first from $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$

$$\det\begin{bmatrix} 4-\lambda & 0 \\ -1 & 1-\lambda \end{bmatrix} = (4-\lambda)(1-\lambda) = 0 \quad \Leftrightarrow \quad \lambda = 1 \ \text{ or } \ \lambda = 4 .$$

The corresponding eigenvectors $\mathbf{x}$ follow from $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0$ when the eigenvalues are substituted there

$$\lambda_1 = 1: \ \begin{bmatrix} 4-1 & 0 \\ -1 & 1-1 \end{bmatrix}\begin{Bmatrix} \mathrm{a} \\ 1 \end{Bmatrix} = 0 \quad \Rightarrow \quad \mathrm{a} = 0 \quad \Rightarrow \quad \mathbf{x}_1 = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix},$$

$$\lambda_2 = 4: \ \begin{bmatrix} 4-4 & 0 \\ -1 & 1-4 \end{bmatrix}\begin{Bmatrix} \mathrm{a} \\ 1 \end{Bmatrix} = 0 \quad \Rightarrow \quad \mathrm{a} = -3 \quad \Rightarrow \quad \mathbf{x}_2 = \begin{Bmatrix} -3 \\ 1 \end{Bmatrix}.$$

Therefore

$$\mathbf{A} = \mathbf{X}\boldsymbol{\lambda}\mathbf{X}^{-1} = \begin{bmatrix} 0 & -3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 1/3 & 1 \\ -1/3 & 0 \end{bmatrix}.$$

Let us use the definition: if $\mathbf{A} = \mathbf{X}\boldsymbol{\lambda}\mathbf{X}^{-1}$ then $f(\mathbf{A}) = \mathbf{X}f(\boldsymbol{\lambda})\mathbf{X}^{-1}$. When applied to the present case of a square root

$$\sqrt{\mathbf{A}} = \mathbf{X}\left(\pm\sqrt{\boldsymbol{\lambda}}\right)\mathbf{X}^{-1} = \begin{bmatrix} 0 & -3 \\ 1 & 1 \end{bmatrix}\left(\pm\begin{bmatrix} \sqrt{1} & 0 \\ 0 & \sqrt{4} \end{bmatrix}\right)\begin{bmatrix} 1/3 & 1 \\ -1/3 & 0 \end{bmatrix} = \pm\begin{bmatrix} 2 & 0 \\ -1/3 & 1 \end{bmatrix}.$$

Derive the consistent mass matrix $\mathbf{M}$ of a two-node beam element (bending in *xz*-plane). Assume that density is constant, cross section is constant, and the beam element is thin in the sense $t/h \ll 1$, so that $\delta w_{\Omega}^{\text{ine}} = -\delta w \rho A \ddot{w}$.

Solution

The starting is the virtual work density of inertia forces and the element approximation of the beam model (see the formulae collection)

$$w(x,t) = \begin{Bmatrix} (1-\xi)^2(1+2\xi) \\ h(1-\xi)^2\xi \\ (3-2\xi)\xi^2 \\ h\xi^2(\xi-1) \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} u_{z1}(t) \\ -\theta_{y1}(t) \\ u_{z2}(t) \\ -\theta_{y2}(t) \end{Bmatrix} \Rightarrow \delta w(x,t) = \begin{Bmatrix} \delta u_{z1}(t) \\ \delta\theta_{y1}(t) \\ \delta u_{z2}(t) \\ \delta\theta_{y2}(t) \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} (1-\xi)^2(1+2\xi) \\ -h(1-\xi)^2\xi \\ (3-2\xi)\xi^2 \\ -h\xi^2(\xi-1) \end{Bmatrix},$$

$$\ddot{w}(x,t) = \begin{Bmatrix} (1-\xi)^2(1+2\xi) \\ -h\xi(1-\xi)^2 \\ (3-2\xi)\xi^2 \\ -h\xi^2(\xi-1) \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \ddot{u}_{z1}(t) \\ \ddot{\theta}_{y1}(t) \\ \ddot{u}_{z2}(t) \\ \ddot{\theta}_{y2}(t) \end{Bmatrix} \quad (\text{here } \xi = \frac{x}{h}).$$

Virtual work expression of the inertia forces consists of terms taking into account translation and rotation of the cross-section. Here, rotation part is assumed to be negligible so that

$$\delta w_{\Omega}^{\text{ine}} = -\delta w \rho A \ddot{w}.$$

When the approximation is substituted there, virtual work density takes the form

$$\delta w_{\Omega}^{\text{ine}} = -\begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\mathrm{T}} \rho A \begin{Bmatrix} (1-\xi)^2(1+2\xi) \\ -h\xi(1-\xi)^2 \\ (3-2\xi)\xi^2 \\ -h\xi^2(\xi-1) \end{Bmatrix} \begin{Bmatrix} (1-\xi)^2(1+2\xi) \\ -h\xi(1-\xi)^2 \\ (3-2\xi)\xi^2 \\ -h\xi^2(\xi-1) \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \ddot{u}_{z1} \\ \ddot{\theta}_{y1} \\ \ddot{u}_{z2} \\ \ddot{\theta}_{y2} \end{Bmatrix}.$$

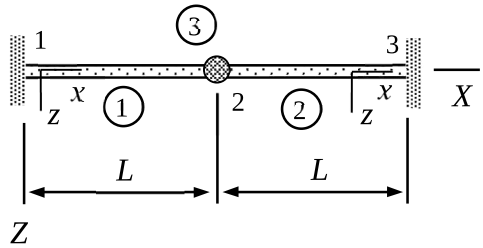
Integration over the spatial domain gives (use Mathematica in this step)

$$\delta W^{\text{ine}} = -\begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\mathrm{T}} \frac{\rho A h}{420} \begin{bmatrix} 156 & -22h & 54 & 13h \\ -22h & 4h^2 & -13h & -3h^2 \\ 54 & -13h & 156 & 22h \\ 13h & -3h^2 & 22h & 4h^2 \end{bmatrix} \begin{Bmatrix} \ddot{u}_{z1} \\ \ddot{\theta}_{y1} \\ \ddot{u}_{z2} \\ \ddot{\theta}_{y2} \end{Bmatrix}.$$

Therefore, the mass matrix

$$\mathbf{M} = \frac{\rho A h}{420}\begin{bmatrix} 156 & -22h & 54 & 13h \\ -22h & 4h^2 & -13h & -3h^2 \\ 54 & -13h & 156 & 22h \\ 13h & -3h^2 & 22h & 4h^2 \end{bmatrix}.$$

The XZ-plane structure shown consists of two *massless* beams and a homogeneous disk considered as a rigid body. Derive the equations of motion in terms of displacements u_{Z2} and θ_{Y2}. Young's modulus of the beam material and the second moment of area are E and I, and the mass and moment of inertia of the disk are m and J, respectively.



Solution

The non-zero displacement/rotation components of the structure are u_{Z2} and θ_{Y2}. Let us start with the element contributions. Since the beam is assumed to be massless, only the virtual work expressions of the internal forces (available in the formulae collection) is needed.

$$\delta W^1 = -\begin{Bmatrix} 0 \\ 0 \\ \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3}\begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ u_{Z2} \\ \theta_{Y2} \end{Bmatrix} = -\begin{Bmatrix} \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3}\begin{bmatrix} 12 & 6L \\ 6L & 4L^2 \end{bmatrix}\begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \end{Bmatrix},$$

$$\delta W^2 = -\begin{Bmatrix} \delta u_{Z2} \\ \delta\theta_{Y2} \\ 0 \\ 0 \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3}\begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix}\begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \\ 0 \\ 0 \end{Bmatrix} = -\begin{Bmatrix} \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3}\begin{bmatrix} 12 & -6L \\ -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \end{Bmatrix}.$$

Element contribution of the rigid body (formulae collection) simplifies to

$$\delta W^3 = -\begin{Bmatrix} 0 \\ 0 \\ \delta u_{Z2} \end{Bmatrix}^{\mathrm{T}} m\begin{Bmatrix} 0 \\ 0 \\ \ddot{u}_{Z2} \end{Bmatrix} - \begin{Bmatrix} 0 \\ \delta\theta_{Y2} \\ 0 \end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix} 0 \\ J\ddot{\theta}_{Y2} \\ 0 \end{Bmatrix} = -\begin{Bmatrix} \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}}\begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix}\begin{Bmatrix} \ddot{u}_{Z2} \\ \ddot{\theta}_{Y2} \end{Bmatrix}.$$

Virtual work expression of structure is the sum of element contributions.

$$\delta W = \delta W^1 + \delta W^2 + \delta W^3 = -\begin{Bmatrix} \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EI}{L^3}\begin{bmatrix} 24 & 0 \\ 0 & 8L^2 \end{bmatrix}\begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \end{Bmatrix} + \begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix}\begin{Bmatrix} \ddot{u}_{Z2} \\ \ddot{\theta}_{Y2} \end{Bmatrix}\right).$$

Finally, principle of virtual work and the fundamental lemma of variation calculus imply a set of ordinary differential equations:

$$\frac{EI}{L^3}\begin{bmatrix} 24 & 0 \\ 0 & 8L^2 \end{bmatrix}\begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \end{Bmatrix} + \begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix}\begin{Bmatrix} \ddot{u}_{Z2} \\ \ddot{\theta}_{Y2} \end{Bmatrix} = 0. \quad \leftarrow$$

The angular speeds of free vibrations can be deduced from the stiffness and mass matrix of the equation system

$$\mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix} \text{ and } \mathbf{K} = \frac{EI}{L^3}\begin{bmatrix} 24 & 0 \\ 0 & 8L^2 \end{bmatrix} \Rightarrow \mathbf{\Omega}^2 = \mathbf{M}^{-1}\mathbf{K} = \frac{EI}{L^3}\begin{bmatrix} 24/m & 0 \\ 0 & 8L^2/J \end{bmatrix}.$$

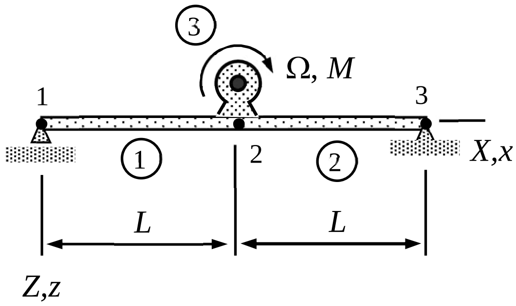
The angular speeds of free vibrations are the eigenvalues of $\mathbf{\Omega}$. Let us start with the eigenvalues of $\mathbf{\Omega}^2 = \mathbf{M}^{-1}\mathbf{K}$

$$\det\left(\frac{EI}{L^3}\begin{bmatrix} 24/m & 0 \\ 0 & 8L^2/J \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \left(24\frac{EI}{mL^3} - \lambda\right)\left(8\frac{EI}{JL} - \lambda\right) = 0 \quad \Rightarrow \lambda \in \left\{24\frac{EI}{mL^3}, 8\frac{EI}{JL}\right\}.$$

Eigenvalues of $\mathbf{\Omega}$ are square roots of eigenvalues of $\mathbf{\Omega}^2$

$$\omega_1 = \sqrt{\lambda_1} = \sqrt{24\frac{EI}{mL^3}} \quad \text{and} \quad \omega_2 = \sqrt{\lambda_2} = \sqrt{8\frac{EI}{JL}}.$$

The rotor of the machine shown rotates with angular speed Ω. Determine the bending stiffness EI so that the (smallest) angular speed of free vibrations of the foundation-machine system coincides with Ω. The foundation is modeled as two *massless* beams and the machine as particle of mass M. Assume that $\theta_{Y1} = -\theta_{Y3}$ and $\theta_{Y2} = 0$.



Solution

The non-zero displacement/rotation components of the structure are u_{Z2}, θ_{Y1}, and $\theta_{Y3} = -\theta_{Y1}$. Let us start with the element contributions. Since the beam is assumed to be massless, only the virtual work expressions of the internal forces (available in formulae collection) is needed.

$$\delta W^1 = -\begin{Bmatrix} 0 \\ \delta\theta_{Y1} \\ \delta u_{Z2} \\ 0 \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3}\begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ \theta_{Y1} \\ u_{Z2} \\ 0 \end{Bmatrix} = -\begin{Bmatrix} \delta\theta_{Y1} \\ \delta u_{Z2} \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3}\begin{bmatrix} 4L^2 & 6L \\ 6L & 12 \end{bmatrix}\begin{Bmatrix} \theta_{Y1} \\ u_{Z2} \end{Bmatrix},$$

$$\delta W^2 = -\begin{Bmatrix} \delta u_{Z2} \\ 0 \\ 0 \\ -\delta\theta_{Y1} \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3}\begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix}\begin{Bmatrix} u_{Z2} \\ 0 \\ 0 \\ -\theta_{Y1} \end{Bmatrix} = -\begin{Bmatrix} \delta\theta_{Y1} \\ \delta u_{Z2} \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3}\begin{bmatrix} 4L^2 & 6L \\ 6L & 12 \end{bmatrix}\begin{Bmatrix} \theta_{Y1} \\ u_{Z2} \end{Bmatrix}.$$

Element contribution of the rigid body (formulae collection) simplifies to

$$\delta W^3 = -\begin{Bmatrix} 0 \\ 0 \\ \delta u_{Z2} \end{Bmatrix}^{\mathrm{T}} m\begin{Bmatrix} 0 \\ 0 \\ \ddot{u}_{Z2} \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} = -\begin{Bmatrix} \delta\theta_{Y1} \\ \delta u_{Z2} \end{Bmatrix}^{\mathrm{T}}\begin{bmatrix} 0 & 0 \\ 0 & m \end{bmatrix}\begin{Bmatrix} \ddot{\theta}_{Y1} \\ \ddot{u}_{Z2} \end{Bmatrix}.$$

Virtual work expression of structure is the sum of element contributions.

$$\delta W = \delta W^1 + \delta W^2 + \delta W^3 = -\begin{Bmatrix} \delta\theta_{Y1} \\ \delta u_{Z2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EI}{L^3}\begin{bmatrix} 8L^2 & 12L \\ 12L & 24 \end{bmatrix}\begin{Bmatrix} \theta_{Y1} \\ u_{Z2} \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & m \end{bmatrix}\begin{Bmatrix} \ddot{\theta}_{Y1} \\ \ddot{u}_{Z2} \end{Bmatrix}\right).$$

Finally, principle of virtual work and the fundamental lemma of variation calculus imply a differential algebraic system (DAE):

$$\left(\frac{EI}{L^3}\begin{bmatrix} 8L^2 & 12L \\ 12L & 24 \end{bmatrix}\begin{Bmatrix} \theta_{Y1} \\ u_{Z2} \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & m \end{bmatrix}\begin{Bmatrix} \ddot{\theta}_{Y1} \\ \ddot{u}_{Z2} \end{Bmatrix} = 0.\right.$$

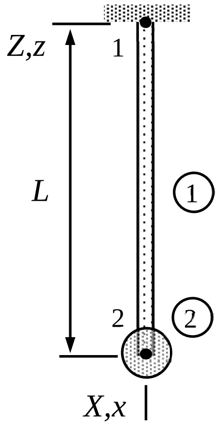
Let us eliminate the rotation from the differential equation by using the algebraic equation $8L^2\theta_{Y1} + 12Lu_{Z2} = 0 \quad \Leftrightarrow \quad \theta_{Y1} = -u_{Z2}3/(2L)$. Therefore

$$\frac{EI}{L^3}(12L\theta_{Y1}+24u_{Z2})+m\ddot{u}_{Z2}=0 \quad \Leftrightarrow \quad \frac{EI}{L^3}6u_{Z2}+m\ddot{u}_{Z2}=0 \quad \text{or} \quad \ddot{u}_{Z2}+6\frac{EI}{mL^3}u_{Z2}=0\,.$$

The angular speed of free vibrations should match the angular speed of rotor (the condition of resonance and increasing amplitude in vibrations)

$$\omega=\sqrt{6\frac{EI}{mL^3}}=\Omega \quad \Rightarrow \quad EI=\frac{1}{6}mL^3\Omega^2\,.$$

The XZ-plane structure shown consists of a beam and a homogeneous disk considered as a rigid body. Derive the equations of motion in terms of u_{Z2}, θ_{Y2} and determine the angular speeds of free vibrations. Assume that mass of the beam is negligible compared to that of the disk and that the beam is inextensible in the axial direction. Young's modulus E of the beam material and the second moment of area I are constants. Mass and moment of inertia of the disk are m and $\mathbf{J} = \frac{1}{5} mL^2 \mathbf{I}$, respectively.



Solution

Virtual work expressions of the beam and rigid body elements are given by (inertia contribution is omitted from the beam contribution and rigid body has only the inertia part)

$$\delta W^1 = -\begin{Bmatrix} 0 \\ 0 \\ \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3} \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_{Z2} \\ \theta_{Y2} \end{Bmatrix} = -\begin{Bmatrix} \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 12\dfrac{EI}{L^3} & 6\dfrac{EI}{L^2} \\ 6\dfrac{EI}{L^2} & 4\dfrac{EI}{L} \end{bmatrix} \begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \end{Bmatrix},$$

$$\delta W^2 = -\begin{Bmatrix} 0 \\ 0 \\ \delta u_{Z2} \end{Bmatrix}^{\mathrm{T}} m \begin{Bmatrix} 0 \\ 0 \\ \ddot{u}_{Z2} \end{Bmatrix} - \begin{Bmatrix} 0 \\ \delta\theta_{Y2} \\ 0 \end{Bmatrix}^{\mathrm{T}} \mathbf{J} \begin{Bmatrix} 0 \\ \ddot{\theta}_{Y2} \\ 0 \end{Bmatrix} = -\begin{Bmatrix} \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} m & 0 \\ 0 & \dfrac{mL^2}{5} \end{bmatrix} \begin{Bmatrix} \ddot{u}_{Z2} \\ \ddot{\theta}_{Y2} \end{Bmatrix}.$$

Principle of virtual work $\delta W = \delta W^1 + \delta W^2 = 0 \ \forall \delta \mathrm{a}$ gives

$$\delta W = -\begin{Bmatrix} \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \left(\begin{bmatrix} 12\dfrac{EI}{L^3} & 6\dfrac{EI}{L^2} \\ 6\dfrac{EI}{L^2} & 4\dfrac{EI}{L} \end{bmatrix} \begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \end{Bmatrix} + \begin{bmatrix} m & 0 \\ 0 & mL^2/5 \end{bmatrix} \begin{Bmatrix} \ddot{u}_{Z2} \\ \ddot{\theta}_{Y2} \end{Bmatrix} \right) = 0 \quad \Rightarrow$$

$$\begin{bmatrix} 12\dfrac{EI}{L^3} & 6\dfrac{EI}{L^2} \\ 6\dfrac{EI}{L^2} & 4\dfrac{EI}{L} \end{bmatrix} \begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \end{Bmatrix} + \begin{bmatrix} m & 0 \\ 0 & \dfrac{mL^2}{5} \end{bmatrix} \begin{Bmatrix} \ddot{u}_{Z2} \\ \ddot{\theta}_{Y2} \end{Bmatrix} = 0. \quad \leftarrow$$

The angular speeds ω_1 and ω_2 of free vibrations can be obtained (as square roots of the eigenvalues) from the eigenvalue decomposition $\mathbf{\Omega}^2 = \mathbf{M}^{-1}\mathbf{K} = \mathbf{X}\boldsymbol{\omega}^2\mathbf{X}^{-1}$. Let us start with

$$\mathbf{\Omega}^2 = \mathbf{M}^{-1}\mathbf{K} = \begin{bmatrix} m & 0 \\ 0 & \dfrac{mL^2}{5} \end{bmatrix}^{-1} \begin{bmatrix} 12\dfrac{EI}{L^3} & 6\dfrac{EI}{L^2} \\ 6\dfrac{EI}{L^2} & 4\dfrac{EI}{L} \end{bmatrix} = \begin{bmatrix} 12\dfrac{EI}{mL^3} & 6\dfrac{EI}{mL^2} \\ 30\dfrac{EI}{mL^4} & 20\dfrac{EI}{mL^3} \end{bmatrix}.$$

and continue with the characteristic equation

$$\det(\mathbf{\Omega}^2 - \lambda\mathbf{I}) = (12\frac{EI}{mL^3} - \lambda)(20\frac{EI}{mL^3} - \lambda) - 180\frac{EI}{mL^2}\frac{EI}{mL^4} = 0$$

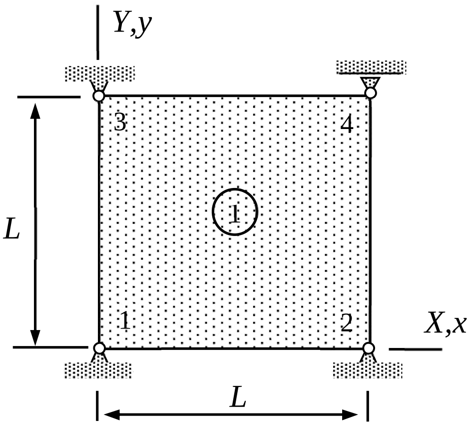
giving the eigenvalue solutions

$$\lambda_1 = 2\frac{EI}{mL^3} \quad \text{and} \quad \lambda_2 = 30\frac{EI}{mL^3}.$$

Finally, the angular speeds are square roots of the eigenvalues

$$\omega_1 = \sqrt{\lambda_1} = \sqrt{2}\sqrt{\frac{EI}{mL^3}} \quad \text{and} \quad \omega_2 = \sqrt{\lambda_2} = \sqrt{30}\sqrt{\frac{EI}{mL^3}}.$$

Node 4 of a thin rectangular slab (assume plane stress conditions) is allowed to move horizontally and nodes 1, 2, and 3 are fixed. Derive the initial value problem giving as its solution the horizontal displacement $u_{X4}(t)$ of node 4 as function of time, if $u_{X4}(0)=U$ and $\dot{u}_{X4}(0)=0$. Use just one bilinear element. Material parameters E, $\nu=0$, ρ and thickness h of the slab are constants.



Solution

Let us use the xy – coordinate system of the figure as the material coordinate system for the thin slab element 1. Only the shape function of node 4 is needed in the approximations:

$$u=\frac{x}{L}\frac{y}{L}u_{X4},\quad \frac{\partial u}{\partial x}=\frac{1}{L}\frac{y}{L}u_{X4},\quad \frac{\partial u}{\partial y}=\frac{x}{L}\frac{1}{L}u_{X4},\quad \ddot{u}=\frac{x}{L}\frac{y}{L}\ddot{u}_{X4}\ \text{ and }\ v=0.$$

When the approximations are substituted there, virtual work densities of internal and inertia forces simplify to (here $\nu=0$)

$$\delta w_{\Omega}^{\text{int}}=-\begin{Bmatrix}\dfrac{y}{L^2}\delta u_{X4}\\ 0\\ \dfrac{x}{L^2}\delta u_{X4}\end{Bmatrix}^{\mathrm{T}}\frac{hE}{2}\begin{bmatrix}2&0&0\\0&2&0\\0&0&1\end{bmatrix}\begin{Bmatrix}\dfrac{y}{L^2}u_{X4}\\ 0\\ \dfrac{x}{L^2}u_{X4}\end{Bmatrix}=-\delta u_{X4}u_{X4}\frac{hE}{L^4}(y^2+\frac{1}{2}x^2),$$

$$\delta w_{\Omega}^{\text{ine}}=-\begin{Bmatrix}\delta u\\ \delta v\end{Bmatrix}^{\mathrm{T}}h\rho\begin{Bmatrix}\ddot{u}\\ \ddot{v}\end{Bmatrix}=-\begin{Bmatrix}\delta u_{X4}\dfrac{x}{L}\dfrac{y}{L}\\ 0\end{Bmatrix}^{\mathrm{T}}h\rho\begin{Bmatrix}\ddot{u}_{X4}\dfrac{x}{L}\dfrac{y}{L}\\ 0\end{Bmatrix}=-\delta u_{X4}\ddot{u}_{X4}\frac{h\rho}{L^4}x^2y^2.$$

Virtual work expressions are obtained by integrating the densities over the domain occupied by the element

$$\delta W^{\text{int}}=\int_0^L\int_0^L\delta w_{\Omega}^{\text{int}}dydx=-\delta u_{X4}u_{X4}\frac{hE}{2},$$

$$\delta W^{\text{ine}}=\int_0^L\int_0^L\delta w_{\Omega}^{\text{ine}}dxdy=-\delta u_{X4}\ddot{u}_{X4}\frac{1}{9}h\rho L^2.$$

Virtual work expression is the sum of the terms

$$\delta W=\delta W^{\text{int}}+\delta W^{\text{ine}}=-\delta u_{X4}(\frac{hE}{2}u_{X4}+\frac{1}{9}h\rho L^2\ddot{u}_{X4}).$$

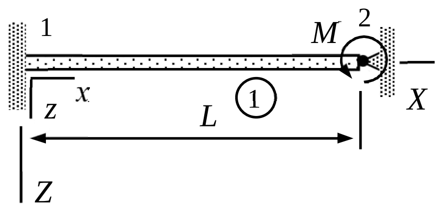
Principle of virtual work $\delta W=0\quad\forall\delta\mathbf{a}$ and the fundamental lemma of variation calculus $\delta\mathbf{a}^{\mathrm{T}}\mathbf{F}=0\ \forall\delta\mathbf{a}\quad\Leftrightarrow\quad\mathbf{F}=0$ imply the ordinary differential equation

$$\frac{hE}{2}u_{X4}+\frac{1}{9}h\rho L^2\ddot{u}_{X4}=0\,.$$

Initial value problem consists of the second order ordinary differential equation above and additional conditions at $t=0$

$$\ddot{u}_{X4}+\frac{9}{2}\frac{E}{L^2\rho}u_{X4}=0 \quad t>0 \quad \text{and} \quad u_{X4}=U\,,\ \dot{u}_{X4}=0 \quad \text{at} \quad t=0\,.$$

The beam of the figure is subjected to moment M when $t<0$. At $t=0$, the moment is suddenly removed and the beam starts to vibrate. Derive the initial value problem giving $\theta_{Y2}(t)$ for $t>0$. The beam is thin so that the rotational part of the inertia term is negligible. The geometrical quantities of the cross-section are A, I and the material constants E and ρ.



Solution

Virtual work expression consists of parts coming from internal and inertial forces. Finding the equation of motion is the first thing to do. The beam element contributions needed in the problem are (the term having to do with rotational inertia is omitted)

$$\delta W^{\text{int}} = -\begin{Bmatrix} 0 \\ 0 \\ 0 \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3} \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \theta_{Y2} \end{Bmatrix} = -\delta\theta_{Y2} 4\frac{EI}{L}\theta_{Y2},$$

$$\delta W^{\text{ine}} = -\begin{Bmatrix} 0 \\ 0 \\ 0 \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{\rho AL}{420} \begin{bmatrix} 156 & -22L & 54 & 13L \\ -22L & 4L^2 & -13L & -3L^2 \\ 54 & -13L & 156 & 22L \\ 13L & -3L^2 & 22L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \ddot{\theta}_{Y2} \end{Bmatrix} = -\delta\theta_{Y2} \frac{\rho AL^3}{105}\ddot{\theta}_{Y2}$$

giving

$$\delta W^1 = -\delta\theta_{Y2}(4\frac{EI}{L}\theta_{Y2} + \frac{\rho AL^3}{105}\ddot{\theta}_{Y2}).$$

In terms of moment $P(t)$ (positive in the positive direction of Y-axis) which is piecewise constant in time so that $P(t)=M \quad t\le 0$ and $P(t)=0 \quad t>0$, the element contribution of the moment is

$$\delta W^2 = \delta\theta_{Y2}P.$$

Virtual work expression is the sum of element contributions:

$$\delta W = \delta W^1 + \delta W^2 = -\delta\theta_{Y2}(4\frac{EI}{L}\theta_{Y2} + \frac{\rho AL^3}{105}\ddot{\theta}_{Y2} - P) = 0.$$

Principle of virtual work and the fundamental lemma of variation calculus imply the ordinary differential equation

$$\delta W = -\delta\theta_{Y2}(4\frac{EI}{L}\theta_{Y2} + \frac{\rho AL^3}{105}\ddot{\theta}_{Y2} - P) = 0 \quad \forall\delta\theta_{Y2} \quad \Leftrightarrow$$

$$4\frac{EI}{L}\theta_{Y2}+\frac{\rho AL^3}{105}\ddot{\theta}_{Y2}-P=0.$$

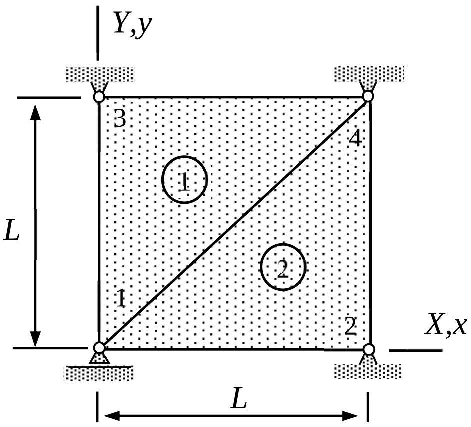
When $t \le 0$, external moment $P = M$ is acting on node 2 and the system is at rest. Therefore, the equation of motion becomes an equilibrium equation giving as its solution the initial rotation

$$\theta_{Y2}=\frac{1}{4}\frac{ML}{EI}.$$

When $t > 0$, external moment is zero and acceleration does not vanish. The initial value problem giving as its solution $\theta_{Y2}(t)$ for $t > 0$ takes the form

$$4\frac{EI}{L}\theta_{Y2}+\frac{\rho AL^3}{105}\ddot{\theta}_{Y2}=0 \quad t>0, \quad \theta_{Y2}(0)=\frac{1}{4}\frac{ML}{EI}, \quad \text{and} \quad \dot{\theta}_{Y2}(0)=0 .$$

Node 1 of a thin rectangular slab (assume plane stress conditions) is allowed to move horizontally at node 1 whereas nodes 2, 3 and 4 are fixed. Derive the expression of horizontal displacement $u_{X1}(t)$ of node 1 as function of time, if $u_{X1}(0)=U$ and $\dot{u}_{X1}(0)=0$. Use two linear triangle elements. Material parameters E, ν, ρ and thickness h of the slab are constants.



Solution

Let us use the $xy-$coordinate system of the figure as the material coordinate system for the thin slab elements 1 and 2. Only the displacement $u_{X1}(t)$ of node 1 in the $X-$direction matters.

Shape functions of element 1 can be deduced from the figure. However, only the shape function $N_1 = 1 - y/L$ is needed as the other nodes are fixed. Approximations to the in-plane displacement components are $v \equiv 0$ and

$$u = (1-\frac{y}{L})u_{X1} \;\Rightarrow\; \frac{\partial u}{\partial x} = 0,\; \frac{\partial u}{\partial y} = -\frac{1}{L}u_{X1},\; \text{ and } \; \ddot{u} = (1-\frac{y}{L})\ddot{u}_{X1}.$$

When the approximations above are substituted there, virtual work densities of internal and inertia forces simplify to

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} 0 \\ 0 \\ -\delta u_{X1}/L \end{Bmatrix}^{\mathrm{T}} \frac{hE}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ -u_{X1}/L \end{Bmatrix} = -\delta u_{X1}\frac{hE}{2L^2(1+\nu)}u_{X1},$$

$$\delta w_\Omega^{\text{ine}} = -\begin{Bmatrix} \delta u_{X1}(1-y/L) \\ 0 \end{Bmatrix}^{\mathrm{T}} h\rho \begin{Bmatrix} \ddot{u}_{X1}(1-y/L) \\ 0 \end{Bmatrix} = -\delta u_{X1}(1-\frac{y}{L})^2 h\rho \ddot{u}_{X1}.$$

Integration over the domain occupied by the element gives the virtual work expression. The limits of the double integral over a triangle are not constants (equation of the tilted edge is $y = x$)

$$\delta W^1 = \int_0^L \int_x^L (\delta w_\Omega^{\text{int}} + \delta w_\Omega^{\text{ine}}) dy dx \quad \Rightarrow$$

$$\delta W^1 = \int_0^L [-\delta u_{X1}\frac{hE}{2L^2(1+\nu)}u_{X1}(L-x) - \frac{L}{3}\delta u_{X1}(1-\frac{x}{L})^3 h\rho \ddot{u}_{X1}]dx \quad \Rightarrow$$

$$\delta W^1 = -\delta u_{X1}\frac{h}{12}(3\frac{E}{1+\nu}u_{X1} + \rho L^2 \ddot{u}_{X1}).$$

In the same manner, shape functions of element 2 can be deduced from the figure. Only $N_1 = 1 - x/L$ is needed as the other nodes are fixed. Approximations to the in-plane displacement components are $v \equiv 0$ and

$$u = (1-\frac{x}{L})u_{X1} \;\Rightarrow\; \frac{\partial u}{\partial y} = 0\,,\; \frac{\partial u}{\partial x} = -\frac{1}{L}u_{X1}\,,\;\text{ and }\; \ddot{u} = (1-\frac{x}{L})\ddot{u}_{X1}.$$

When the approximations are substituted there, virtual work densities of internal and inertia forces simplify to

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} -\delta u_{X1}/L \\ 0 \\ 0 \end{Bmatrix}^{\mathrm{T}} \frac{hE}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}\begin{Bmatrix} -u_{X1}/L \\ 0 \\ 0 \end{Bmatrix} = -\delta u_{X1}\frac{hE}{L^2(1-\nu^2)}u_{X1},$$

$$\delta w_{\Omega}^{\text{ine}} = -\begin{Bmatrix} (1-x/L)\delta u_{X1} \\ 0 \end{Bmatrix}^{\mathrm{T}} h\rho\begin{Bmatrix} (1-x/L)\ddot{u}_{X1} \\ 0 \end{Bmatrix} = -\delta u_{X1}(1-\frac{x}{L})^2 h\rho\ddot{u}_{X1}.$$

Integration over the domain occupied by the element gives the virtual work expression (notice the limits of the double integral and the order of the integrations)

$$\delta W^2 = \int_0^L \int_0^x (\delta w_{\Omega}^{\text{int}} + \delta w_{\Omega}^{\text{ine}})dydx \quad\Rightarrow$$

$$\delta W^2 = \int_0^L [-\delta u_{X1}\frac{hE}{L^2(1-\nu^2)}u_{X1}x - \frac{1}{L}\delta u_{X1}(1-\frac{x}{L})^2 h\rho\ddot{u}_{X1}x]dx \quad\Rightarrow$$

$$\delta W^2 = -\delta u_{X1}\frac{h}{12}(6\frac{E}{1-\nu^2}u_{X1} + \rho L^2\ddot{u}_{X1}).$$

Virtual work expression of a structure is sum over the element contributions

$$\delta W = \delta W^1 + \delta W^2 = -\delta u_{X1}\frac{h}{12}(3\frac{E}{1+\nu}u_{X1} + \rho L^2\ddot{u}_{X1}) - \delta u_{X1}\frac{h}{12}(6\frac{E}{1-\nu^2}u_{X1} + \rho L^2\ddot{u}_{X1}) \quad\Leftrightarrow$$

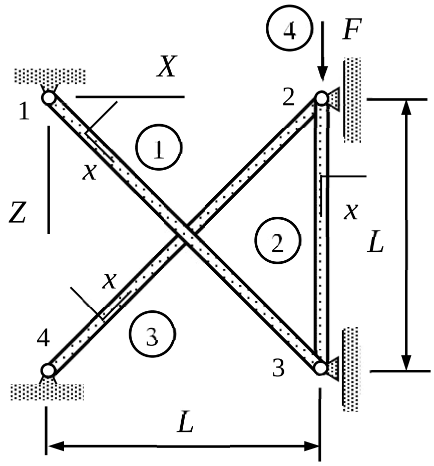
$$\delta W = -\delta u_{X1}\frac{h}{12}(\frac{3-\nu}{1-\nu^2}3Eu_{X1} + 2\rho L^2\ddot{u}_{X1}).$$

Principle of virtual work $\delta W = 0 \;\; \forall\delta\mathbf{a}$ and the fundamental lemma of variation calculus $\delta\mathbf{a}^{\mathrm{T}}\mathbf{F} = 0 \;\forall\delta\mathbf{a} \;\Leftrightarrow\; \mathbf{F} = 0$ imply

$$\frac{3-\nu}{1-\nu^2}3Eu_{X1} + 2\rho L^2\ddot{u}_{X1} = 0 \quad\text{or}\quad \ddot{u}_{X1} + \Omega^2 u_{X1} = 0 \quad\text{in which}\quad \Omega^2 = \frac{3}{2}\frac{3-\nu}{1-\nu^2}\frac{E}{\rho L^2}.$$

What remains, is solving for the displacement from the ordinary differential equation above for $t > 0$ and the initial conditions $u_{X1}(0) = U$ and $\dot{u}_{X1}(0) = 0$. Solution to equations is (this can be shown, e.g., by substituting the solution in the equations above)

$$u_{X1}(t) = U\cos(\sqrt{\frac{3}{2}\frac{3-\nu}{1-\nu^2}\frac{E}{\rho L^2}}t) \quad t > 0.$$



Bars 1 and 3 of the structure shown are massless and bar 2 is rigid. Force F is acting on node 2. Determine the displacement $u_{Z2}(t)$ of node 2 for $t>0$, if the force is removed at $t=0$. Young's modulus of bars 1 and 3 is E and density of bar 2 is ρ. Cross-sectional area is constant A.

Solution

Only the displacement of nodes 2 and 3 in the Z-direction matter. As bar 2 is known to be rigid, vertical displacements of nodes 2 and 3 coincide i.e. $u_{Z2}=u_{Z3}$. Bar element contributions of the formulae collection are

$$\delta W^{\text{int}}=-\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}^{\mathrm{T}}\frac{EA}{h}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_{x1}\\ u_{x2}\end{Bmatrix} \text{ and } \delta W^{\text{ine}}=-\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}^{\mathrm{T}}\frac{\rho Ah}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\begin{Bmatrix}\ddot{u}_{x1}\\ \ddot{u}_{x2}\end{Bmatrix}.$$

From the figure, the nodal displacement and length of bar 1 are $u_{x1}=0$, $u_{x2}=u_{Z2}/\sqrt{2}$ and $h=\sqrt{2}L$. As the bar is assumed to be massless, inertia term vanishes and

$$\delta W^{1}=-\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}^{\mathrm{T}}\frac{EA}{h}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_{x1}\\ u_{x2}\end{Bmatrix}=-\begin{Bmatrix}0\\ \delta u_{Z2}\end{Bmatrix}^{\mathrm{T}}\frac{EA}{\sqrt{8}L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}0\\ u_{Z2}\end{Bmatrix}=-\delta u_{Z2}\frac{EA}{\sqrt{8}L}u_{Z2}.$$

The relationships for bar 2 are $u_{x1}=u_{Z2}$, $u_{x2}=u_{Z2}$ and $h=L$. As the axial displacements coincide, internal part vanishes and

$$\delta W^{2}=-\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}^{\mathrm{T}}\frac{\rho Ah}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\begin{Bmatrix}\ddot{u}_{x1}\\ \ddot{u}_{x2}\end{Bmatrix}=-\begin{Bmatrix}\delta u_{Z2}\\ \delta u_{Z2}\end{Bmatrix}^{\mathrm{T}}\frac{\rho AL}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\begin{Bmatrix}\ddot{u}_{Z2}\\ \ddot{u}_{Z2}\end{Bmatrix}=-\rho AL\delta u_{Z2}\ddot{u}_{Z2}.$$

The relationships for bar 3 are $u_{x1}=0$, $u_{x2}=-u_{Z2}/\sqrt{2}$ and $h=\sqrt{2}L$. As the bar is assumed to be massless

$$\delta W^{3}=-\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}^{\mathrm{T}}\frac{EA}{h}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_{x1}\\ u_{x2}\end{Bmatrix}=-\begin{Bmatrix}0\\ -\delta u_{Z2}\end{Bmatrix}^{\mathrm{T}}\frac{EA}{\sqrt{8}L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}0\\ -u_{Z2}\end{Bmatrix}=-\delta u_{Z2}\frac{EA}{\sqrt{8}L}u_{Z2}.$$

Point force $P(t)$ acting on node 2 is piecewise constant in time so that $P(t)=F$ $t\le 0$ and $P(t)=0$ $t>0$. Virtual work expression is

$$\delta W^{4}=\delta u_{Z2}P.$$

Virtual work expression of the structure is the sum of element contributions

$$\delta W = \delta W^1 + \delta W^2 + \delta W^3 + \delta W^4 = -\delta u_{Z2}\left(\frac{EA}{\sqrt{2}L} u_{Z2} + \rho A L \ddot{u}_{Z2} - P\right).$$

Principle of virtual work $\delta W = 0 \quad \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus $\delta \mathbf{a}^{\mathrm{T}} \mathbf{F} = 0 \ \forall \delta \mathbf{a} \quad \Leftrightarrow \quad \mathbf{F} = 0$ imply that

$$\frac{EA}{\sqrt{2}L} u_{Z2} + \rho A L \ddot{u}_{Z2} - P = 0.$$

When $t \le 0$, u_{Z2} does not depend on time and therefore $\ddot{u}_{Z2} = \dot{u}_{Z2} = 0$. As the second derivative vanishes and $P = F$, the ordinary differential equation simplifies to an algebraic one giving

$$\frac{EA}{\sqrt{2}L} u_{Z2} - F = 0 \quad \Leftrightarrow \quad u_{Z2} = \frac{\sqrt{2}L}{EA} F \quad \text{when} \quad t \le 0.$$

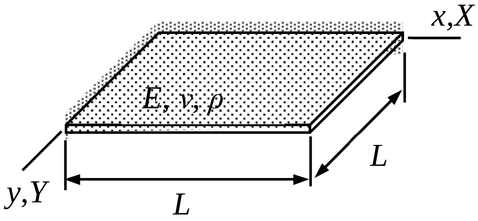
When $t > 0$, $P = 0$ and the initial value problem for the displacement becomes (notice that the initial conditions are taken from the solution for $t \le 0$)

$$\frac{EA}{\sqrt{2}L} u_{Z2} + \rho A L \ddot{u}_{Z2} = 0 \quad t > 0, \quad u_{Z2}(0) = \frac{\sqrt{2}L}{EA} F \quad \text{and} \quad \dot{u}_{Z2}(0) = 0.$$

Solution to the equations is given by

$$u_{Z2}(t) = F \frac{\sqrt{2}L}{EA} \cos\left(\frac{t}{L} \sqrt{\frac{E}{\sqrt{2}\rho}}\right) \quad t > 0.$$

A plate is simply supported on two edges and free on the other two edges as shown. Use the approximation $w(x,y,t)=a(t)xy/L^2$ to determine the transverse displacement as function of time $t>0$. Material properties E, ν, and ρ are constants and thickness of the plate is h. At $t=0$, initial conditions are $\dot{w}(x,y,0)=0$ and $w(x,y,0)=Uxy/L^2$. Assume that the plate is thin so that the rotation part of the inertia term is negligible.



Solution

Only the bending mode of the plate matters. When the approximation $w=a(t)xy/L^2$ is substituted there, virtual work densities of internal and inertia forces (without the rotation part) of the plate simplify to (shear modulus $G=E/(2+2\nu)$)

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} \partial^2\delta w/\partial x^2 \\ \partial^2\delta w/\partial y^2 \\ 2\partial^2\delta w/\partial x\partial y \end{Bmatrix}^{\mathrm{T}} \frac{h^3}{12}\frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}\begin{Bmatrix} \partial^2 w/\partial x^2 \\ \partial^2 w/\partial y^2 \\ 2\partial^2 w/\partial x\partial y \end{Bmatrix} = -\delta a\frac{1}{L^4}\frac{h^3}{3}Ga,$$

$$\delta w_\Omega^{\text{ine}} = -\begin{Bmatrix} \partial\delta w/\partial x \\ \partial\delta w/\partial y \end{Bmatrix}^{\mathrm{T}} \frac{t^3}{12}\rho\begin{Bmatrix} \partial\ddot{w}/\partial x \\ \partial\ddot{w}/\partial y \end{Bmatrix} - \delta w t\rho\ddot{w} = -\delta a(\frac{x}{L})^2(\frac{y}{L})^2 h\rho\ddot{a}$$

in which h is thickness of the plate. Integration over the domain occupied by the element gives the virtual work expressions

$$\delta W^{\text{int}} = \int_0^L\int_0^L \delta w_\Omega^{\text{int}} dydx = \int_0^L\int_0^L -\delta a\frac{1}{L^4}\frac{h^3}{3}Gadydx = -\delta a\frac{1}{L^2}\frac{h^3}{3}Ga,$$

$$\delta W^{\text{ine}} = \int_0^L\int_0^L \delta w_\Omega^{\text{ine}} dydx = \int_0^L\int_0^L -\delta a(\frac{x}{L})^2(\frac{y}{L})^2 h\rho\ddot{a}dxdy = -\delta a\frac{L^2}{9}h\rho\ddot{a}.$$

Virtual work expression of the structure consists of the internal and inertia parts

$$\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = -\delta a(\frac{1}{L^2}\frac{h^3}{3}Ga + \frac{L^2}{9}h\rho\ddot{a}).$$

Principle of virtual work $\delta W=0 \quad \forall\delta\mathbf{a}$ and the fundamental lemma of variation calculus $\delta\mathbf{a}^{\mathrm{T}}\mathbf{F}=0 \; \forall\delta\mathbf{a} \quad \Leftrightarrow \quad \mathbf{F}=0$ imply

$$\frac{1}{L^2}\frac{h^3}{3}Ga + \frac{L^2}{9}h\rho\ddot{a} = 0.$$

What remains, is solving for the displacement from the initial value problem

$$\ddot{a} + 3\frac{Gh^2}{\rho L^4} a = 0 \quad t > 0, \quad a(0) = U, \quad \dot{a}(0) = 0.$$

Solution to equations is (this can be shown e.g. by substituting the solution in the equations above)

$$a(t) = U \cos(\sqrt{3\frac{G}{\rho}} \frac{h}{L^2} t) \quad t > 0.$$

Finally, substituting the solution to parameter $a(t)$ into the approximation gives

$$w(x, y, t) = U \cos(\sqrt{3\frac{G}{\rho}} \frac{h}{L^2} t) \frac{xy}{L^2}. \quad \Leftarrow$$