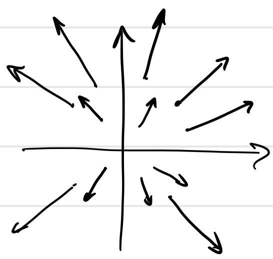
Vector fields

We have studied $f: \mathbb{R}^n \to \mathbb{R}$. It is also necessary to study $f: \mathbb{R}^n \to \mathbb{R}^m$ (vector-valued functions). These maps are called vector fields if $n=m$.

Ex $F(x,y) = x\vec{e_1} + y\vec{e_2} = (x,y) = (F_1(x,y), F_2(x,y))$



Notation and terminology

$F: \mathbb{R}^n \to \mathbb{R}^n$

$$F(x_1, \dots, x_n) = (F_1(x_1, \dots, x_n), F_2(x_1, \dots, x_n), \dots, F_n(x_1, \dots, x_n))$$

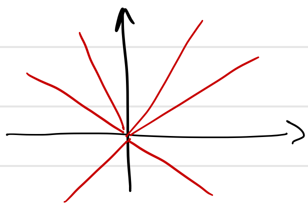
$$= F_1(\vec{x})\vec{e_1} + \dots + F_n(\vec{x})\vec{e_n}$$

C^k-vector field if $F_i \in C^k$ for $i = 1, \dots, n$.
smooth / C^∞-vector field if F_i are.

Integral curves / Field lines / Trajectories

An integral curve for a vector field is a curve to which the vector field is tangent at all points on the curve.

<u>Ex</u> $F(x,y) = (x,y)$



Integral curves = half-rays.

What is a curve?

$$r : \mathbb{R} \to \mathbb{R}^n$$

Can we easily find tangent vectors for r?

$$\frac{dr}{dt} = \dot{r}(t) = \left(\frac{dx_1}{dt}, \dots, \frac{dx_n}{dt} \right)$$

For an integral curve we have
$$\dot{r}(t) = \lambda(t)\, F(r(t))$$

When $n=3$ (or $n=2$) we find

$$\frac{dx}{dt} = \lambda(t)\, F_1(x,y,z) \quad , \quad \frac{dy}{dt} = \lambda(t)\, F_2(x,y,z)$$

and $\frac{dz}{dt} = \lambda(t)\, F_3(x,y,z)$

$$\Longrightarrow \lambda(t)\, dt = \frac{dx}{F_1(x,y,z)} = \frac{dy}{F_2(x,y,z)} = \frac{dz}{F_3(x,y,z)}$$

If we can multiply these equations by a function so we get

$$P(x)\, dx = Q(y)\, dy = R(z)\, dz$$

then we can integrate to find the integral curves.

Ex $F(x,y) = (x,y)$. Integral curves?

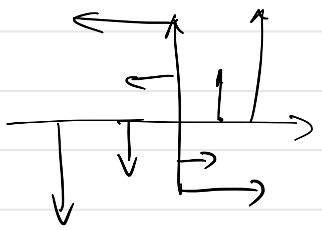
$$\frac{dx}{x} = \frac{dy}{y}$$

$$\ln|x| = \ln|y| + C$$

$$|y| = A|x| \Longrightarrow y = Ax \, , \; x>0$$
$$\text{or } y = Ax \, , \; x<0$$

We can also check that $x=0$ and $y=0$ works

Ex $F(x,y) = (-y, x)$



$$\frac{dx}{-y} = \frac{dy}{x} \Rightarrow$$

$$\Rightarrow x\,dx = -y\,dy$$

$$\Rightarrow \frac{x^2}{2} = -\frac{y^2}{2} + \frac{C}{2}$$

$$\Rightarrow x^2 + y^2 = C$$

The integral curves are circles.

Conservative fields

Given a function $f: \mathbb{R}^n \to \mathbb{R}$ it's gradient $\nabla f = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)$ is a vector field. When is a given vector field the gradient of a function? When the vector field is the gradient of a function it is called conservative. The function(s) are called the potential of the vector field. It is easy to find a necessary condition for a planar vector field to be conservative.

Assume that $\nabla \phi = F$

$$\left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}\right) = (F_1, F_2)$$

$$\frac{\partial F_1}{\partial y} = \frac{\partial^2 \phi}{\partial y \partial x} \qquad \frac{\partial F_2}{\partial x} = \frac{\partial^2 \phi}{\partial x \partial y}$$

$$\Longrightarrow \quad \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$$

So if this doesn't hold then F is not conservative.

Ex $F(x,y) = (x,y)$

Is the vector field conservative?

$$\frac{\partial F_1}{\partial y} = 0 \qquad \frac{\partial F_2}{\partial x} = 0$$

So the field can be conservative.

We try to construct the potential ϕ

$$\frac{\partial \phi}{\partial x} = x \quad \Longrightarrow \quad \phi(x,y) = \frac{x^2}{2} + C(y)$$

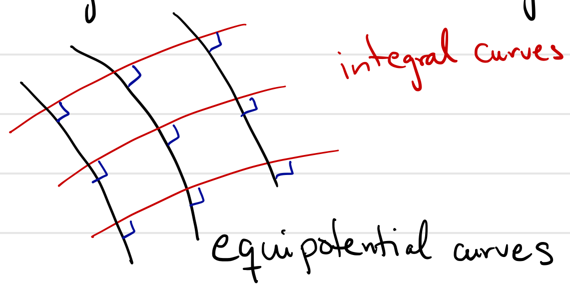
$$\Rightarrow \frac{\partial \phi}{\partial y} = C'(y) = y \Rightarrow C(y) = \frac{y^2}{2} + D$$

$$\phi(x,y) = \frac{x^2}{2} + \frac{y^2}{2} + D$$

$F(x,y)$ is conservative since $\nabla \phi = F$.

The sets $\phi(\vec{x}) = C$ are called equipotential curves (in $\mathbb{R}^2$) / surfaces (in $\mathbb{R}^3$) / hypersurfaces (in $\mathbb{R}^n$, $n \geq 4$)

Fact Equipotential curves are orthogonal trajectories for the integral curves



Ex $F(x,y) = (y,x) = y\vec{e_1} + x\vec{e_2}$

First integral curves

$$\frac{dx}{y} = \frac{dy}{x}$$

$$\Rightarrow \int x\,dx = \int y\,dy \Rightarrow \frac{x^2}{2} = \frac{y^2}{2} + C$$

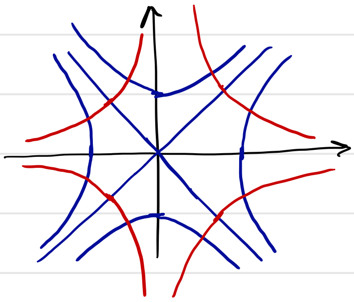
$\Rightarrow$ Integral curves $x^2 - y^2 = A$
hyperbolas

Now equipotential curves

$$\frac{\partial \phi}{\partial x} = y \Rightarrow \phi(x,y) = xy + \alpha(y)$$

$$\frac{\partial \phi}{\partial y} = x + \alpha'(y) \Rightarrow \alpha'(y) = 0$$

$$\Rightarrow \phi(x,y) = xy + C$$



Integral curves

Equipotential curves
(also coordinate axes)

<u>Ex</u> $F(x,y) = (x,y)$

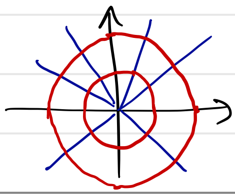
We already know that the integral curves for this vector field are half-rays starting at the origin. Lets find the equipotential curves.

$$\frac{\partial \phi}{\partial x} = x \implies \phi(x,y) = \frac{x^2}{2} + \alpha(y)$$

$$\frac{\partial \phi}{\partial y} = \alpha'(y) = y \implies \alpha(y) = \frac{y^2}{2} + A$$

$$\phi(x,y) = \frac{x^2}{2} + \frac{y^2}{2} + A$$

$$\implies x^2 + y^2 = C$$ Circles around the origin.



Integral curves

Equipotential curves