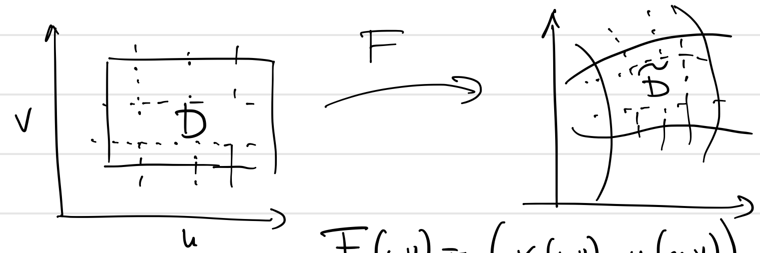
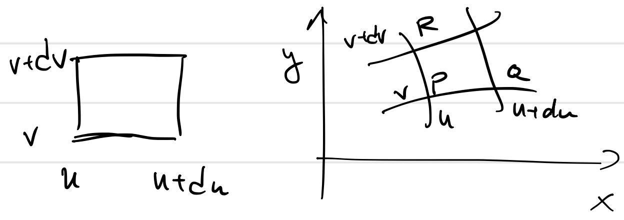
Change of variables in multiple integrals

A mapping $F: U \subset \mathbb{R}^n \to W \subset \mathbb{R}^n$ is called a change of variables if it is bijective (of the right class). First $\mathbb{R}^2$.



$$F(u,v) = (x(u,v), y(u,v))$$

If we want to simplify $\iint_{\tilde{D}} f(x,y)\,dx\,dy$ we need to understand how F changes the area scale.



level curves for u & v.

$$\vec{PQ} = dx\,\vec{e_1} + dy\,\vec{e_2} \qquad \vec{PR} \text{ in the same way}$$

$$dx = \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv \quad ; \quad dy = \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv$$

On PQ v is constant so $dv = 0$

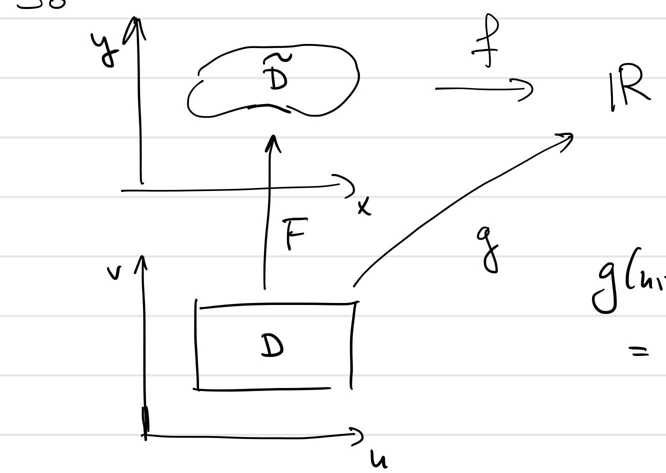
$$\Rightarrow \vec{PQ} = \frac{\partial x}{\partial u}du\,\vec{e_1} + \frac{\partial y}{\partial u}du\,\vec{e_2}$$

Also $\overrightarrow{PR} = \frac{\partial x}{\partial v} dv\, \vec{e_1} + \frac{\partial y}{\partial v} dv\, \vec{e_2}$

$$dx\,dy \approx |\overrightarrow{PQ} \times \overrightarrow{PR}| = \left| \begin{vmatrix} \vec{e_1} & \vec{e_2} & \vec{e_3} \\ \frac{\partial x}{\partial u} du & \frac{\partial y}{\partial u} du & 0 \\ \frac{\partial x}{\partial v} dv & \frac{\partial y}{\partial v} dv & \partial \end{vmatrix} \right| =$$

$$= \left| \underbrace{\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}}_{\text{Jacobian of } F} \right| du\,dv = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

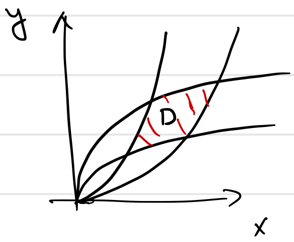
So

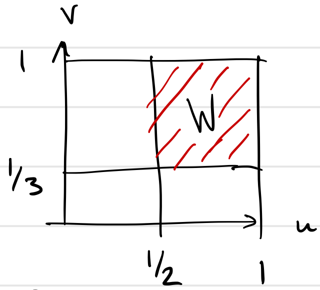


$$g(u,v) = f(F(u,v)) = f(x(u,v), y(u,v))$$

$$\iint_{\tilde{D}} f(x,y)\,dx\,dy = \iint_{D} g(u,v) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

<u>Ex</u> Find the area of the region bounded by the four parabolas $y = x^2$, $y = 2x^2$, $x = y^2$ and $x = 3y^2$.





$u = \frac{x^2}{y}$ $\qquad v = \frac{y^2}{x}$

$$\left|\frac{\partial(u,v)}{\partial(x,y)}\right| = 1 \Big/ \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \longleftarrow \text{Observe}$$

$$\frac{\partial u}{\partial x} = \frac{2x}{y} \; ; \; \frac{\partial u}{\partial y} = -\frac{x^2}{y^2} \; ; \; \frac{\partial v}{\partial x} = -\frac{y^2}{x^2} \text{ and } \frac{\partial v}{\partial y} = \frac{2y}{x}$$

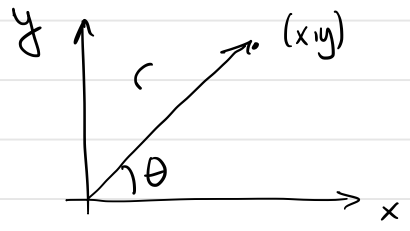
$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{2x}{y} & -\frac{y^2}{x^2} \\ -\frac{x^2}{y^2} & \frac{2y}{x} \end{vmatrix} = 4-1 = 3$$

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{3}$$

$$\iint_D 1 \, dx\,dy = \iint_W \frac{1}{3} \, du\,dv = \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{9} \text{ a.u.}$$

$\boxed{\text{Very important substitution}}$

Polar coordinates



$r > 0$

$0 \le \theta < 2\pi$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad r^2 = x^2 + y^2$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r$$

$$\boxed{dx\,dy = r\,dr\,d\theta}$$

So $\displaystyle\iint_D f(x,y)\,dx\,dy = \iint_{\hat{D}} g(r,\theta)\,r\,dr\,d\theta$

Ex

Let $D = \{(x,y) \in \mathbb{R}^2 ;\ 1 < x^2+y^2 < 4\}$

Calculate

$$I = \iint_D \frac{1}{x^2+y^2}\,dx\,dy$$

In polar coordinates

$$I = \int_0^{2\pi}\int_1^2 \frac{1}{r^2}\, r\, dr\, d\theta = 2\pi \int_1^2 \frac{1}{r}\, dr = 2\pi \Big[\ln r\Big]_1^2 =$$
$$= 2\pi \ln 2$$

Change of variables in higher dimensions works the same

$$\begin{cases} x = x(u,v,w) \\ y = y(u,v,w) \\ z = z(u,v,w) \end{cases} \qquad dx\,dy\,dz = \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du\,dv\,dw$$

Ex Calculate the volume of the ellipsoid E

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1$$

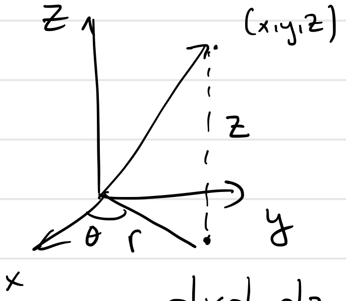
$$u = \frac{x}{a}\;;\quad v = \frac{y}{b}\;;\quad w = \frac{z}{c}$$

$\Longrightarrow u^2 + v^2 + w^2 \le 1$ a sphere S with radius 1.

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$$

$$\iiint_E 1\, dx\,dy\,dz = \iiint_S abc\, du\,dv\,dw =$$
$$= abc \cdot (\text{Volume of sphere with radius } 1)$$
$$= \frac{4\pi}{3} abc$$

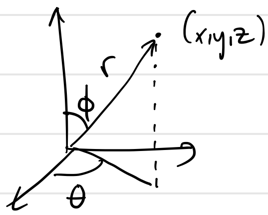
Cylindrical coordinates



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \qquad \begin{matrix} 0 < r \\ 0 \le \theta < 2\pi \\ z \in \mathbb{R} \end{matrix}$$

$$dx\,dy\,dz = r\,dr\,d\theta\,dz$$

Spherical coordinates



$0 < r$

$0 \le \theta < 2\pi$

$0 \le \phi < \pi$

$$x = r\sin\phi\cos\theta$$
$$y = r\sin\phi\sin\theta$$
$$z = r\cos\phi$$

If you calculate the Jacobian you get

$$dx\,dy\,dz = r^2\sin\phi\,dr\,d\phi\,d\theta$$

Ex Calculate the volume of a sphere with radius R.

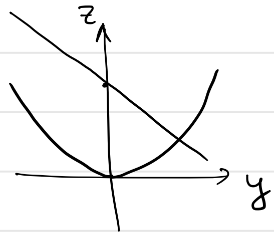
$$S_R = \{ (x,y,z) \in \mathbb{R}^3 ;\ x^2+y^2+z^2 \leq R^2 \}$$

$$\iiint_{S_R} 1\, dx\,dy\,dz = \int_0^R \int_0^{2\pi} \int_0^{\pi} r^2 \sin\phi \, d\phi\, d\theta\, dr =$$

$$= \int_0^R \int_0^{2\pi} \left[-r^2 \cos\phi \right]_0^{\pi} d\theta\, dr = \int_0^R \int_0^{2\pi} 2r^2\, d\theta\, dr =$$

$$= \int_0^R 4\pi r^2\, dr = \left[\frac{4\pi r^3}{3} \right]_0^R = \frac{4\pi R^3}{3}$$

Ex Find the volume of the solid S lying below the plane $z = 3-2y$ and above the paraboloid $z = x^2+y^2$.



Intersect when

$$x^2+y^2 = 3-2y$$

$$x^2+y^2+2y-3 = 0$$

$$x^2+(y+1)^2-4 = 0 \iff x^2+(y+1)^2 = 4$$

$$D = \{ (x,y) \in \mathbb{R}^2 ;\ x^2+(y+1)^2 \leq 4$$

$$\text{Volume of } S = \iint_D 3-2y-x^2-y^2 \, dx\,dy =$$

$$= \iint_D 4-x^2-(y+1)^2 dx\,dy \overset{T}{=}$$

$x = r\cos\theta$
$y = -1 + r\sin\theta$
$0 \leq r < 2 \ ;\ 0 \leq \theta < 2\pi$

$$= \int_0^{2\pi} \int_0^2 (4-r^2)\, r\, dr\, d\theta = \int_0^{2\pi} \left[\frac{4r^2}{2} - \frac{r^4}{4} \right]_0^2 d\theta =$$

$$= 2\pi(8-4) = 8\pi \text{ cubic units}$$