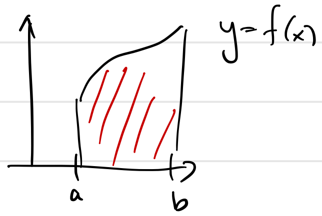
Line integrals

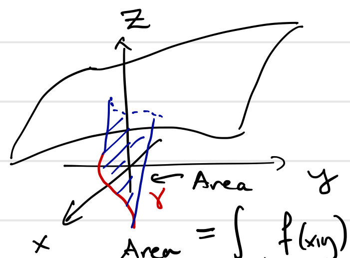
When we first meet $\int_a^b f(x)\,dx$ we think of this as an area calculation



However the applications of integrals goes beyond area calculations. The same is true for line integrals but we begin by think about these integrals as area calculations. We talk a little about curves first.

$\gamma : (a,b) \to \mathbb{R}^n$ is a C^1-curve if $\frac{d\gamma}{dt}$ is continuous and $\frac{d\gamma}{dt} \neq \vec{0}$ for all $t \in (a,b)$

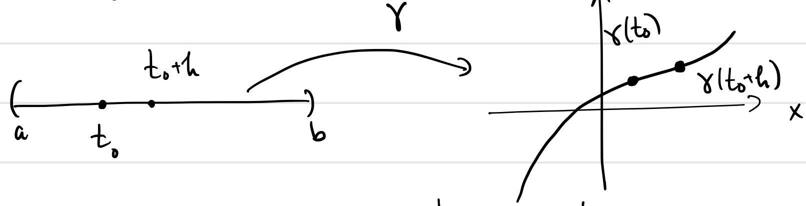
Assume $z = f(x,y)$



$$\text{Area} = \int_\gamma f(x,y)\,ds$$

We use the parametrization to reduce this calculation to an ordinary integral.

Change of length scale

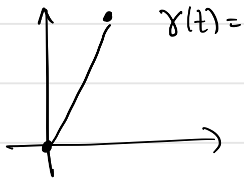


$$\left|\gamma(t_0+h)-\gamma(t_0)\right| \approx \left|\frac{d\gamma}{dt}(t_0)\right| h$$

$$\int_\gamma f(x,y)\, ds = \int_a^b f(\gamma(t)) \left|\frac{d\gamma}{dt}\right| dt$$

Ex Calculate $\int_\gamma (x^2+y^2)\, ds$ where γ is the straight line from $(0,0)$ to $(1,3)$.

Solution: First we parametrize γ.



$\gamma(t) = (t, 3t) \qquad 0 \le t \le 1$

$$\frac{d\gamma}{dt} = (1,3)$$

$$\left|\frac{d\gamma}{dt}\right| = \sqrt{1^2+3^2} = \sqrt{10}$$

$$\int_\gamma x^2+y^2\, ds = \int_0^1 \left(t^2+(3t)^2\right)\sqrt{10}\, dt = 10\sqrt{10}\int_0^1 t^2\, dt$$

$$= \frac{10\sqrt{10}}{3}$$

<u>Ex</u> Calculate the length of a circle with radius r (>0).

$$\gamma(t) = (r\cos t, r\sin t) \qquad 0 \le t \le 2\pi$$

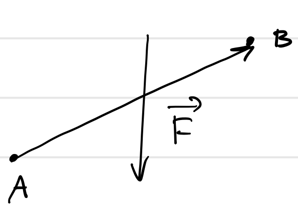
$$\text{length of } \gamma = \int_\gamma 1\, ds = \int_0^{2\pi} 1 \cdot \left|\frac{d\gamma}{dt}\right| dt =$$

$$= \left[\left|\frac{d\gamma}{dt}\right| = r \right] = \int_0^{2\pi} r\, dt = 2\pi r$$

One thing to notice. The chain rule gives that different parametrizations of γ gives the same value for the integrals. Therefore $\int_\gamma f(x,y)\, ds$ is independent of the parametrization. Another thing to note: going from a to b or from b to a gives same value ($\gamma(t)$ or $\gamma(-t)$ have the same $\left|\frac{d\gamma}{dt}\right|$)

<u>Line integrals of vector fields</u>

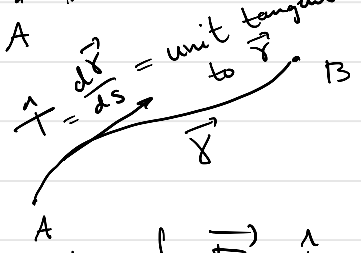
In physics work done is "force times distance"



$$W = \vec{F} \cdot \overrightarrow{AB}$$

$\vec{F}$ varies in space

How to calculate work?



$$W = \int_\gamma \vec{F} \cdot \hat{T}\, ds = \int_\gamma \vec{F} \cdot d\vec{r} = \int_\gamma F_1\, dx + F_2\, dy$$

<u>Notice</u> $\displaystyle\int_\gamma \vec{F} \cdot d\vec{r} = -\int_{-\gamma} \vec{F} \cdot d\vec{r}$

<u>Ex</u> Let $F(x,y) = (y^2, 2xy)$. Calculate

$$\int_\gamma F \cdot d\vec{r} \quad \text{when } \gamma \text{ is}$$

a) a straight line from $(0,0)$ to $(1,1)$.

b) the curve $y = x^2$ from $(0,0)$ to $(1,1)$.

<u>Solutions</u>: a) $\gamma(t) = (t,t) \quad 0 \le t \le 1$

$$\frac{d\gamma}{dt} = (1,1) \qquad \hat{T} = \frac{1}{\sqrt{2}}(1,1)$$

$$F \cdot \hat{T} = (y^2, 2xy) \cdot \frac{1}{\sqrt{2}}(1,1) = \frac{1}{\sqrt{2}}(y^2 + 2xy)$$

$$\int_0^1 \frac{1}{\sqrt{2}} \cdot 3t^2 \cdot \sqrt{2}\, dt = \left[t^3\right]_0^1 = 1$$

Notice that

$$\text{"}\ \hat{T}\, ds = \hat{T}\left|\frac{d\vec{\gamma}}{dt}\right| dt = \frac{d\vec{\gamma}}{dt}\, dt\ \text{"}$$

b) $\gamma(t) = (t, t^2)$ $\qquad \dfrac{d\vec{\gamma}}{dt} = (1, 2t) \qquad F(t) = (t^4, 2t^3)$

$$\int_\gamma F \cdot d\vec{r} = \int_0^1 (t^4, 2t^3) \cdot (1, 2t)\, dt =$$

$$= \int_0^1 t^4 + 4t^4\, dt = \int_0^1 5t^4\, dt = \left[t^5\right]_0^1 = 1.$$

Alternative: $\displaystyle\int_\gamma y^2\, dx + 2xy\, dy = \int_0^1 (t^2)^2 \frac{dx}{dt} + 2t \cdot t^2 \frac{dy}{dt}\, dt$

$$= \int_0^1 t^4 \cdot 1 + 2t^3 \cdot 2t\, dt = \int_0^1 5t^4\, dt = 1$$

We will now investigate when $\int_\gamma F \cdot d\vec{r}$ is dependent only on the end-points of γ.

First let γ be a closed curve

$$\oint_\gamma F \cdot d\vec{r} = \text{circulation of } F \text{ around } \gamma.$$

$\oint_\gamma$ indicates that the curve is closed.

Theorem If D is an open connected domain and F is a smooth vector field then the following are equivalent

1) F is conservative in D.
2) $\oint_\gamma F\cdot d\vec{r} = 0$ for every piecewise smooth closed curve γ in D.
3) $\int_\gamma F\cdot d\vec{r}$ depends only on the end-points of γ.

Proof: 1) $\Rightarrow$ 2) (in $\mathbb{R}^2$)

$$\nabla\phi = F$$

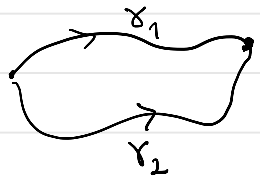
$$F\cdot d\vec{r} = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}\right)\cdot\left(\frac{dx}{dt}, \frac{dy}{dt}\right)dt = \frac{d\phi(\gamma(t))}{dt}\,dt$$

$$\int_\gamma F\cdot d\vec{r} = \int_a^b \frac{d\phi(\gamma(t))}{dt}\,dt = \phi(\gamma(b)) - \phi(\gamma(a))$$

γ closed $\Rightarrow \gamma(b) = \gamma(a)$

$$\Rightarrow \oint_\gamma F\cdot d\vec{r} = 0$$

2) $\Rightarrow$ 3)



$$\int_{\gamma_1} F\cdot d\vec{r} \neq \int_{\gamma_2} F\cdot d\vec{r}$$

$$\Rightarrow \int_{\gamma_1-\gamma_2} F\cdot d\vec{r} \neq 0 \qquad \Big(\neg 3) \Rightarrow \neg 2)\Big)$$

3) $\Rightarrow$ 1) Fix $(x_0, y_0) \in D$. For $(x,y) \in D$ choose γ from (x_0, y_0) to (x,y).

Define $\phi(x,y) = \int_\gamma F \cdot d\vec{r}$

$\phi(x,y)$ is well-defined (independent of γ)

$$\phi(x+h,y) - \phi(x,y) = \int_x^{x+h} F_1(\xi, y)\, d\xi$$

$$\frac{\partial \phi}{\partial x} = \lim_{h \to 0} \frac{\phi(x+h,y) - \phi(x,y)}{h} = F_1(x,y)$$

In the same way we get $\frac{\partial \phi}{\partial y} = F_2(x,y)$ $\otimes$

This shows why conservative vector fields are so pleasant to work with.

<u>Ex</u> Find the work done when moving an object from $(-1,0,1)$ to $(0,-2,3)$ along any smooth curve in the force field

$$F(x,y,z) = (x+y)\vec{e_1} + (x-z)\vec{e_2} + (z-y)\vec{e_3}.$$

<u>Solutions</u>: We try to construct a potential function

$$\frac{\partial \phi}{\partial x} = x+y \implies \phi(x,y,z) = \frac{x^2}{2} + xy + \alpha(y,z)$$