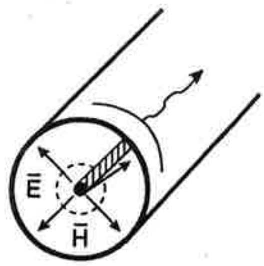
Transmission lines

- Electromagnetic waves can propagate in free space ← review this! based on Maxwell's equations!
  * but also they can be guided by conducting or dielectric boundaries
- Transmission-line behaviour: occurs when $\lambda \ll$ length of transmission line

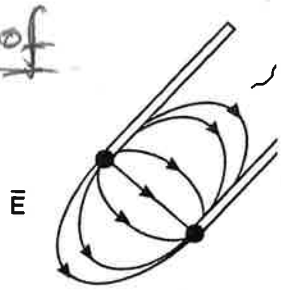
Transmission lines = guiding devices for the electromagnetic field

- The electromagnetic fields are TEM (transverse electromagnetic mode) if the conductors are ideal (zero-resistance); otherwise there will be a small axial component of the electromagnetic field.

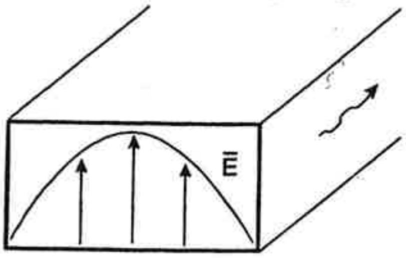
Types of transmission lines



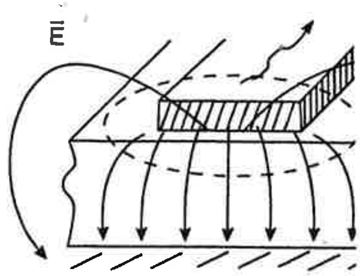
(a) = COAXIAL LINE



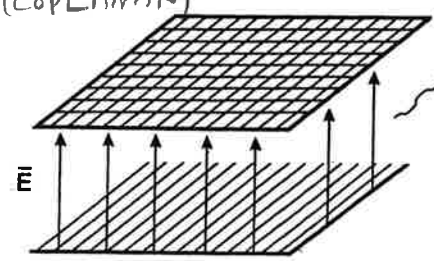
(b) = TWO-WIRE TRANSMISSION LINE



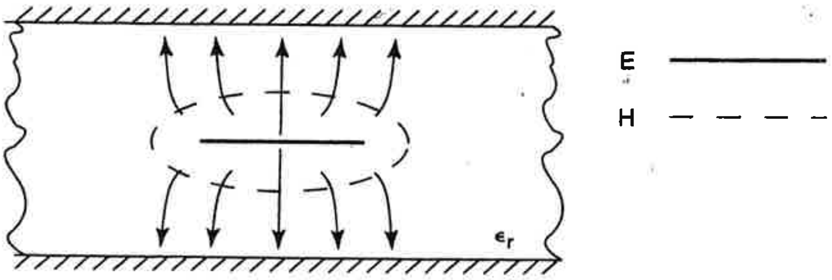
(c) = WAVEGUIDE (COPLANAR)



(d) = MICROSTRIPLINE

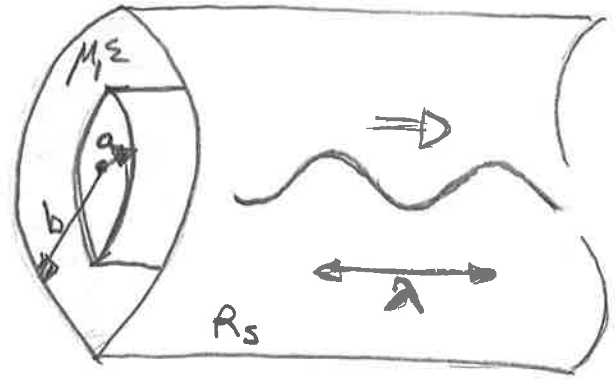


(e) = PARALLEL-PLATE WAVEGUIDE

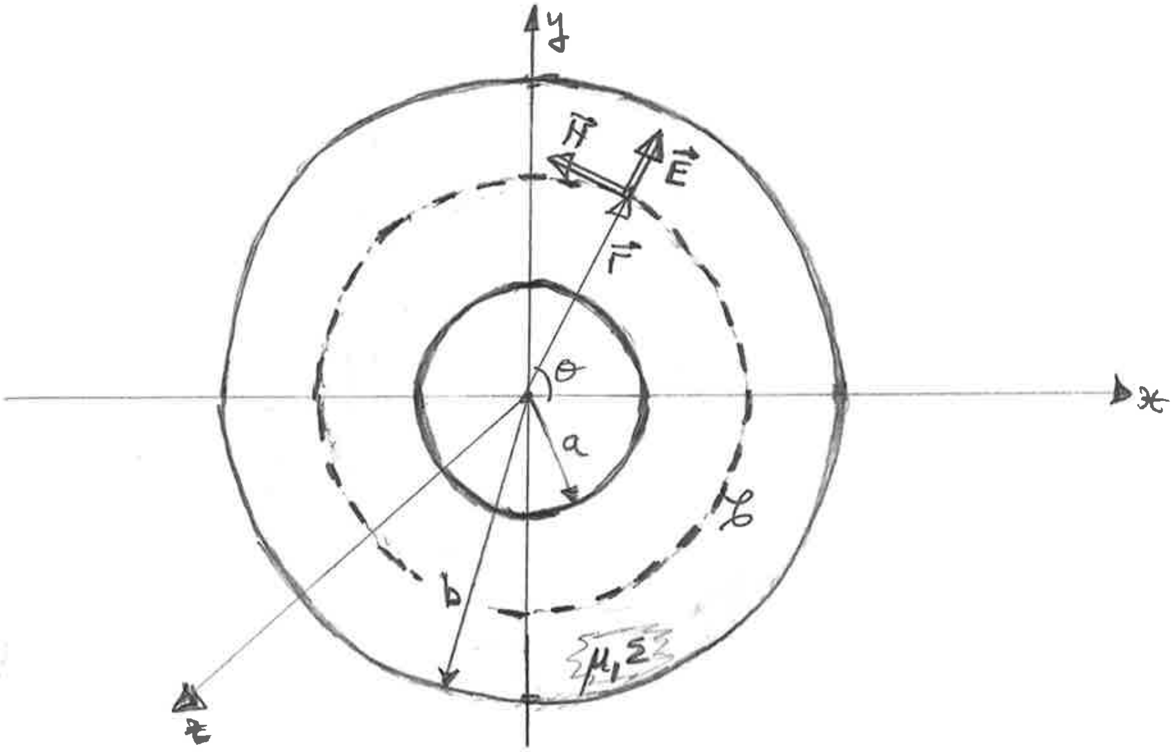


(f) = STRIPLINE

EXAMPLE: the coaxial line

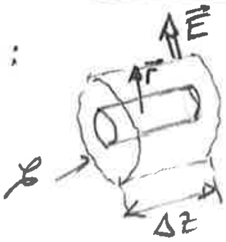


- How to calculate the $\vec{E}$, $\vec{H}$ fields inside?



Electric field $\boxed{\vec{E} = \dfrac{V_0}{\ln\frac{b}{a}}\,\dfrac{\hat{\vec{r}}}{r}}$

Proof:



$\underline{\nabla\cdot\vec{D} = \rho}$ $\Rightarrow$ $\int d\vec{S}\cdot\vec{E} = \int \frac{\rho}{\varepsilon}dV$ (Gauss' law)

$$\underbrace{\int d\vec{S}\cdot\vec{E}}_{2\pi r(\Delta z)\cdot E} = \underbrace{\int \frac{\rho}{\varepsilon}dV}_{\frac{1}{\varepsilon}(\Delta z)\cdot\rho\cdot\pi a^2}$$

Therefore

$$E = \frac{1}{r}\cdot\frac{\rho a^2}{2\varepsilon}$$

Also $V_0 = \int_a^b dr\cdot E$

$$= \int_a^b \frac{dr}{r}\cdot\frac{\rho a^2}{2\varepsilon} = \frac{\rho a^2}{2\varepsilon}\ln\frac{b}{a} \rightarrow \frac{\rho a^2}{2\varepsilon} = \frac{V_0}{\ln\frac{b}{a}}$$

So $\vec{E} = \dfrac{V_0}{\ln\frac{b}{a}}\cdot\dfrac{\hat{\vec{r}}}{r}$

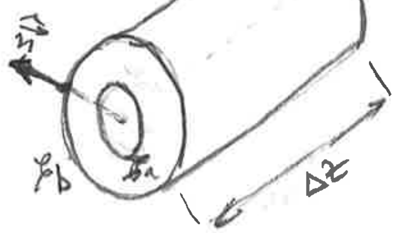
Magnetic field $\boxed{\vec{H} = \dfrac{I_0}{2\pi r}\cdot\hat{\vec{e}}_\theta}$

$\nabla\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t}$ $\longrightarrow$ $\oint_{\ell}\vec{H}\cdot d\vec{\ell} = \int_S\vec{J}\cdot d\vec{S} = I_0$ or $2\pi r\cdot H = I_0$ $\Rightarrow$ $\vec{H} = \dfrac{I_0}{2\pi r}\cdot\hat{\vec{e}}_\theta$

Towards a distributed model of inductances, capacitances, resistances, conductances

Problem: how to connect the electric and magnetic fields to circuit-elements?

Answer: via stored or dissipated energy



① Inductance per unit length

magnetic energy $= \frac{\mu}{4}\int ds\cdot(\Delta z)\, H^2 = \frac{(L'\Delta z) I_0^2}{4} \Rightarrow L' = \frac{\mu}{I_0^2}\int ds\, H^2$

$$L' = \frac{\mu}{I_0^2}\int ds\, H^2 = \frac{\mu}{I_0^2}\cdot I_0^2 \int_0^{2\pi} d\theta \int_a^b dr\cdot r \cdot \frac{1}{(2\pi r)^2} = \frac{\mu}{2\pi}\ln\frac{b}{a}$$

$$\boxed{L' = \frac{\mu}{2\pi}\ln\frac{b}{a}}$$

(measured in H/m)

② Capacitance per unit length

electrostatic energy $= \frac{\varepsilon}{4}\int ds\cdot(\Delta z)\cdot E^2 = \frac{(C'\Delta z) V_0^2}{4} \Rightarrow C' = \frac{\varepsilon}{V_0^2}\cdot\int ds\cdot E^2$

$$C' = \frac{\varepsilon}{V_0^2}\int ds\, \vec{E}^2 = \frac{\varepsilon}{V_0^2}\cdot V_0^2 \cdot \frac{1}{\ln^2\frac{b}{a}} \int_0^{2\pi} d\theta \int_0^b dr\cdot r\cdot \frac{1}{r^2} \Rightarrow$$

$$\boxed{C' = \frac{2\pi\varepsilon}{\ln\frac{b}{a}}}$$

(measured in F/m)

③ Resistance per unit length

power dissipated in the lossy conductors $= \frac{R_s}{2}\int_{\rho_a+\rho_b} dl\cdot\Delta z\cdot J_s^2 = \frac{R_s}{2}\Delta z\cdot\int_{\rho_a+\rho_b} dl\cdot H^2 = \frac{R'\Delta z}{2} I_0^2$

$$\Downarrow$$

$$R' = \frac{R_s}{I_0^2}\int_{\rho_a+\rho_b} dl\cdot \vec{H}^2$$

Here R_s = surface resistance

$\vec{J}_s = \hat{n}\times\vec{H}$ = surface current

$\hat{n}$ = vector unit pointing outwards (normal to the conducting surface)

$$R' = \frac{R_s}{I_0^2}\int_{\rho_a+\rho_b} dl\cdot H^2 = \frac{R_s}{(2\pi)^2}\left[\int_0^{2\pi} d\theta\cdot a\cdot\frac{1}{a^2} + \int_0^{2\pi} d\theta\cdot b\cdot\frac{1}{b^2}\right] = \frac{R_s}{2\pi}\left(\frac{1}{a}+\frac{1}{b}\right)$$

$$\Rightarrow \boxed{R' = \frac{R_s}{2\pi}\left(\frac{1}{a}+\frac{1}{b}\right)}$$

(measured in Ω/m)

④ Conductance (radial) per unit length

$$\varepsilon = \varepsilon' - i\varepsilon'' = \varepsilon_0 \varepsilon_r (1 - i\tan\delta)$$

$\varepsilon' = \varepsilon_0 \varepsilon_r$

$\varepsilon'' = \varepsilon \tan\delta$ → dissipation in the dielectric between the core metal and the outside shield.

$$\text{power dissipated} = \frac{\omega \varepsilon''}{2} \int ds \cdot \Delta z \cdot E^2 = \frac{G' V_0^2}{2} \longrightarrow G' = \frac{\omega \varepsilon''}{V_0^2} \cdot \int ds \cdot E^2$$

$$\Rightarrow G' = \frac{\omega \varepsilon''}{V_0^2} \int ds \cdot E^2 = \frac{\omega \varepsilon''}{V_0^2} \cdot \int_0^{2\pi} d\theta \int_a^b dr \cdot r \cdot \frac{V_0^2}{r^2 \ln\frac{b}{a}} \Rightarrow \boxed{G' = \frac{2\pi\omega\varepsilon''}{\ln\frac{b}{a}}}$$

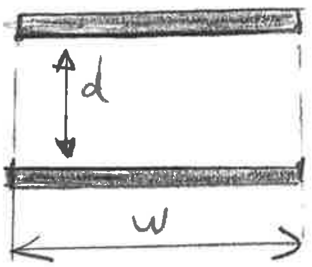
(measured in S/m)

- Examples of materials used in coaxes

Conductor	Copper Cu	Aluminum Al	Silver Ag	Gold Au
Resistivity ρ [nΩ·m] ↓ 10^{-9} "nano"	16.9	26.7	16.3	22.0

Dielectric	Dry air	Polyethylene	PTFE	PVC
ε_r	1,0006	2,2	2,1	3,2
$\tan\delta$	low	0,0002	0,0002	0,001
Resistivity (Ω·m)	high	10^{15}	10^{15}	10^{15}
Breakdown voltage (MV/m)	3	47	59	34

Other transmission lines:

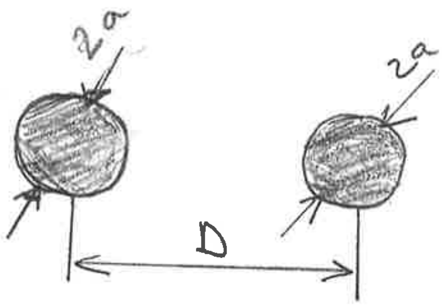


$$L' = \frac{\mu d}{w}$$

$$C' = \frac{\varepsilon' w}{d}$$

$$R' = \frac{2R_s}{w}$$

$$G' = \frac{\omega \varepsilon'' w}{d}$$



$$L' = \frac{\mu}{\pi} \cosh^{-1} \frac{D}{2a}$$

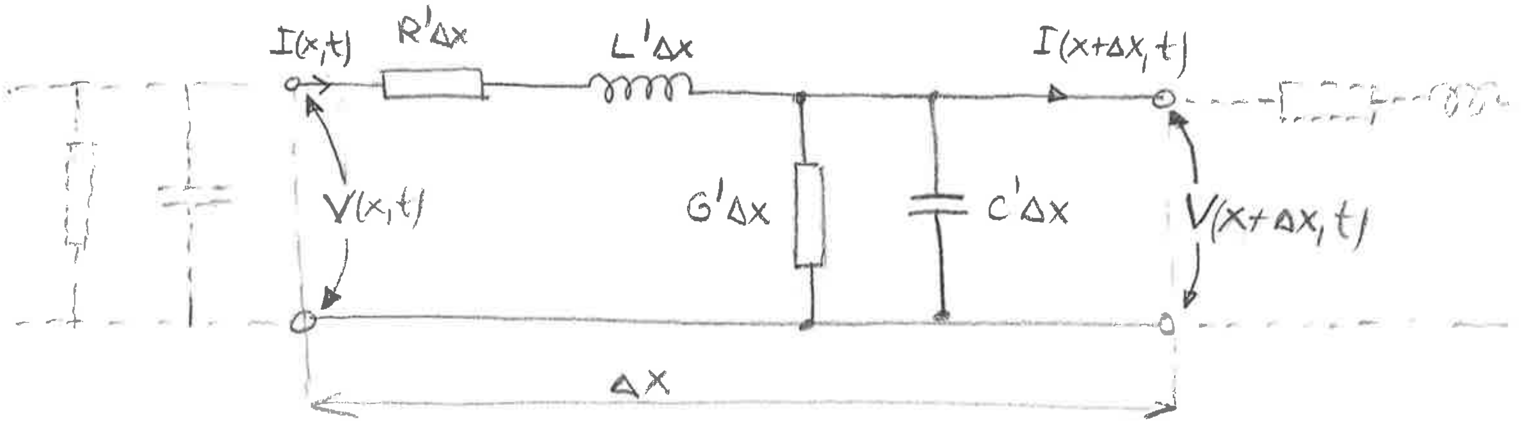
$$C' = \frac{\pi \varepsilon'}{\cosh^{-1} \frac{D}{2a}}$$

$$R' = \frac{R_s}{\pi a}$$

$$G' = \frac{\pi \omega \varepsilon''}{\cosh^{-1} \frac{D}{2a}}$$

TRANSMISSION LINES – general models

- If the length of a circuit is $\gtrsim \lambda$ we have to use either a simulator of Maxwell's equations or a distributed model of lumped elements.
- TLs = two parallel conductors that guide the electromagnetic field
  examples: two-wire lines, striplines, microstrip lines



R', L', G', C' = resistance, inductance, conductance, capacitance per unit length.

KIRCHHOFF says:

$$\begin{cases} V(x,t) = I(x,t)\,R'\Delta x + L'\Delta x \cdot \dfrac{\partial I(x,t)}{\partial t} + V(x+\Delta x, t) \\ I(x,t) = V(x+\Delta x,t)\,G'\Delta x + C'\Delta x\, \dfrac{\partial V(x+\Delta x,t)}{\partial t} + I(x+\Delta x,t) \end{cases}$$

$\Delta x \to 0$

$$\begin{cases} -\dfrac{\partial V(x,t)}{\partial x} = R' I(x,t) + L' \dfrac{dI(x,t)}{dt} \\ -\dfrac{\partial I(x,t)}{\partial x} = G' V(x,t) + C' \dfrac{\partial V(x,t)}{\partial t} \end{cases}$$

Therefore,

$$\begin{cases} -\dfrac{\partial^2 V(x,t)}{\partial x^2} = -R'\left(G'V(x,t) + C'\dfrac{\partial V(x,t)}{\partial t}\right) - L'\left(G'\dfrac{\partial V(x,t)}{\partial t} + C'\dfrac{\partial^2 V(x,t)}{\partial t^2}\right) \\ -\dfrac{\partial^2 I(x,t)}{\partial x^2} = -G'\left(R'I(x,t) + L'\dfrac{\partial I(x,t)}{\partial t}\right) - C'\left(R'\dfrac{\partial I(x,t)}{\partial t} + L'\dfrac{\partial^2 I(x,t)}{\partial t^2}\right) \end{cases}$$

or

$$\begin{cases} \dfrac{\partial^2 V(x,t)}{\partial x^2} = L'C'\dfrac{\partial^2 V(x,t)}{\partial t^2} + (R'C' + L'G')\dfrac{\partial V(x,t)}{\partial t} + R'G'V(x,t) \\ \dfrac{\partial^2 I(x,t)}{\partial x^2} = L'C'\dfrac{\partial^2 I(x,t)}{\partial t^2} + (R'C' + L'G')\dfrac{\partial I(x,t)}{\partial t} + R'G'I(x,t) \end{cases}$$

Harmonic signals:

$$V(x,t) = V(x)\, e^{i\omega t} \qquad V(x), I(x) = \text{phasors}$$
$$I(x,t) = I(x)\, e^{i\omega t}$$

$$\Rightarrow \begin{cases} \dfrac{d^2V(x)}{dx^2} - \gamma^2 V(x) = 0 \\ \dfrac{d^2I(x)}{dx^2} - \gamma^2 I(x) = 0 \end{cases}$$

$$\gamma = \alpha + i\beta = \sqrt{(R' + i\omega L')(G' + i\omega C')}$$

γ = propagation constant
α = attenuation constant
β = phase constant

$$\Rightarrow \boxed{V(x) = V^+ e^{-\gamma x} + V^- e^{\gamma x}} \quad \text{general solution}$$

From $-\dfrac{\partial V(x,t)}{\partial x} = R I(x,t) + L \dfrac{\partial I(x,t)}{\partial t}$ we get $I(x) = -\dfrac{1}{R + i\omega L}\dfrac{dV(x)}{dx}$

or

$$I(x) = \frac{1}{Z_0} V^+ e^{-\gamma x} - \frac{1}{Z_0} V^- e^{\gamma x} = I^+ e^{-\gamma x} + I^- e^{\gamma x}$$

where $I^\pm = \pm \dfrac{V^\pm}{Z_0}$

where $Z_0 = \sqrt{\dfrac{R' + i\omega L'}{G' + i\omega C'}}$ = characteristic impedance of the transmission line

Lossless transmission case: $R' = G' = 0$

$$\gamma = i\beta = i\omega\sqrt{L'C'}$$

$$Z_0 = \frac{1}{Y_0} = \sqrt{\frac{L'}{C'}} \rightarrow \text{now independent of frequency!}$$

Note: free-space impedance $= 377\,\Omega$

$$v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{L'C'}} = \text{phase velocity}$$

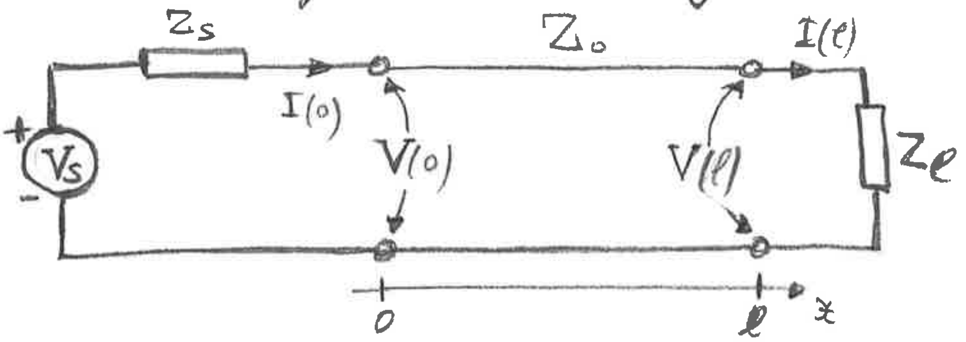
Exercise: Show that for the low-loss case $R \ll \omega L$, $G \ll \omega C$ we have

$$\beta \simeq \omega\sqrt{L'C'}$$
$$\alpha \simeq \frac{1}{2}\sqrt{L'C'}\left(\frac{R'}{L'} + \frac{G'}{C'}\right)$$

- Standardized values:

Z_0	Application
$50\,\Omega$	Instrumentation, communication
$75\,\Omega$	TV, VHF radio
$300\,\Omega$	RF
$600\,\Omega$	audio

- Incident and reflected waves along a loaded transmission line



line terminated in a load impedance Z_ℓ

$$V(x) = V^+ e^{-\gamma x} + V^- e^{\gamma x}$$
$$I(x) = I^+ e^{-\gamma x} + I^- e^{\gamma x} \quad , \; I^\pm = \pm \frac{V^\pm}{Z_0}$$

$$\begin{cases} V(0) = V_S - Z_S I(0) & \text{— Kirchhoff's law} \\ V(\ell) = Z_\ell I(\ell) \end{cases}$$

or:

$$\begin{cases} V^+ + V^- = V_S - \dfrac{Z_S}{Z_0}(V^+ + V^-) \\ V^+ e^{-\gamma\ell} + V^- e^{\gamma\ell} = \dfrac{Z_\ell}{Z_0}\left(V^+ e^{-\gamma\ell} - V^- e^{\gamma\ell}\right) \end{cases}$$

- Define a reflection coefficient at the load $x = \ell$ $\quad \Gamma_V^\ell = \dfrac{V^- e^{\gamma\ell}}{V^+ e^{-\gamma\ell}}$

$$\rightarrow \; 1 + \Gamma_V^\ell = \frac{Z_\ell}{Z_0}\left(1 - \Gamma_V^\ell\right) \;\Rightarrow\; \boxed{\Gamma_V^\ell = \frac{Z_\ell - Z_0}{Z_\ell + Z_0}}$$

We can also define a current reflection coefficient at the load

$$\boxed{\Gamma_I^\ell \equiv \frac{I^- e^{\gamma\ell}}{I^+ e^{-\gamma\ell}} = -\Gamma_V^\ell}$$

- Define the Transmission coefficient at the load $x = \ell$ $\quad T_V = \dfrac{V^+ e^{-\gamma\ell} + V^- e^{\gamma\ell}}{V^+ e^{-\gamma\ell}}$

or $\boxed{T_V = 1 + \Gamma_V^\ell}$

and for current $\boxed{T_I = \dfrac{I^+ e^{-\gamma\ell} + I^- e^{\gamma\ell}}{I^+ e^{-\gamma\ell}} = 1 + \Gamma_I^\ell}$

• **Average power delivered to the load**

$$\overline{P}_\ell = \frac{1}{2} \mathrm{Re}\left[V(\ell) I^*(\ell)\right]$$

$\frac{1}{2}$ — comes from the fact that the field is harmonic

Now
$$1 - \Gamma_V = \frac{I^- e^{\gamma \ell} + I^+ e^{-\gamma \ell}}{I^+ e^{-\gamma \ell}} = \frac{I(\ell)}{I^+ e^{-\gamma \ell}}$$
$$1 + \Gamma_V = \frac{V^+ e^{-\gamma \ell} + V^- e^{\gamma \ell}}{V^+ e^{\gamma \ell}} = \frac{V(\ell)}{V^+ e^{-\gamma \ell}}$$
$\rightarrow$

$$(1 - \Gamma_V^*)(1 + \Gamma_V) = \frac{V(\ell) I^*(\ell)}{I^{+*} V^+ e^{-\gamma \ell} (e^{-\gamma \ell})^*}$$

but $I^+ = \dfrac{V^+}{Z_0}$

$$V(\ell) I^*(\ell) = \frac{1}{Z_0} \left|V^+ e^{-\gamma \ell}\right|^2 \cdot \underbrace{(1 - \Gamma_V^*)(1 + \Gamma_V)}_{= 1 - \underbrace{\Gamma_V^* + \Gamma_V}_{\text{imaginary!}} - |\Gamma_V|^2}$$

$$\Rightarrow \quad \overline{P}_\ell = \frac{1}{2 Z_0} \cdot \left|V^+ e^{-\gamma \ell}\right|^2 \left(1 - |\Gamma_V|^2\right)$$

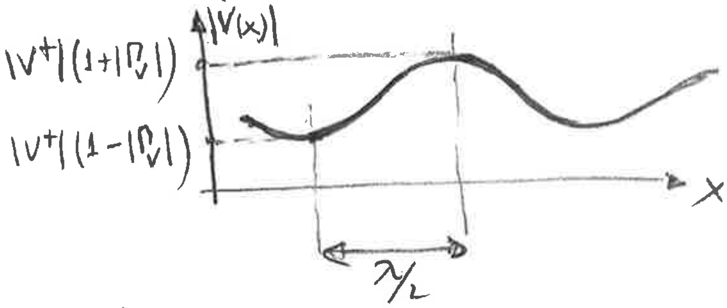
• **VSWR** (voltage standing-wave ratio)

$$V(x) = V^+ e^{-\gamma x} + V^- e^{\gamma x} = V^+ e^{-\gamma x}\left[1 + \Gamma_V e^{-2\gamma(\ell - x)}\right]$$

remember that $\Gamma_V = \dfrac{V^- e^{\gamma \ell}}{V^+ e^{-\gamma \ell}}$

Let's consider a lossless line $\alpha = 0$ $\quad \gamma = i\beta = \dfrac{2\pi i}{\lambda}$

$$|V(x)| = |V^+| \cdot \left|1 + \Gamma_V e^{-2i\beta(\ell - x)}\right|$$ — oscillates, min. and max. separated by $\dfrac{\pi}{\beta} = \dfrac{\lambda}{2}$



$$\mathrm{VSWR} = \frac{1 + |\Gamma_V|}{1 - |\Gamma_V|}$$ = ratio between the max. line voltage and min. line voltage

- Impedance along the line

$$Z(x) \equiv \frac{V(x)}{I(x)} = Z_0 \frac{V^+ e^{-\gamma x} + V^- e^{\gamma x}}{V^+ e^{-\gamma x} - V^- e^{\gamma x}} = Z_0 \frac{1 + \Gamma_V e^{-2\gamma(l-x)}}{1 - \Gamma_V e^{-2\gamma(l-x)}}$$

take $x=0 \rightarrow$ we get $Z(0) \equiv Z_{in}$ = input impedance of the line, i.e. the impedance seen when looking toward the load.

$$\boxed{Z_{in} = Z_0 \cdot \frac{Z_l + Z_0 \tanh \gamma l}{Z_0 + Z_l \tanh \gamma l}}$$

— can be verified immediately by recalling that $\Gamma_V = \frac{Z_l - Z_0}{Z_l + Z_0}$

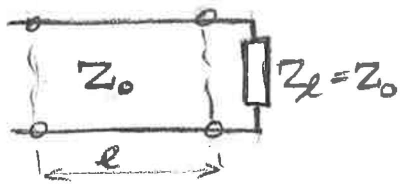
Note that in general $Z_{in} \neq Z_0$, so the termination matters!

and also Z_{in} is frequency-dependent

EXAMPLES OF LOADS (TERMINATIONS)

1) Matched load

$Z_l = Z_0$



$\Rightarrow \Gamma_V \equiv \frac{Z_l - Z_0}{Z_l + Z_0} = 0$

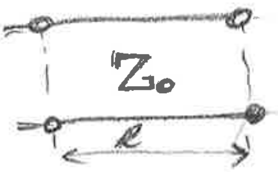
No reflection!

VSWR = 1 , $P_l = \frac{1}{2 Z_0} |V^+|^2 e^{-2\alpha l}$ — powered delivered is maximal, it is only attenuated if $\alpha \neq 0$

$Z_{in} = Z_0$

2) Open-circuit

$Z_l = \infty$



$\Gamma_V = \frac{Z_l - Z_0}{Z_l + Z_0} = 1$

VSWR = ∞

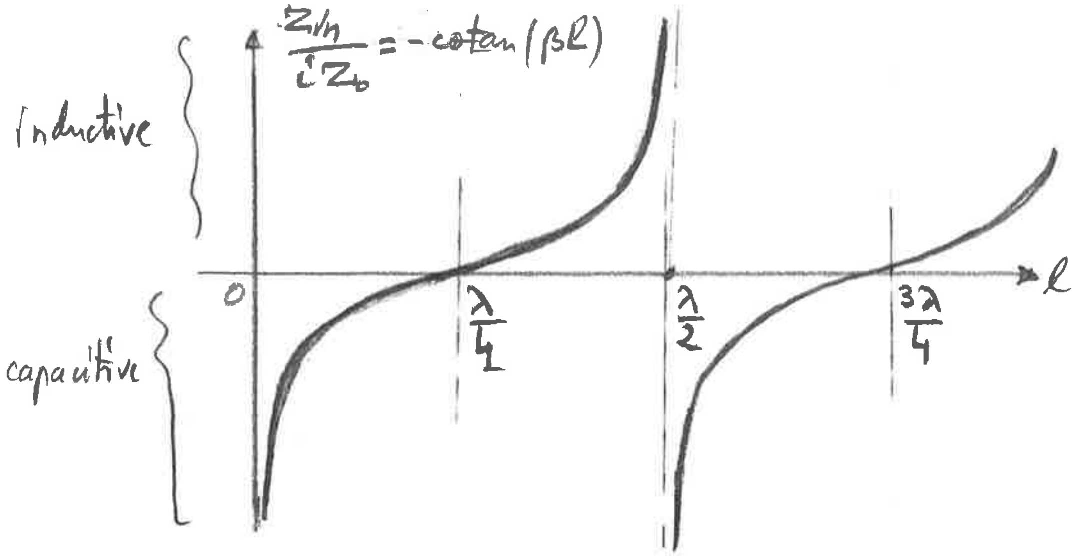
$P_l = 0$ — compare this with the DC-case where all input power is delivered!

$Z_{in} = Z_0 \coth \gamma l$

For $\alpha = 0$ (lossless) $Z_{in} = -i Z_0 \cot\!an \frac{2\pi l}{\lambda}$

if $l = \frac{\lambda}{4}$, $Z_{in} = 0$ so the open line will look as a shortcut!

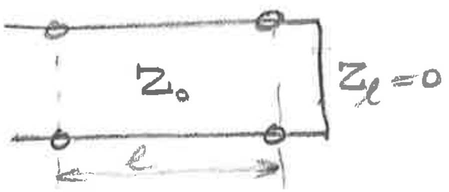
$$\frac{Z_{in}}{iZ_0} = -\cot an(\beta l) \qquad \beta = \frac{2\pi}{\lambda}$$



3) Short-circuit

$Z_l = 0$

$\Gamma_v = \dfrac{Z_l - Z_0}{Z_l + Z_0} = -1$



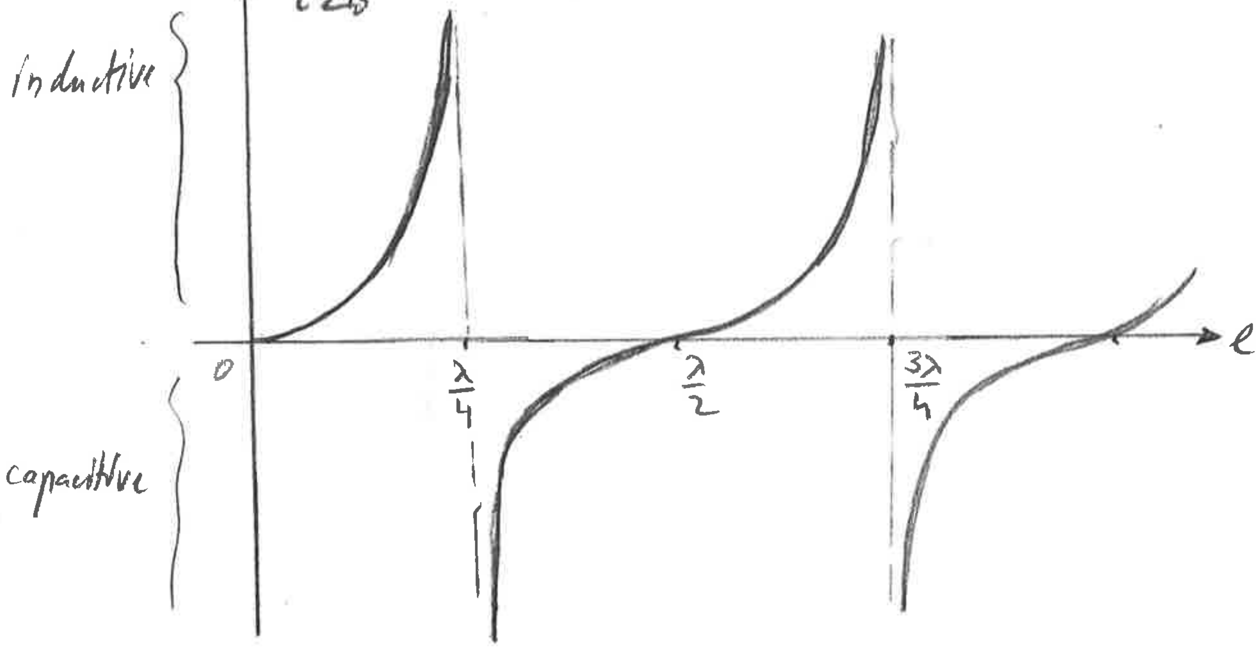
VSWR $= \infty$ $\qquad$ $P_l = 0$

$Z_{in} = Z_0 \tanh \gamma l$ $\qquad$ For $\alpha = 0$ (lossless) $\qquad$ $\beta = \dfrac{2\pi}{\lambda}$

$$Z_{in} = i Z_0 \tan \frac{2\pi l}{\lambda}$$

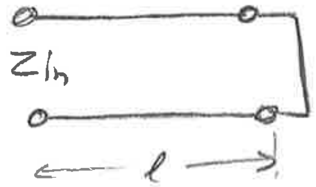
if $l = \frac{\lambda}{4}$, $Z_{in} = \infty$
so the shorted line looks like an infinite impedance to a source!
(even if the resistance of the wires is zero!)

$$\frac{Z_{in}}{iZ_0} = \tan(\beta l)$$



Resonators from transmission lines

- It is possible to make resonators from transmission lines, 3D cavities, etc.
- The most usual case is the short-circuited transmission-line resonator.



$$Z_\ell = 0$$

$$Z_{in} = Z_0 \tanh(\alpha l + i\beta l)$$

$$= Z_0 \frac{\tanh \alpha l + i \tan \beta l}{1 + i \tan\beta l \tan \alpha l}$$

If losses are not too large, $\alpha l \ll 1$, we have $\tan \alpha l \simeq \alpha l$

so
$$Z_{in} = Z_0 \frac{\alpha l + i \tan \beta l}{1 + i \alpha l \tan \beta l}$$

Now, recall that $\beta = \frac{\omega}{v_p} = \omega\sqrt{L'C'}$ $\quad v_p = \frac{1}{\sqrt{L'C'}}$ $\quad Z_0 = \sqrt{\frac{L'}{C'}}$

$$\alpha = \frac{R'}{2}\sqrt{\frac{C'}{L'}}$$

We will consider $\underline{\beta_0 l = \pi}$ or $l = \frac{\lambda_0}{2}$

as the resonance condition, leading to a resonance frequency ω_0 $\quad \frac{\omega_0}{v_p} l = \omega_0 \sqrt{L'C'}\, l = \pi$

so $\omega_0 = \frac{\pi}{l\sqrt{L'C'}}$

We can expand around this point

$$\tan \beta l \approx \tan\left(\pi + \pi \frac{\omega-\omega_0}{\omega_0}\right) = \tan \pi \frac{\omega-\omega_0}{\omega_0} \simeq \pi \frac{\omega-\omega_0}{\omega_0}$$

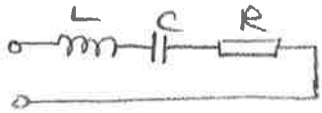
if $|\omega-\omega_0| \ll \omega_0$

So

$$Z_{in} = Z_0 \frac{\alpha l + i\pi \frac{\omega-\omega_0}{\omega_0}}{1+\alpha l \pi \frac{\omega-\omega_0}{\omega_0}} \simeq Z_0 \left(\alpha l + i\pi \frac{\omega-\omega_0}{\omega_0}\right)$$

$$= \sqrt{\frac{L'}{C'}} \left(\frac{l}{2} R' \sqrt{\frac{C'}{L'}} + i l \sqrt{L'C'}\,(\omega-\omega_0)\right) = \frac{1}{2} R' l + i L' l\,(\omega-\omega_0)$$

- Suppose now that we look back at a series RLC circuit



$$Z = R + i\frac{L}{\omega}(\omega^2 - \omega_0^2)$$

$$\simeq R + 2iL(\omega-\omega_0) \text{ near resonance}, \quad \omega \simeq \omega_0$$

Therefore we can identify $\underline{R = \frac{1}{2} R' l}$ and $\underline{L = \frac{1}{2} L' l}$

Quality factor $\underline{Q = \dfrac{\omega_0 L}{R} = \dfrac{\omega_0 L'}{R'} = \dfrac{\beta_0}{2\alpha}}$

- Interesting question to think about: why do we get the factor 1/2 in the RLC equivalent above?

Because the current in the short-circuited line is half a sinusoid, therefore we obtain only half of the total resistance and inductance of the full length l.

To see this explicitly, let us write the solution

$$\begin{cases} V(x) \simeq V^+ e^{-i\beta x} + V^- e^{i\beta x} \\ I(x) \simeq -\dfrac{\beta}{\omega L}\left(-V^+ e^{-i\beta x} + V^- e^{i\beta x}\right) \end{cases}$$

— here we neglect α

Since $I(0) = 0 \Rightarrow V^+ \equiv V^-$ at this point

(also you can see that $\Gamma_V \equiv \frac{V^-}{V^+} e^{+2i\beta_0 l} = -1$ and $\beta_0 l = \pi$)

So $\begin{cases} V(x) = 2V^{+}\cos\beta_0 x \\ I(x) = -\dfrac{2j\beta}{\omega L}\, V^{+}\sin\beta_0 x = I^{+}\sin\beta_0 x \end{cases}$

Therefore the magnetic field energy

$$\overline{W}_{L'} = \int_0^{\lambda_0/2} dx \cdot \frac{1}{4} L' |I(x)|^2 = \frac{1}{4} |I^{+}|^2 L' \int_0^{\frac{\lambda_0}{2}} \sin^2\beta_0 x \, dx$$

$$= \frac{\lambda_0}{16} \cdot |I^{+}|^2 L'$$

$\overline{W}_{C'} = \overline{W}_{L'}$ at resonance   so $\overline{W} = \overline{W}_{C'} + \overline{W}_{L'} = \dfrac{\lambda_0}{8} L' |I^{+}|^2$

$$\overline{P} = \frac{1}{2} \int_0^{\lambda_0/2} dx \cdot R' |I(x)|^2 = \frac{R'}{2} |I^{+}|^2 \int_0^{\frac{\lambda_0}{2}} \sin^2\beta_0 x \, dx$$

so $\overline{P} = \dfrac{\lambda_0}{8} R' |I^{+}|^2$

Therefore $\underline{Q = \dfrac{\omega_0 \overline{W}}{\overline{P}} = \dfrac{\omega_0 L'}{R'}}$

or: $Q = \dfrac{\pi}{\ell R'} \sqrt{\dfrac{L'}{C'}} = \dfrac{\pi Z_0}{\ell R'} = \dfrac{\pi}{2\ell\alpha}$

where we used $\alpha \simeq \dfrac{R}{2Z_0}$, $\omega_0 = \dfrac{\pi}{\ell\sqrt{L'C'}}$

REFERENCES

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