Lecture 3

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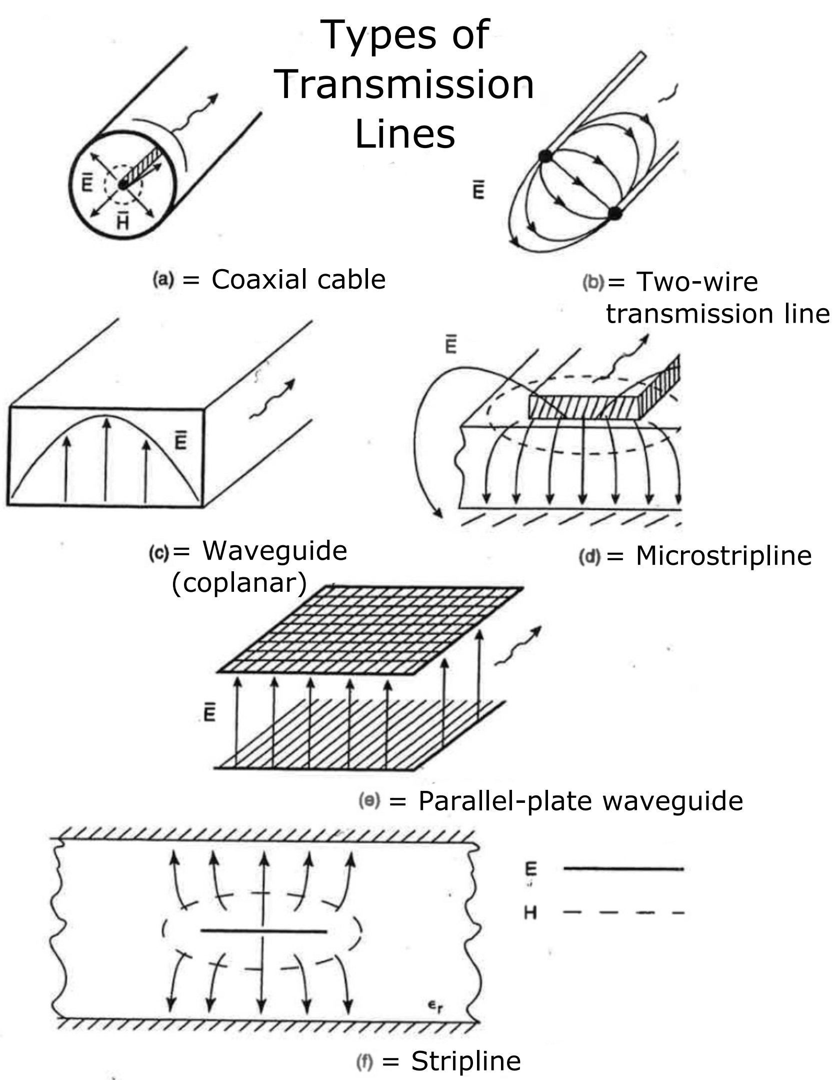
I. TRANSMISSION LINES

– Electromagnetic waves can propagate in free space (Review this! Based on Maxwell's equations!). But also they can be guided by conducting or dielectric boundaries.

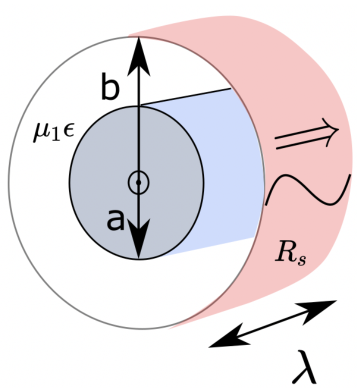
– Transmission line behavior: occurs when $\lambda \ll$ length of transmission line.

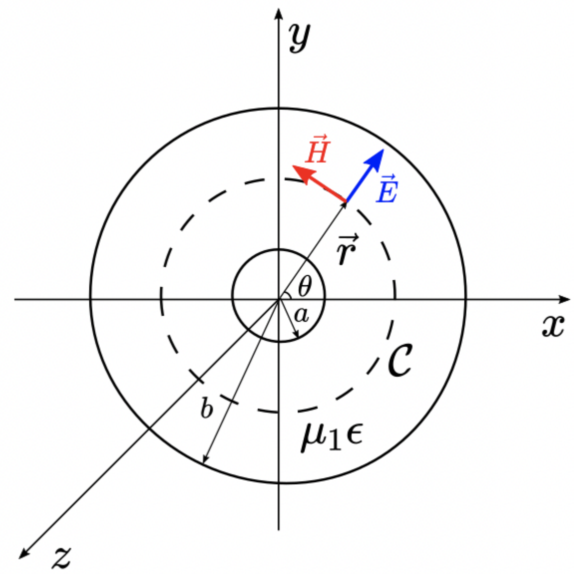
– Transmission lines = guiding devices for the electromagnetic field.

– The electromagnetic fields are TEM (transverse electromagnetic mode) if the conductors are ideal (zero-resistance); otherwise there will be a small axial component of the electromagnetic field.



EXAMPLE: The coaxial line





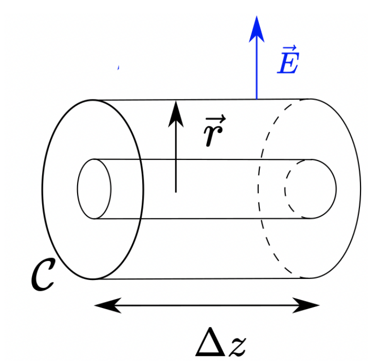
* How to calculate the $\vec{E}$, $\vec{H}$ fields inside?

$$\text{Electric Field:} \qquad \vec{E} = \frac{V_0}{\ln\frac{b}{a}} \frac{\hat{\vec{r}}}{r} \tag{1}$$

Proof:

$\vec{\nabla}\cdot\vec{D} = \rho \implies \int d\vec{S}\cdot\vec{E} = \int \frac{\rho}{2} dV \implies 2\pi r\cdot(\Delta z)\cdot E = \frac{1}{\epsilon}(\Delta z)\cdot\rho\cdot\pi a^2 \quad \therefore E = \frac{1}{r}\cdot\frac{\rho a^2}{2\epsilon}$

Also $V_0 = \int_a^b dr\cdot E = \int_a^b \frac{dr}{r}\cdot\rho\frac{a^2}{2\epsilon}\ln\frac{b}{a} \to \rho\frac{a^2}{2\epsilon} = \frac{V_0}{\ln\frac{b}{a}}$, so $\vec{E} = \frac{V_0}{\ln\frac{b}{a}}\frac{\hat{\vec{r}}}{r}$.



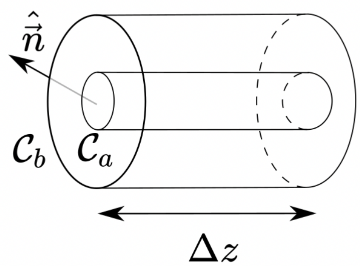
$$\text{Magnetic Field:} \qquad \vec{H} = \frac{I_0}{2\pi r}\cdot\hat{\vec{e}_\theta} \tag{2}$$

Proof:

$\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t} \implies \int_{\mathcal{C}}\vec{H}\cdot d\vec{\ell} = \int\vec{J}\cdot d\vec{S} = I_0$, or $2\pi r\cdot H = I_0 \implies \vec{H} = \frac{I_0}{2\pi r}\cdot\hat{\vec{e}_\theta}$.

II. TOWARDS A DISTRIBUTED MODEL OF INDUCTORS, CAPACITANCES, RESISTANCES, CONDUCTANCES

Problem: How to connect the electric and magnetic fields to circuit elements.



Answer: Via stored or dissipated energy.

1. Inductance per unit length

   Magnetic energy $= \frac{\mu}{4} \int ds \cdot (\Delta z) \vec{H}^2 = \frac{(L' \Delta t) I_0^2}{4} \implies L' = \frac{\mu}{I_0^2} \int ds \vec{H}^2$

   $L' = \frac{\mu}{I_0^2} \int ds H^2 = \frac{\mu}{I_0^2} \cdot I_0^2 \int_0^{2\pi} d\theta \int_a^b dr \cdot r \cdot \frac{1}{(2\pi r)^2} = \frac{\mu}{2\pi} \ln \frac{b}{a}$.

   Therefore,

$$L' = \frac{\mu}{2\pi} \ln \frac{b}{a} \quad \text{(measured in units of H/m)} \ . \tag{3}$$

2. Capacitance per unit length

   Electrostatic energy$= \frac{\epsilon}{4} \int ds \cdot (\Delta z) \cdot E^2 = \frac{(C' \Delta z) V_0^2}{4} \implies C' = \frac{\epsilon}{V_0^2} \cdot \int ds \cdot E^2$

   $C' = \frac{\epsilon}{V_0^2} \int ds E^2 = \frac{\epsilon}{V_0^2} \cdot V_0^2 \cdot \frac{1}{\ln^2 \frac{b}{a}} \int_0^{2\pi} d\theta \int_0^b dr \cdot r \cdot \frac{1}{r^2}$

$$\implies \qquad C' = \frac{2\pi\epsilon}{\ln \frac{b}{a}} \quad \text{(measured in units of F/m)} \ . \tag{4}$$

3. Resistance per unit length

   Power dissipated in the lossy conductors $= \frac{R_s}{2} \int_{\mathcal{C}_a + \mathcal{C}_b} d\ell \cdot \Delta z \cdot \mathcal{J}_s^2 = \frac{R_s}{2} \Delta z \cdot \int_{\mathcal{C}_a + \mathcal{C}_b} d\ell \cdot H^2 = \frac{R' \Delta z}{2} I_0^2$. Here R_s = surface resistance, $\vec{\mathcal{J}}_s = \hat{\vec{n}} \times \vec{H}$ = surface current, $\hat{\vec{n}}$ = vector unit pointing outwards (normal to the conducting surface), and $R' = \frac{R_s}{I_0^2} \int_{\mathcal{C}_a+\mathcal{C}_b} d\ell \cdot \vec{H}^2$.

$R' = \frac{R_s}{I_0^2} \int_{\mathcal{C}_a+\mathcal{C}_b} d\ell \cdot H^2 = \frac{R_s}{(2\pi)^2} \left[\int_0^{2\pi} d\theta \cdot a \cdot \frac{1}{a^2} + \int_0^{2\pi} d\theta \cdot b \cdot \frac{1}{b^2} \right] = \frac{R_s}{2\pi} \left(\frac{1}{a} + \frac{1}{b} \right)$

$$R^{'} = \frac{R_s}{2\pi} \left(\frac{1}{a} + \frac{1}{b} \right) \quad \text{(measured in units of } \Omega/\text{m)} \ . \tag{5}$$

4. <u>Conductance (radial) per unit length</u>

$\epsilon = \epsilon^{'} - i\epsilon^{''} = \epsilon_0 \epsilon_r (1 - i \tan\delta)$

$\epsilon' = \epsilon_0 \epsilon_r$

$\underline{\epsilon^{''} = \epsilon \tan\delta} \rightarrow$ dissipation in the dielectric between the core metal and the outside shield.

Power dissipated $= \frac{\omega\epsilon^{''}}{2} \int ds \cdot \Delta z \cdot E^2 = \frac{G' V_0^2}{2} \rightarrow G'' = \frac{\omega\epsilon^{''}}{V_0^2} \int ds \cdot E^2$

$\Longrightarrow \ G' = \frac{\omega\epsilon^{''}}{V_0^2} \int ds \cdot E^2 = \frac{\omega\epsilon^{''}}{V_0^2} \cdot \int_0^{2\pi} d\theta \int_a^b dr \cdot r \cdot \frac{V_0^2}{r^2 \ln\frac{b}{a}}$

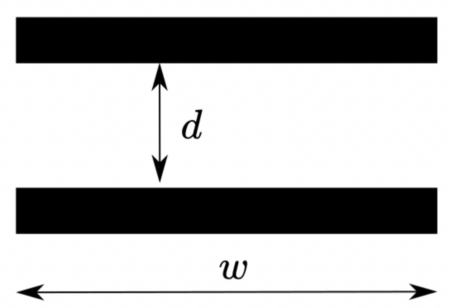
$$\Longrightarrow \qquad G' = \frac{2\pi\omega\epsilon^{''}}{\ln\frac{b}{a}} \quad \text{(measured in units of S/m)} \ . \tag{6}$$

– <u>Examples of materials used in coaxes:</u>

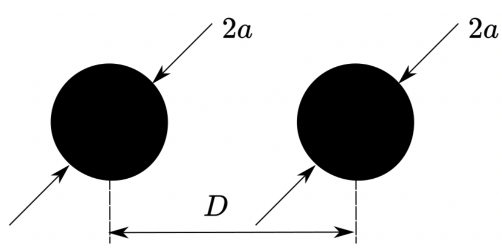
Conductor	Copper Cu	Aluminum Al	Silver Ag	Gold Au
Resistivity $\rho[\text{n}\Omega \cdot \text{m}]$	16.9	26.7	16.3	22.0

Dielectric	Dry Air	Polyethylene	PTFE	PVC
$\boldsymbol{\epsilon_r}$	1.0006	2.2	2.1	3.2
$\tan\delta$	low	0.0002	0.0002	0.001
Resistivity $(\Omega \cdot \text{m})$	high	10^{15}	10^{15}	10^{15}
Breakdown voltage (mV/m)	3	47	59	34

Other transmission lines:



$$L' = \frac{\mu d}{w} \tag{7}$$
$$C' = \frac{\epsilon' w}{d} \tag{8}$$
$$R' = \frac{2R_s}{w} \tag{9}$$
$$G' = \frac{\omega \epsilon'' w}{d} \tag{10}$$

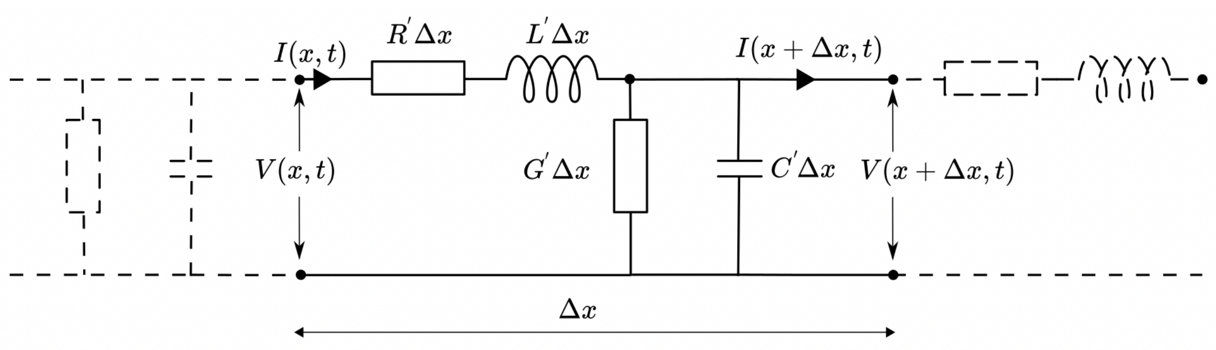


$$L' = \frac{\mu}{\pi} \cosh^{-1} \frac{D}{2a} \tag{11}$$
$$C' = \frac{\pi \epsilon'}{\cosh^{-1} \frac{D}{2a}} \tag{12}$$
$$R' = \frac{R_s}{\pi a} \tag{13}$$
$$G' = \frac{\pi \omega \epsilon''}{\cosh^{-1} \frac{D}{2a}} \tag{14}$$

III. TRANSMISSION LINES: GENERAL MODELS

– If the length of a circuit is $\gtrsim \lambda$ we have to use either a simulator of Maxwell's equations or a distributed model of lumped elements.

– Transmission lines: Two parallel conductors that guide the electromagnetic field. Examples: two-wire lines, striplines, microstrip lines.



R', L', G', C' = resistance, inductance, conductance, capacitance per unit length.

Kirchoff says:

$$\begin{cases} V(x,t) = I(x,t)R'\Delta x + L'\Delta x \cdot \frac{\partial I(x,t)}{\partial t} + V(x+\Delta x,t) \\ I(x,t) = V(x+\Delta x,t)G'\Delta x + C'\Delta x \frac{\partial V(x+\Delta x,t)}{\partial t} + I(x+\Delta x,t) \end{cases} \tag{15}$$

$$\Delta x \longrightarrow 0 \begin{cases} -\frac{\partial V(x,t)}{\partial x} = R'I(x,t) + L'\frac{dI(x,t)}{dt} \\ -\frac{\partial I(x,t)}{\partial x} = G'V(x,t) + C'\frac{\partial V(x,t)}{\partial t} \end{cases} \tag{16}$$

Therefore,

$$\begin{cases} -\frac{\partial^2 V(x,t)}{\partial x^2} = -R'(G'V(x,t) + C'\frac{\partial V(x,t)}{\partial t}) - L'(G'\frac{\partial V(x,t)}{\partial t} + C'\frac{\partial^2 V(x,t)}{\partial t^2}) \\ -\frac{\partial^2 I(x,t)}{\partial x^2} = -G'(R'I(x,t) + L'\frac{\partial I(x,t)}{\partial t}) - C'(R'\frac{\partial I(x,t)}{\partial t} + L'\frac{\partial^2 I(x,t)}{\partial t^2}) \end{cases} \tag{17}$$

or

$$\begin{cases} \frac{\partial^2 V(x,t)}{\partial x^2} = L'C'\frac{\partial^2 V(x,t)}{\partial t^2} + (R'C' + L'G')\frac{\partial V(x,t)}{\partial t} + R'G'V(x,t) \\ \frac{\partial^2 I(x,t)}{\partial x^2} = L'C'\frac{\partial^2 I(x,t)}{\partial t^2} + (R'C' + L'G')\frac{\partial I(x,t)}{\partial t} + R'G'I(x,t)\ . \end{cases} \tag{18}$$

Harmonic signals:

$$V(x,t) = V(x)e^{i\omega t}\ , \qquad V(x), I(x) = \text{phasors}, \qquad I(x,t) = I(x)e^{i\omega t}$$

$$\Longrightarrow \begin{cases} \frac{d^2V(x)}{dx^2} - \gamma^2 V(x) = 0, \quad \text{where} \quad \gamma = \alpha + i\beta = \sqrt{(R' + i\omega L')(G' + i\omega C')} \\ \frac{d^2I(x)}{dx^2} - \gamma^2 I(x) = 0, \quad \gamma = \text{propagation constant},\ \alpha = \text{attenuation constant},\ \beta = \text{phase constant} \end{cases}$$

$$\text{General Solution:} \qquad V(x) = V^{\dagger}e^{-\gamma x} + V^{-}e^{\gamma x}\ . \tag{19}$$

From $-\frac{\partial V(x,t)}{\partial x} = RI(x,t) + L\frac{\partial I(x,t)}{\partial t}$, we get $I(x) = -\frac{1}{R+i\omega L}\frac{dV(x)}{dx}$ or $I(x) = \frac{1}{Z_0}V^{\dagger}e^{\gamma x} - \frac{1}{Z_0}V^{-}e^{\gamma x} = I^{+}e^{-\gamma x} + I^{-}e^{\gamma x}$, where

$Z_0 = \sqrt{\frac{R'+i\omega L'}{G'+i\omega C'}}$ = characteristic impedance of the transmission line,

and where $I^{\pm} = \pm\frac{V^{\pm}}{Z_0}$.

Lossless transmission case: $R' = G' = 0$

$\gamma = i\beta = i\omega\sqrt{L'C'}$

$Z_0 = \frac{1}{Y_0} = \sqrt{\frac{L'}{C'}} \longrightarrow$ now independent of frequency!

Note: Free-space impedance $= 377\ \Omega$

$v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{L'C'}}$ = phase velocity.

Exercise: Show that for the loss-less case $R \ll \omega L$, $G \ll \omega C$, we have $\beta \simeq \omega\sqrt{L'C'}$ and $\alpha \simeq \frac{1}{2}\sqrt{L'C'}(\frac{R'}{L'} + \frac{G'}{C'})$.

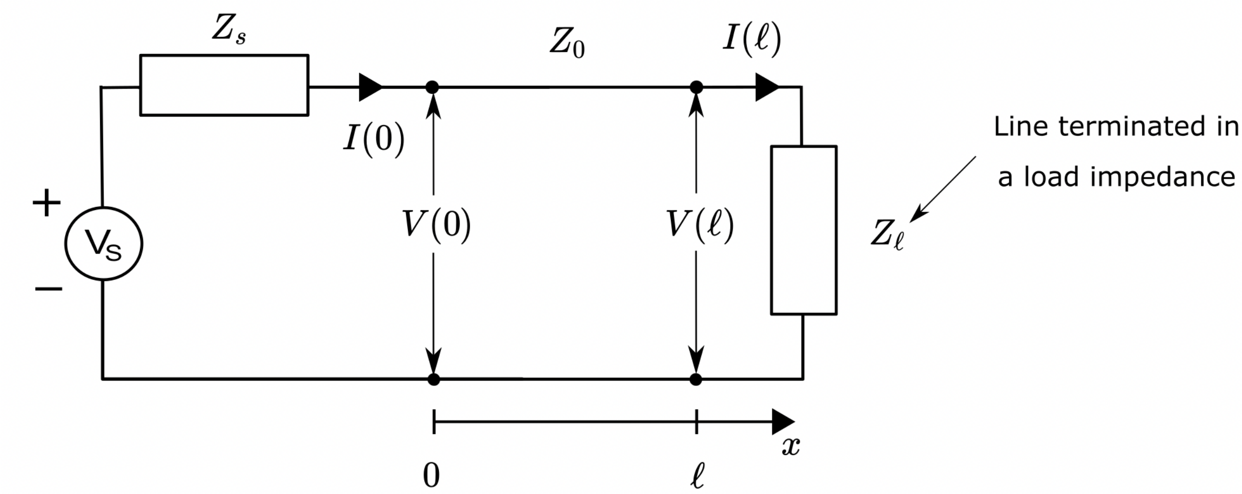
- Standardized values:

Z_0	**Application**
50 Ω	Instrumentation, communication
75 Ω	TV, VHF radio
300 Ω	RF
600 Ω	Audio

- Incident and Reflected Waves Along a Loaded Transmission Line

  $V(x) = V^{+}e^{-\gamma x} + V^{-}e^{\gamma x}$

  $I(x) = I^{+}e^{-\gamma x} + I^{-}e^{\gamma x}, \quad I^{\pm} = \pm\frac{V^{\pm}}{Z_0}$



$$\begin{cases} V(0) = V_s - Z_s I_0 \text{ — Kirchoff's law} \\ V(\ell) = Z_\ell I(\ell) \ , \end{cases} \tag{20}$$

or

$$\begin{cases} V^+ + V^- = V_s - \frac{Z_s}{Z_0}(V^+ + V^-) \\ V^+ e^{-\gamma\ell} + V^- e^{\gamma\ell} = \frac{Z_\ell}{Z_0}(V^+ e^{-\gamma\ell} - V^- e^{\gamma\ell}) \end{cases} \tag{21}$$

- Define a reflection coefficient of the load at $x = \ell$: $\Gamma_V = \frac{V^- e^{\gamma\ell}}{V^+ e^{-\gamma\ell}}$.

$\rightarrow 1 + \Gamma_V = \frac{Z_\ell}{Z_0}(1 - \Gamma_V)$.

$$\Longrightarrow \qquad \Gamma_V = \frac{Z_\ell - Z_0}{Z_\ell + Z_0} \tag{22}$$

We can also define a current reflection coefficient at the load

$$\Gamma_I = \frac{I^- e^{\gamma\ell}}{I^+ e^{-\gamma\ell}} = -\Gamma_V \tag{23}$$

- Define the transmission coefficient at the load $x = \ell$: $T_V = \frac{V^+ e^{+\ell} + V^- e^{\gamma\ell}}{V^+ e^{-\gamma\ell}}$.

$$\therefore \qquad T_V = 1 + \Gamma_V \ , \tag{24}$$

and for the current

$$T_I = \frac{I^+ e^{-\gamma\ell} + I^- e^{\gamma\ell}}{I^+ e^{-\gamma\ell}} = 1 + \Gamma_I \ . \tag{25}$$

- <u>Average power delivered to the load</u>

$\overline{P_\ell} = \frac{1}{2}\mathrm{Re}[V(\ell)I^*(\ell)]$, where the 1/2 comes from the fact that the field is harmonic.

Now,

$$\begin{cases} 1-\Gamma_V = \frac{I^- e^{\gamma\ell} + I^+ e^{-\gamma\ell}}{I^+ e^{-\gamma\ell}} = \frac{I(\ell)}{I^+ e^{-\gamma\ell}} \\ 1+\Gamma_V = \frac{V^+ e^{-\gamma\ell} + V^- e^{-\gamma\ell}}{V^+ e^{\gamma\ell}} = \frac{V(\ell)}{V^+ e^{-\gamma\ell}} \ . \end{cases} \tag{26}$$

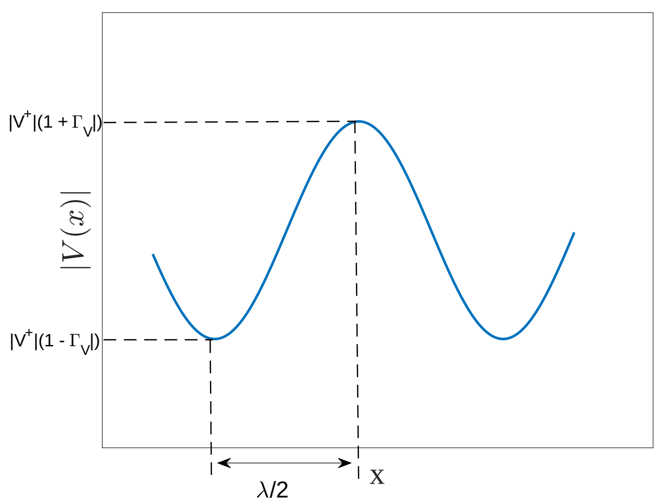
$\therefore \qquad (1+\Gamma_V^*)(1+\Gamma_V) = \frac{V(\ell)I^*(\ell)}{I^{+*}V^+ e^{-\gamma\ell}(e^{-\gamma\ell})^*}$, but $I^+ = \frac{V^+}{Z_0}$

$V(\ell)I^*(\ell) = \frac{1}{Z_0}|V^+ e^{-\gamma\ell}|^2 \cdot (1-\Gamma_V^*)(1+\Gamma_V) \equiv 1 - \Gamma_V^* + \Gamma_V - |\Gamma_V|^2$, where $\Gamma_V^* + \Gamma_V =$ Imaginary!.

$$\overline{P_\ell} = \frac{1}{2Z_0} \cdot |V^+ e^{-\gamma\ell}|^2 (1 - |\Gamma_V|^2) \ . \tag{27}$$

- <u>VSWR</u> (Voltage standing-wave ratio)

$V(x) = V^+ e^{-\gamma x} + V^- e^{\gamma x} = V^+ e^{-\gamma x}[1 + \Gamma_V e^{-2\gamma(\ell - x)}]$ (Remember that $\Gamma_V \equiv \frac{V^- e^{\gamma\ell}}{V^+ e^{-\gamma\ell}}$.)



Let's consider a lossless line $\alpha = 0$, $\quad \gamma = i\beta = \frac{2\pi i}{\lambda}$

$|V(x)| = |V^+| \cdot |1 + \Gamma_V e^{-2i\beta(\ell - x)}|$ — oscillates, min. and max. separated by $\frac{\pi}{\beta} = \frac{\lambda}{2}$.

$VSWR = \frac{1+|\Gamma_V|}{1-|\Gamma_V|}$ = ratio between the max. line voltage and min. line voltage.

- Impedance along the line

$Z(x) = \frac{V(x)}{I(x)} = Z_0 \frac{V^+ e^{-\gamma x} + V^- e^{\gamma x}}{V^+ e^{-\gamma x} - V^- e^{\gamma x}} = \frac{1+\Gamma_V e^{-2\gamma(\ell - x)}}{1-\Gamma_V e^{-2\gamma(\ell - x)}}.$

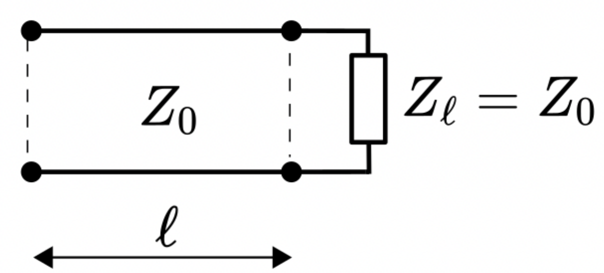
Take $x = 0 \to$ we get $Z(0) \equiv Z_{in} =$ input impedance of the line, i.e., the impedance seen when looking toward the load.

$$Z_{in} = Z_0 \cdot \frac{Z_\ell + Z_0 \tanh \gamma\ell}{Z_0 + Z_\ell \tanh \gamma\ell} \tag{28}$$

Note that this can be verified immediately by recalling that $\Gamma_V = \frac{Z_\ell - Z_0}{Z_\ell + Z_0}$, and that in general, $Z_{in} \neq Z_0$, so the termination matters! Also, Z_{in} is frequency-dependent.

IV. EXAMPLES OF LOADS (TERMINATIONS)

1. Matched Load



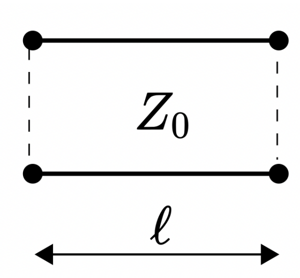
$Z_\ell = Z_0 \implies \Gamma_V \equiv \frac{Z_\ell - Z_0}{Z_\ell + Z_0} = 0$ No reflection!

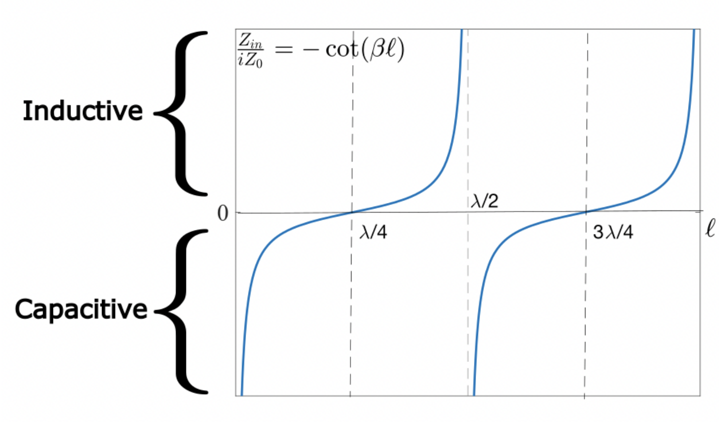
VSWR $= 1$, $Z_{in} = Z_0$, $P_\ell = \frac{1}{2Z_0}|V^+|^2 e^{-2\alpha\ell}$ — power delivered is maximum. This is only obtained if $\alpha \neq 0$.

2. Open-Circuit $Z_\ell = \infty \implies \Gamma_V = \frac{Z_\ell - Z_0}{Z_\ell + Z_0} = 1$

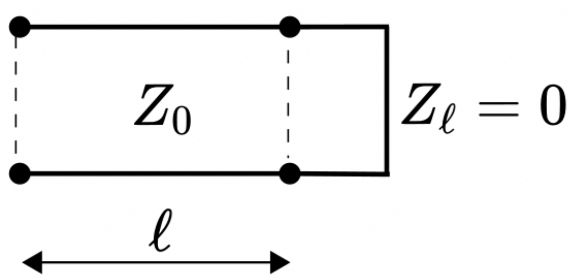
VSWR $= \infty$, $Z_{in} = Z_0 \coth \gamma\ell$, $P_\ell = 0$ — Compare this with the DC-case where all the input power is delivered!

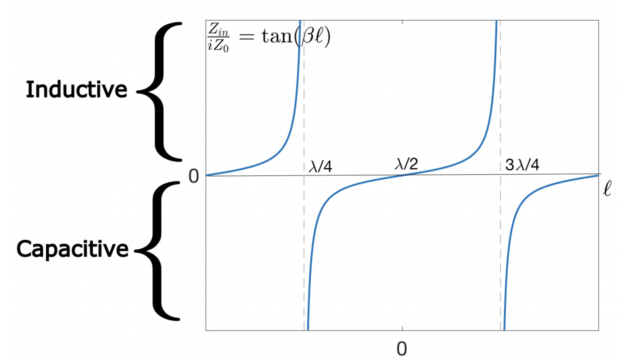
For $\alpha = 0$ (lossless), $Z_{in} = -iZ_0 \cot \frac{2\pi\ell}{\lambda}$ if $\ell = \frac{\lambda}{4}$, $Z_{in} = 0$, so the open line will look as a shortcut!





3. <u>Short-circuit</u> $Z_\ell = 0 \implies \Gamma_V = \frac{Z_\ell - Z_0}{Z_\ell + Z_0} = -1$,





VSWR $= \infty$, $\quad Z_{in} = Z_0 \tanh \gamma\ell$, $\quad P_\ell = 0$.

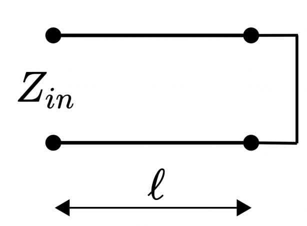
For $\alpha = 0$ (lossless), $\beta = \frac{2\pi}{\lambda}$ quad $Z_{in} = iZ_0 \tan\frac{2\pi\ell}{\lambda}$ — If $\ell = \frac{\lambda}{4}$, $\quad Z_{in} = \infty$, so the shorted line looks like an infinite impedance to a source! (even if the resistance of the wire is zero!)

V. RESONATORS FROM TRANSMISSION LINES

– It is possible to make resonators from transmission lines, 3D cavities, etc.

– The most usual case is the short-circuited transmission-line resonator.

$Z_\ell = 0, \quad Z_{in} = Z_0 \tanh(\alpha\ell - i\beta\ell) = Z_0 \frac{\tanh\alpha\ell + i\tan\beta\ell}{1 + i\tan\beta\ell\tan\alpha\ell}.$



If losses are not too large, $\alpha\ell \ll 1$, we have $\tan\alpha\ell \approx \alpha\ell$, so

$$Z_{in} = Z_0 \frac{\alpha\ell + i\tan\beta\ell}{1 + i\alpha\ell\tan\beta\ell}\ . \tag{29}$$

Now, recall that $\beta = \omega/v_p = \omega\sqrt{L'C'}, \quad v_p = 1/\sqrt{L'C'}, \quad Z_0 = \sqrt{L'/C'}$,
$\alpha = \frac{R'}{2}\sqrt{C'/L'}$.

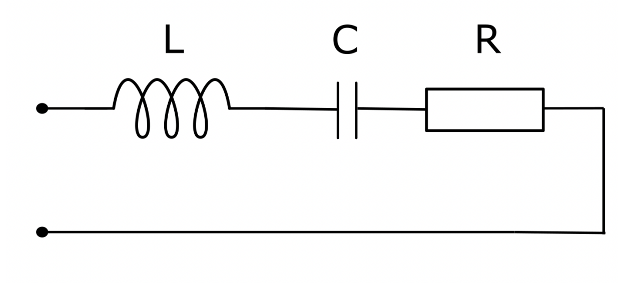
We will consider $\underline{\beta_0\ell = \pi}$, or $\ell = \lambda_0/2$ as the resonance condition, leading to a resonance frequency ω_0.
$\frac{\omega_0}{v_p}\ell = \omega_0\sqrt{L'C'}\ell = \pi$, so $\omega = \frac{\pi}{\ell\sqrt{L'C'}}$.

We can expand this around this point: $\tan\beta\ell \simeq \tan(\pi + \pi\frac{\omega-\omega_0}{\omega_0}) = \tan\pi\frac{\omega-\omega_0}{\omega_0} \simeq \pi\frac{\omega-\omega_0}{\omega_0}$, if $|\omega - \omega_0| \ll \omega_0$.

So, $\quad Z_{in} = Z_0 \frac{\alpha\ell + i\pi\frac{\omega-\omega_0}{\omega_0}}{1 + \alpha\ell\pi\frac{\omega-\omega_0}{\omega_0}} \simeq Z_0(\alpha\ell + i\pi\frac{\omega-\omega_0}{\omega_0})$
$= \sqrt{L'/C'}(\frac{\ell}{2}R'\sqrt{C'/L'} + i\ell\sqrt{L'C'}(\omega - \omega_0)) = \frac{1}{2}R'\ell + iL'\ell(\omega - \omega_0)$.

- Suppose now that we look back at the series RLC circuit

$Z = R + i\frac{L}{\omega}(\omega^2 - \omega_0^2) \simeq R + 2iL(\omega - \omega_0)$ near resonance, $\omega \simeq \omega_0$.



Therefore, we can identify $\underline{R = \frac{1}{2}R'\ell}$ and $\underline{L = \frac{1}{2}L'\ell}$.

$$\text{Quality Factor} \quad Q = \frac{\omega_0 L}{R} = \frac{\omega_0 L'}{R'} = \frac{\beta_0}{2\alpha}. \tag{30}$$

– Interesting question to think about: Why do we get the factor 1/2 in the RLC equivalent above?

–Answer: Because the current in the short-circuited line is half a sinusoid , therefore we obtain only half of the total resistance and inductance of the full length ℓ.

To see this explicitly, let us write the solution

$$\begin{cases} V(x) \simeq V^+ e^{-i\beta x} + V^- e^{i\beta x} \text{ — here we neglect } \alpha. \\ I(x) \simeq -\frac{\beta}{\omega L}(-V^+ e^{-i\beta x} + V^- e^{i\beta x}) \end{cases} \tag{31}$$

Since $I(0) = 0 \implies V^+ \equiv V^-$ at this point (also you can see that $\Gamma_V \equiv \frac{V^-}{V^+}e^{2i\beta_0\ell} = -1$ and $\beta_0\ell = \pi$).

So

$$\begin{cases} V(x) = 2V^+ \cos\beta_0 x \\ I(x) = -\frac{2i\beta}{\omega L}V^+ \sin\beta_0 x = I^+ \sin\beta_0 x\ . \end{cases} \tag{32}$$

Therefore the magnetic-field energy:

$$\overline{W_{L'}} = \int_0^{\lambda_0/2} dx \cdot \frac{1}{4}L^{'}|I(x)|^2 = \frac{1}{4}|I^+|^2 L' \int_0^{\lambda_0/2} \sin^2\beta_0 x dx = \frac{\lambda_0}{16} \cdot |I^+|^2 L'. \tag{33}$$

At resonance: $\overline{W_{C'}} = \overline{W_{L'}}$, so

$$\overline{W} = \overline{W_{C'}} + \overline{W_{L'}} = \frac{\lambda_0}{8} L' |I^+|^2 \; . \tag{34}$$

$\overline{P} = \frac{1}{2} \int_0^{\lambda_0} dx \cdot R' |I(x)|^2 = \frac{R'}{2} |I^+|^2 \int_0^{\lambda_0} \sin^2 \beta_0 x dx,$

so

$$\overline{P} = \frac{\lambda_0}{8} R' |I^+|^2 \; . \tag{35}$$

Therefore,

$$Q = \frac{\omega_0 \overline{W}}{\overline{P}} = \frac{\omega L^{'}}{R'} \; , \tag{36}$$

or: $Q = \frac{\pi}{\ell R'} \sqrt{\frac{L'}{C'}} = \frac{\pi Z_0}{\ell R'} = \frac{\pi}{2 \ell \alpha}$, where we used $\alpha \simeq \frac{R}{2 Z_0}$, $\omega_0 = \frac{\pi}{\ell \sqrt{L' C'}}$.

References

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- R.E. Collin — Foundations for Microwave Engineering.