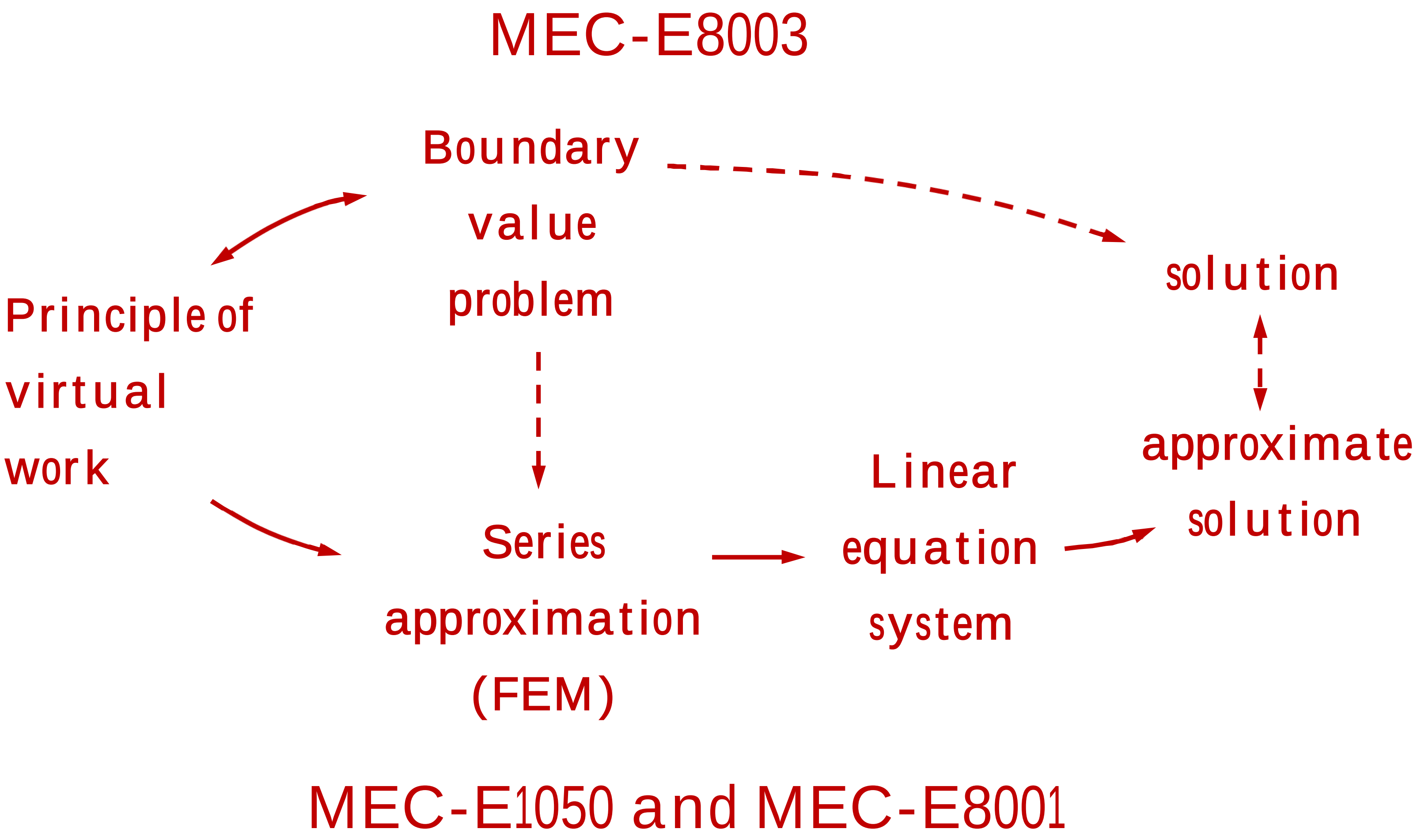
3 KINETICS

LEARNING OUTCOMES

Students are able to solve the weekly lecture problems, home problems, and exercise problems on kinetics:

- ☐ Quantities and equations of classical elasticity
- ☐ Constitutive equation of linearly elastic isotropic material
- ☐ Principle of virtual work in solid mechanics
- ☐ Derivation of engineering models by using the principle of virtual work, integration by parts, and the fundamental lemma of variation calculus

DERIVATION OF ENGINEERING MODELS



3.1 CLASSICAL LINEAR ELASTICITY

Balance of mass (def. of a body or a material volume) Mass of a body is constant. ←

Balance of linear momentum (Newton 2) The rate of change of linear momentum within a material volume equals the external force resultant acting on the material volume. ←

Balance of angular momentum (Cor. of Newton 2) The rate of change of angular momentum within a material volume equals the external moment resultant acting on the material volume. ←

Balance of energy (Thermodynamics 1)

Entropy growth (Thermodynamics 2)

LOCAL FORMS

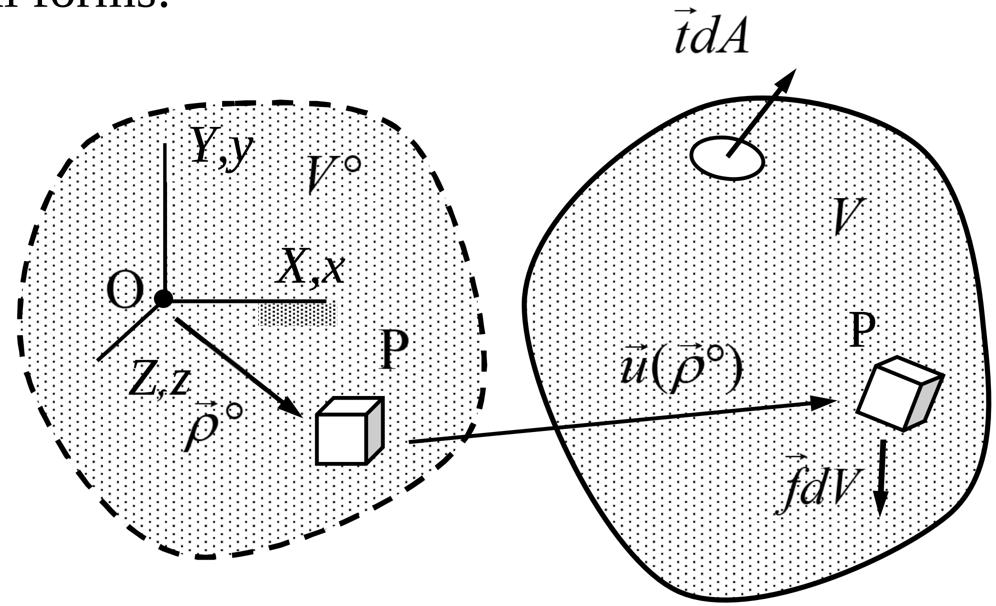
Application of the first principles to a material element inside the body or from its boundary gives the coordinate system invariant local forms:

$\dot{m}=0 \quad : \quad \rho^{\circ}=\rho J \quad \text{in } V$

$\dot{\vec{p}}=\vec{F} \quad : \quad \nabla\cdot\vec{\vec{\sigma}}+\vec{f}=0 \quad \text{in } V$

$\dot{\vec{p}}=\vec{F} \quad : \quad \vec{\sigma}\equiv\vec{n}\cdot\vec{\vec{\sigma}}=\vec{t} \quad \text{on } \partial V_t$

$\dot{\vec{L}}=\vec{M} \quad : \quad \vec{\vec{\sigma}}=\vec{\vec{\sigma}}_{\mathrm{c}} \quad \text{in } V$



Assuming an equilibrium setting (geometry, stress, loading etc.) the local forms can be used to find a new equilibrium setting (actually, displacements of the particles) when, e.g., external given forces are changed in some manner.

TRACTION AND STRESS

Material elements of a body interact with a surface force (force per unit area) called as the traction vector. Stress $\overleftrightarrow{\sigma}$ describes the surface forces acting on (all edges of) a material element. In a Cartesian $(x, y, z)-$coordinate system, the second order stress tensor

$$\overleftrightarrow{\sigma} = \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{k}\vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix} + \begin{Bmatrix} \vec{i}\vec{j} \\ \vec{j}\vec{k} \\ \vec{k}\vec{i} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix} + \begin{Bmatrix} \vec{j}\vec{i} \\ \vec{k}\vec{j} \\ \vec{i}\vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \sigma_{yx} \\ \sigma_{zy} \\ \sigma_{xz} \end{Bmatrix}$$

Traction acting on an edge of unit outward normal $\vec{n}$ is given by $\vec{\sigma} = \vec{n} \cdot \overleftrightarrow{\sigma}$ and the force (element) $d\vec{F} = \vec{\sigma} dA = \vec{n} \cdot \overleftrightarrow{\sigma} dA = d\vec{A} \cdot \overleftrightarrow{\sigma}$ where the last form uses the directed area concept $d\vec{A} = \vec{n} dA$. The representation in $(\alpha, \beta, \gamma)-$coordinate system follows by changing the basis vectors and indices of the components.

LINEAR STRAIN

Shape deformation measure of material element is the symmetric part of displacement gradient, i.e., $\vec{\vec{\varepsilon}} = [\nabla\vec{u} + (\nabla\vec{u})_{\mathrm{c}}]/2$. In a Cartesian $(x, y, z)-$coordinate system, the second order linear strain tensor

$$\vec{\vec{\varepsilon}} = \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{k}\vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{Bmatrix} + \begin{Bmatrix} \vec{i}\vec{j} \\ \vec{j}\vec{k} \\ \vec{k}\vec{i} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \end{Bmatrix} + \begin{Bmatrix} \vec{j}\vec{i} \\ \vec{k}\vec{j} \\ \vec{i}\vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \varepsilon_{yx} \\ \varepsilon_{zy} \\ \varepsilon_{xz} \end{Bmatrix} \quad \text{where}$$

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{Bmatrix} = \begin{Bmatrix} \partial u_x / \partial x \\ \partial u_y / \partial y \\ \partial u_z / \partial z \end{Bmatrix} \text{ and } \begin{Bmatrix} \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{yx} \\ \varepsilon_{zy} \\ \varepsilon_{xz} \end{Bmatrix} = \frac{1}{2} \begin{Bmatrix} \partial u_y / \partial x + \partial u_x / \partial y \\ \partial u_z / \partial y + \partial u_y / \partial z \\ \partial u_x / \partial z + \partial u_z / \partial x \end{Bmatrix}.$$

The representation in $(\alpha, \beta, \gamma)-$coordinate system follows from the definition.

LINEARLY ELASTIC MATERIAL

The generalized Hooke's law for an isotropic (properties do not depend on direction) and homogeneous (properties do not depend on position) can be expressed in tensor form $\vec{\vec{\sigma}} = \overset{\leftrightarrow}{\overset{\leftrightarrow}{E}} : \nabla\vec{u}$. In a Cartesian $(x, y, z)-$coordinate system, the fourth order elasticity tensor

$$\overset{\leftrightarrow}{\overset{\leftrightarrow}{E}} = \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{k}\vec{k} \end{Bmatrix}^{\mathrm{T}} [E] \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{k}\vec{k} \end{Bmatrix} + \begin{Bmatrix} \vec{i}\vec{j} + \vec{j}\vec{i} \\ \vec{j}\vec{k} + \vec{k}\vec{j} \\ \vec{k}\vec{i} + \vec{i}\vec{k} \end{Bmatrix}^{\mathrm{T}} [G] \begin{Bmatrix} \vec{i}\vec{j} + \vec{j}\vec{i} \\ \vec{j}\vec{k} + \vec{k}\vec{j} \\ \vec{k}\vec{i} + \vec{i}\vec{k} \end{Bmatrix}$$

depends on the 3×3 elasticity matrices $[E]$ and $[G]$ given material experiments. Representation in a $(\alpha, \beta, \gamma)-$coordinate system follows by replacing the Cartesian $(x, y, z)-$coordinate system basis vectors by their representations in terms of the basis vectors of the $(\alpha, \beta, \gamma)-$coordinate system.

The generalized Hooke's law in its component form and linear strain components (not engineering strains) according to, e.g., literature is given by

$$\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\end{Bmatrix}=[E]\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\end{Bmatrix},\quad \begin{Bmatrix}\sigma_{xy}\\ \sigma_{yz}\\ \sigma_{zx}\end{Bmatrix}=2[G]\begin{Bmatrix}\varepsilon_{xy}\\ \varepsilon_{yz}\\ \varepsilon_{zx}\end{Bmatrix},\text{ and }\begin{Bmatrix}\sigma_{yx}\\ \sigma_{zy}\\ \sigma_{xz}\end{Bmatrix}=2[G]\begin{Bmatrix}\varepsilon_{yx}\\ \varepsilon_{zy}\\ \varepsilon_{xz}\end{Bmatrix}.$$

Starting with the stress representation

$$\vec{\vec{\sigma}}=\begin{Bmatrix}\vec{i}\vec{i}\\ \vec{j}\vec{j}\\ \vec{k}\vec{k}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\end{Bmatrix}+\begin{Bmatrix}\vec{i}\vec{j}\\ \vec{j}\vec{k}\\ \vec{k}\vec{i}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\sigma_{xy}\\ \sigma_{yz}\\ \sigma_{zx}\end{Bmatrix}+\begin{Bmatrix}\vec{j}\vec{i}\\ \vec{k}\vec{j}\\ \vec{i}\vec{k}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\sigma_{yx}\\ \sigma_{zy}\\ \sigma_{xz}\end{Bmatrix},$$

Using the component forms of the generalized Hooke's law (and symmetry of strain to get rid of the multiplier 2)

$$\overleftrightarrow{\sigma} = \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{k}\vec{k} \end{Bmatrix}^{\mathrm{T}} [E] \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{Bmatrix} + \begin{Bmatrix} \vec{i}\vec{j} \\ \vec{j}\vec{k} \\ \vec{k}\vec{i} \end{Bmatrix}^{\mathrm{T}} [G] (\begin{Bmatrix} \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \end{Bmatrix} + \begin{Bmatrix} \varepsilon_{yx} \\ \varepsilon_{zy} \\ \varepsilon_{xz} \end{Bmatrix}) + \begin{Bmatrix} \vec{j}\vec{i} \\ \vec{k}\vec{j} \\ \vec{i}\vec{k} \end{Bmatrix}^{\mathrm{T}} [G] (\begin{Bmatrix} \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \end{Bmatrix} + \begin{Bmatrix} \varepsilon_{yx} \\ \varepsilon_{zy} \\ \varepsilon_{xz} \end{Bmatrix}).$$

Finally substituting the representations

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{Bmatrix} = \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{k}\vec{k} \end{Bmatrix} : \overleftrightarrow{\varepsilon}, \quad \begin{Bmatrix} \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \end{Bmatrix} = \begin{Bmatrix} \vec{j}\vec{i} \\ \vec{k}\vec{j} \\ \vec{i}\vec{k} \end{Bmatrix} : \overleftrightarrow{\varepsilon}, \text{ and } \begin{Bmatrix} \varepsilon_{yx} \\ \varepsilon_{zy} \\ \varepsilon_{xz} \end{Bmatrix} = \begin{Bmatrix} \vec{i}\vec{j} \\ \vec{j}\vec{k} \\ \vec{k}\vec{i} \end{Bmatrix} : \overleftrightarrow{\varepsilon}$$

gives

$$\overleftrightarrow{\sigma} = (\begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{k}\vec{k} \end{Bmatrix}^{\mathrm{T}} [E] \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{k}\vec{k} \end{Bmatrix} + \begin{Bmatrix} \vec{i}\vec{j} + \vec{j}\vec{i} \\ \vec{j}\vec{k} + \vec{k}\vec{j} \\ \vec{k}\vec{i} + \vec{i}\vec{k} \end{Bmatrix}^{\mathrm{T}} [G] \begin{Bmatrix} \vec{i}\vec{j} + \vec{j}\vec{i} \\ \vec{j}\vec{k} + \vec{k}\vec{j} \\ \vec{k}\vec{i} + \vec{i}\vec{k} \end{Bmatrix}) : \overleftrightarrow{\varepsilon} \equiv \overset{\leftrightarrow}{\overset{\leftrightarrow}{E}} : \overleftrightarrow{\varepsilon} = \overset{\leftrightarrow}{\overset{\leftrightarrow}{E}} : \nabla \vec{u}. \quad \Leftarrow$$

CONSTITUTIVE EQUATION VARIANTS

Stress-displacement relationship of linearly elastic material model can be expressed in various equivalent forms depending on the symmetry conditions imposed on the fourth order elasticity tensor $\overset{\leftrightarrow}{\overset{\leftrightarrow}{E}}$:

(a) $\overleftrightarrow{\sigma} = \overset{\leftrightarrow}{\overset{\leftrightarrow}{E}} : \frac{1}{2}[\nabla\vec{u} + (\nabla\vec{u})_{\mathrm{c}}]$ and $\overleftrightarrow{\sigma} = \overleftrightarrow{\sigma}_{\mathrm{c}}$ $\Leftrightarrow$

(b) $\overleftrightarrow{\sigma} = \overset{\leftrightarrow}{\overset{\leftrightarrow}{E}} : \nabla\vec{u}$ and $\overleftrightarrow{\sigma} = \overleftrightarrow{\sigma}_{\mathrm{c}}$ and $\overset{\leftrightarrow}{\overset{\leftrightarrow}{E}} = \overset{\leftrightarrow}{\overset{\leftrightarrow}{E}}_{\cdot\mathrm{c}}$ $\Leftrightarrow$ Last index pair conjugate!

(c) $\overleftrightarrow{\sigma} = \overset{\leftrightarrow}{\overset{\leftrightarrow}{E}} : \nabla\vec{u}$ and $\overset{\leftrightarrow}{\overset{\leftrightarrow}{E}} = \overset{\leftrightarrow}{\overset{\leftrightarrow}{E}}_{\cdot\mathrm{c}} = \overset{\leftrightarrow}{\overset{\leftrightarrow}{E}}_{\mathrm{c}\cdot} = \overset{\leftrightarrow}{\overset{\leftrightarrow}{E}}_{\mathrm{cc}}$ $\Leftarrow$

Also, other kinetic conditions like $\sigma_{zz} = 0$ can be satisfied ‘a priori’ by the selection of elasticity tensor. The conditions of (c) are called as the minor and major symmetries.

ISOTROPIC MATERIAL

The generalized Hooke's law for an isotropic material follows with the elasticity matrices

$$[E]=E\begin{bmatrix}1 & -\nu & -\nu\\ -\nu & 1 & -\nu\\ -\nu & -\nu & 1\end{bmatrix}^{-1}=\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & \nu\\ \nu & 1-\nu & \nu\\ \nu & \nu & 1-\nu\end{bmatrix},$$

$$[G]=\begin{bmatrix}G & 0 & 0\\ 0 & G & 0\\ 0 & 0 & G\end{bmatrix}=\frac{E}{2(1+\nu)}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

in which the material parameters E and ν are the Young's modulus and the Poisson's ratio, respectively, and $G=E/(2+2\nu)$ the shear modulus. Using these, one may deduce the elasticity matrices for the engineering models.

In the coordinate system invariant form $\overset{\leftrightarrow}{\sigma} = \overset{\leftrightarrow}{\overset{\leftrightarrow}{E}} : \overset{\leftrightarrow}{\varepsilon} = \overset{\leftrightarrow}{\overset{\leftrightarrow}{E}} : \nabla \vec{u}$, the elasticity tensor (satisfying the major and minor symmetries) is given by

$$\overset{\leftrightarrow}{\overset{\leftrightarrow}{E}} = \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{k}\vec{k} \end{Bmatrix}^{\mathrm{T}} E \begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix}^{-1} \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{k}\vec{k} \end{Bmatrix} + \begin{Bmatrix} \vec{i}\vec{j} + \vec{j}\vec{i} \\ \vec{j}\vec{k} + \vec{k}\vec{j} \\ \vec{k}\vec{i} + \vec{i}\vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} G & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & G \end{bmatrix} \begin{Bmatrix} \vec{i}\vec{j} + \vec{j}\vec{i} \\ \vec{j}\vec{k} + \vec{k}\vec{j} \\ \vec{k}\vec{i} + \vec{i}\vec{k} \end{Bmatrix}.$$

Elasticity tensor of plate model ($\sigma_{zz} = 0$)

$$\overset{\leftrightarrow}{\overset{\leftrightarrow}{E}} = \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{k}\vec{k} \end{Bmatrix}^{\mathrm{T}} \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{k}\vec{k} \end{Bmatrix} + \begin{Bmatrix} \vec{i}\vec{j} + \vec{j}\vec{i} \\ \vec{j}\vec{k} + \vec{k}\vec{j} \\ \vec{k}\vec{i} + \vec{i}\vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} G & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & G \end{bmatrix} \begin{Bmatrix} \vec{i}\vec{j} + \vec{j}\vec{i} \\ \vec{j}\vec{k} + \vec{k}\vec{j} \\ \vec{k}\vec{i} + \vec{i}\vec{k} \end{Bmatrix}.$$

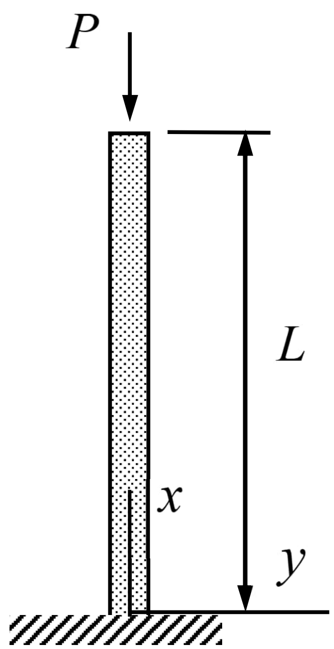
Elasticity tensor of the beam model ($\sigma_{yy} = \sigma_{zz} = 0$)

$$\overset{\leftrightarrow}{E} = \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{k}\vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} E & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{k}\vec{k} \end{Bmatrix} + \begin{Bmatrix} \vec{i}\vec{j} + \vec{j}\vec{i} \\ \vec{j}\vec{k} + \vec{k}\vec{j} \\ \vec{k}\vec{i} + \vec{i}\vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} G & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & G \end{bmatrix} \begin{Bmatrix} \vec{i}\vec{j} + \vec{j}\vec{i} \\ \vec{j}\vec{k} + \vec{k}\vec{j} \\ \vec{k}\vec{i} + \vec{i}\vec{k} \end{Bmatrix}.$$

Representation in some other system can be obtained from the Cartesian $(x, y, z)-$system representation by using the relationships between the basis vectors. For example, in the cylindrical $(r, \phi, z)-$coordinate system

$$\overset{\leftrightarrow}{E} = \begin{Bmatrix} \vec{e}_r\vec{e}_r \\ \vec{e}_\phi\vec{e}_\phi \\ \vec{e}_z\vec{e}_z \end{Bmatrix}^{\mathrm{T}} [E] \begin{Bmatrix} \vec{e}_r\vec{e}_r \\ \vec{e}_\phi\vec{e}_\phi \\ \vec{e}_z\vec{e}_z \end{Bmatrix} + \begin{Bmatrix} \vec{e}_r\vec{e}_\phi + \vec{e}_\phi\vec{e}_r \\ \vec{e}_\phi\vec{e}_z + \vec{e}_z\vec{e}_\phi \\ \vec{e}_z\vec{e}_r + \vec{e}_r\vec{e}_z \end{Bmatrix}^{\mathrm{T}} [G] \begin{Bmatrix} \vec{e}_r\vec{e}_\phi + \vec{e}_\phi\vec{e}_r \\ \vec{e}_\phi\vec{e}_z + \vec{e}_z\vec{e}_\phi \\ \vec{e}_z\vec{e}_r + \vec{e}_r\vec{e}_z \end{Bmatrix}.$$

EXAMPLE The cross section of the column is square of side length h. Density ρ, Young's modulus E, and Poisson's ratio ν are constants. The column is loaded by a constant traction of magnitude P/h^2 at its free end. Determine stress $\vec{\vec{\sigma}}$ and displacement $\vec{u}$ starting from the generic equations for linear elasticity. Assume that the transverse (to the axis) displacement is not constrained by the support.



Answer $\vec{u} = \dfrac{P}{Eh^2}(-x\vec{i} + \nu y\vec{j} + \nu z\vec{k}), \quad \vec{\vec{\sigma}} = -\dfrac{P}{h^2}\vec{i}\vec{i}$

The component forms of the equilibrium equations and constitutive equations of a linearly elastic isotropic material in a Cartesian $(x, y, z)-$coordinate system

$$\begin{Bmatrix} \partial\sigma_{xx}/\partial x + \partial\sigma_{yx}/\partial y + \partial\sigma_{zx}/\partial z + f_x \\ \partial\sigma_{xy}/\partial x + \partial\sigma_{yy}/\partial y + \partial\sigma_{zy}/\partial z + f_y \\ \partial\sigma_{xz}/\partial x + \partial\sigma_{yz}/\partial y + \partial\sigma_{zz}/\partial z + f_z \end{Bmatrix} = 0,$$

$$\begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial w/\partial z \end{Bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix}\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix}, \text{ and } \begin{Bmatrix} \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix} = \begin{Bmatrix} \sigma_{yx} \\ \sigma_{zy} \\ \sigma_{xz} \end{Bmatrix} = G\begin{Bmatrix} \partial u/\partial y + \partial v/\partial x \\ \partial v/\partial z + \partial w/\partial y \\ \partial w/\partial x + \partial u/\partial z \end{Bmatrix}.$$

Let us assume that the only non-zero stress component $\sigma_{xx}(x)$ and displacement components $u_x = u(x)$, $u_y = v(y)$ and $u_z = w(z)$. The axial stress follows from the equilibrium equation and traction is known at the free end $x = L$. Therefore

$$\frac{d\sigma_{xx}}{dx} = 0 \quad 0 < x < L \quad \text{and} \quad \sigma_{xx}(L) = -\frac{P}{h^2} \quad \Rightarrow \quad \sigma_{xx}(x) = -\frac{P}{h^2}.$$

Generalized Hooke's law written for the uniaxial stress implies that

$$\frac{du}{dx} = \frac{\sigma_{xx}}{E} = -\frac{P}{Eh^2}, \quad \frac{dv}{dy} = -\frac{\nu}{E}\sigma_{xx} = \nu\frac{P}{Eh^2}, \quad \frac{dw}{dz} = -\frac{\nu}{E}\sigma_{xx} = \nu\frac{P}{Eh^2}.$$

Axial displacement vanishes at the support and the transverse displacement at the axis:

$$\frac{du}{dx} = -\frac{P}{Eh^2} \quad 0 < x < L \quad \text{and} \quad u(0) = 0 \quad \Rightarrow \quad u(x) = -\frac{P}{Eh^2}x, \quad \Leftarrow$$

$$\frac{dv}{dy} = \nu\frac{P}{Eh^2} \quad -\frac{1}{2}h < y < \frac{1}{2}h \quad \text{and} \quad v(0) = 0 \quad \Rightarrow \quad v(y) = \nu\frac{P}{Eh^2}y, \quad \Leftarrow$$

$$\frac{dw}{dz} = -\nu\frac{P}{Eh^2} \quad -\frac{1}{2}h < z < \frac{1}{2}h \quad \text{and} \quad w(0) = 0 \quad \Rightarrow \quad w(z) = \nu\frac{P}{Eh^2}z. \quad \Leftarrow$$

3.2 PRINCIPLE OF VIRTUAL WORK

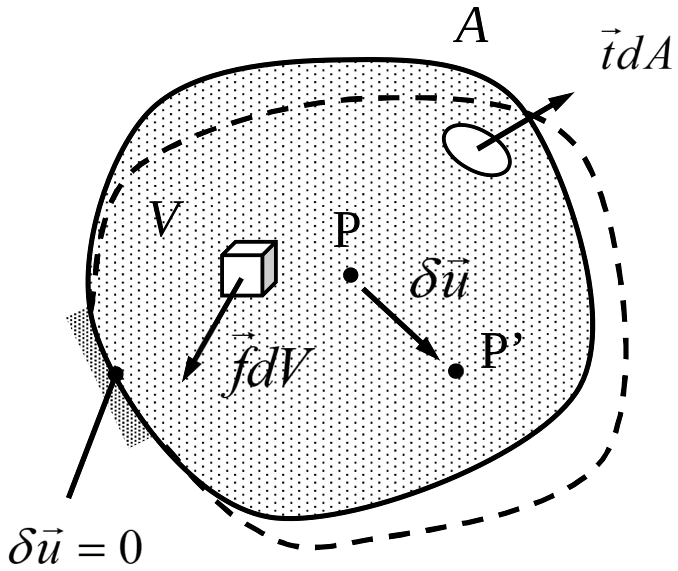
Principle of virtual work $\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = 0 \quad \forall \delta\vec{u} \in U$ is just one representation of the balance laws of continuum mechanics. It is important due to its wide applicability and physical meanings of the terms.

virtual work density

$$\delta W^{\text{int}} = \int_V \delta w_V^{\text{int}} dV = -\int_V (\vec{\vec{\sigma}} : \delta\vec{\vec{\varepsilon}}_{\text{c}}) dV$$

$$\delta W_V^{\text{ext}} = \int_V \delta w_V^{\text{ext}} dV = \int_V (\vec{f} \cdot \delta\vec{u}) dV$$

$$\delta W_A^{\text{ext}} = \int_A \delta w_A^{\text{ext}} dA = \int_A (\vec{t} \cdot \delta\vec{u}) dA$$



The details of the expressions vary case by case, but the principle itself does not!

In what follows, we skip some of the technical details and assume that displacement boundary conditions are satisfied 'a priori'. The local and variational forms of elasticity problem are equivalent, i.e., the local form implies the variational form and the other way around. Let us consider first the derivation of the variational form:

$$\left.\begin{aligned} &\nabla\cdot\vec{\vec{\sigma}}+\vec{f}=0 \quad \text{and} \quad \vec{\vec{\sigma}}=\vec{\vec{\sigma}}_{\mathrm{c}} \quad \text{in } V, \\ &\vec{u}-\underline{\vec{u}}=0 \quad \text{or} \quad \vec{n}\cdot\vec{\vec{\sigma}}-\vec{t}=0 \quad \text{on } \partial V. \end{aligned}\right\} \quad \text{local form}$$

Multiplication of the momentum equation by virtual displacement $\delta\vec{u}$, integration over the solution domain, and integration by parts with $(\nabla\cdot\vec{\vec{a}})\cdot\vec{b}=\nabla\cdot(\vec{\vec{a}}\cdot\vec{b})-\vec{\vec{a}}:(\nabla\vec{b})_{\mathrm{c}}$ (selections $\vec{\vec{a}}=\vec{\vec{\sigma}}$ and $\vec{b}=\delta\vec{u}$), and division of the displacement gradient into its symmetric and anti-symmetric parts according to $\nabla\vec{u}=\vec{\vec{\varepsilon}}+\vec{\vec{\phi}}$ give

$$\int_V (\nabla \cdot \overset{\leftrightarrow}{\sigma} + \vec{f}) \cdot \delta\vec{u} dV = 0 \quad \forall \delta\vec{u} \in U \quad \Rightarrow$$

$$\int_V (-\overset{\leftrightarrow}{\sigma} : \delta\overset{\leftrightarrow}{\varepsilon}_{\mathrm{c}}) dV + \int_V (\vec{f} \cdot \delta\vec{u}) dV + \int_{\partial V} (\vec{n} \cdot \overset{\leftrightarrow}{\sigma} \cdot \delta\vec{u}) dA = 0 \quad \forall \delta\vec{u} \in U .$$

The boundary conditions of the local form imply that either $\delta\vec{u} = 0$ or $\vec{n} \cdot \overset{\leftrightarrow}{\sigma} = \vec{t}$ at all points of ∂V. Therefore, one ends up with

$$\delta W = \int_V (-\overset{\leftrightarrow}{\sigma} : \delta\overset{\leftrightarrow}{\varepsilon}_{\mathrm{c}}) dV + \int_V (\vec{f} \cdot \delta\vec{u}) dV + \int_{\partial V} (\vec{t} \cdot \delta\vec{u}) dA = 0 \quad \forall \delta\vec{u} \in U .$$

variational form

The derivation assumes that $\overset{\leftrightarrow}{\sigma} = \overset{\leftrightarrow}{\sigma}_{\mathrm{c}}$ (where exactly?). In practice, symmetry of stress is satisfied 'a priori' by the form of the constitutive equation.

In derivation to the reverse direction (with the assumption $\overset{\leftrightarrow}{\sigma} = \overset{\leftrightarrow}{\sigma}_{\mathrm{c}}$ for consistency), the starting point is the variational form. One substitutes first division $\overset{\leftrightarrow}{\varepsilon} = \nabla\vec{u} - \overset{\leftrightarrow}{\phi}$ to get

$$\delta W = \int_V [-\vec{\vec{\sigma}} : (\nabla \delta \vec{u})_c] dV + \int_V (\vec{f} \cdot \delta \vec{u}) dV + \int_{\partial V} (\vec{t} \cdot \delta \vec{u}) dA = 0 \quad \forall \delta \vec{u} \ .$$

Integration by parts with $(\nabla \cdot \vec{a}) \cdot \vec{b} = \nabla \cdot (\vec{a} \cdot \vec{b}) - \vec{a} : (\nabla \vec{b})_c$ (selections $\vec{a} = \vec{\vec{\sigma}}$ and $\vec{b} = \delta \vec{u}$) gives an equivalent but more convenient form

$$\delta W = \int_V (\nabla \cdot \vec{\vec{\sigma}} + \vec{f}) \cdot \delta \vec{u} dV + \int_{\partial V} (-\vec{n} \cdot \vec{\vec{\sigma}} + \vec{t}) \cdot \delta \vec{u} dA = 0 \quad \forall \delta \vec{u} \ .$$

The variational form, together with the assumed symmetry of stress and the conditions for the function set U, implies equations

$$\left.\begin{aligned} &\nabla \cdot \vec{\vec{\sigma}} + \vec{f} = 0 \quad \text{and} \quad \vec{\vec{\sigma}} - \vec{\vec{\sigma}}_c = 0 \ \text{ in } V, \\ &\vec{n} \cdot \vec{\vec{\sigma}} - \vec{t} = 0 \quad \text{or} \quad \vec{u} - \underline{\vec{u}} = 0 \quad \text{on } \partial V . \end{aligned}\right\} \text{The starting point}$$

BOUNDARY VALUE PROBLEM

Principle of virtual work is one of the variational forms of equations of mechanics. Given a variational form, the underlying boundary value problems follows with the steps:

First, use integration by parts in the integral over the mathematical solution domain to remove the derivatives acting on the variations of displacement components.

Second, use the fundamental lemma of variation calculus to deduce the differential equation(s) and boundary (natural) conditions. Consider convenient subsets of possible displacement variations to deduce first the equilibrium equation and thereafter the conditions at the boundaries.

Third, deduce the additional (essential) boundary conditions using the set of displacement variations (for example, if variation of a quantity vanishes, the quantity is given).

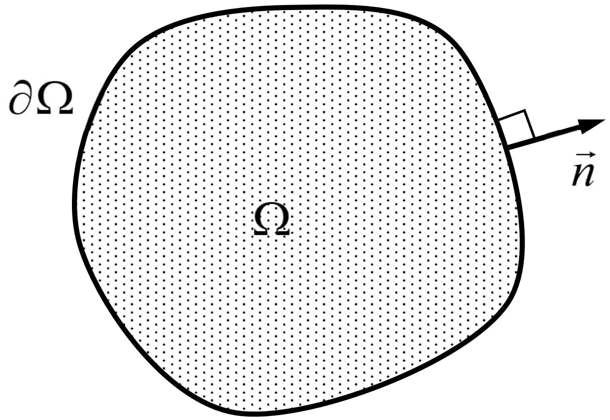
GAUSS'S THEOREM

Divergence theorem is needed in transforming between the local and variational forms of a boundary value problem. For a continuous function $a \in C^0(\Omega)$, the fundamental theorem of calculus implies, e.g.,

1D: $$\int_\Omega \frac{da}{dx} dx = \sum_{\partial\Omega} an$$

2D: $$\int_\Omega (\nabla \cdot \vec{a}) dA = \int_{\partial\Omega} \vec{a} \cdot \vec{n} ds.$$

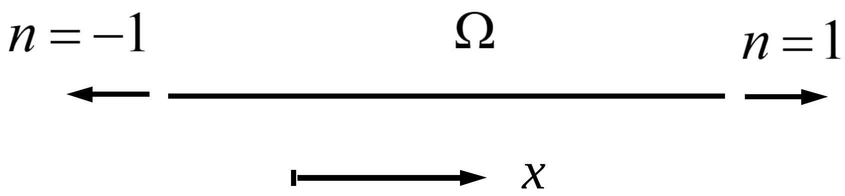
3D: $$\int_\Omega (\nabla \cdot \vec{a}) dV = \int_{\partial\Omega} \vec{a} \cdot \vec{n} dA.$$



The generic theorem implies useful integral identities for various purposes. In derivation of a boundary value problem from its variational form, one uses selections like ab and $\vec{a} \cdot \vec{b}$ with generic vector identities like $(\nabla \cdot \vec{a}) \cdot \vec{b} = \nabla \cdot (\vec{a} \cdot \vec{b}) - \vec{a} : (\nabla \vec{b})_c$.

In the one-dimensional case, the summing on the right-hand side is over the boundary points and the unit normal to the boundary $n = \pm 1$. The integration by parts identity

$$\int_{\Omega} a \frac{db}{dx} dx = \sum_{\partial\Omega} (nab) - \int_{\Omega} b \frac{da}{dx} dx$$



follows with selection ab of the function. Assumption of continuity is essential, and the simple form of integration by parts formula above requires modifications for, e.g., a discontinuity inside Ω. A useful integration by parts identity for several dimension

$$\int_{\Omega} \vec{\vec{a}} : (\nabla \vec{b})_{c} dV = \int_{\partial\Omega} (\vec{n} \cdot \vec{\vec{a}} \cdot \vec{b}) dA - \int_{\Omega} (\nabla \cdot \vec{\vec{a}}) \cdot \vec{b} dV$$

follows with selection $\vec{\vec{a}} \cdot \vec{b}$ and use of vector identity $\nabla \cdot (\vec{\vec{a}} \cdot \vec{b}) = \vec{\vec{a}} : (\nabla \vec{b})_{c} + (\nabla \cdot \vec{\vec{a}}) \cdot \vec{b}$. The various versions of integration by parts identities will be used to move derivatives to act on certain parts of integrand.

FUNDAMENTAL LEMMA OF VARIATION CALCULUS

- ☐ $a, b \in \mathbb{R}$: $ab = 0 \quad \forall b \quad \Leftrightarrow \quad a = 0$
- ☐ $\{a\}, \{b\} \in \mathbb{R}^n$: $\{a\}^{\mathrm{T}} \{b\} = 0 \quad \forall \{b\} \quad \Leftrightarrow \quad \vec{a} = 0$
- ☐ $\vec{a}, \vec{b} \in \mathbb{R}^3$: $\vec{a} \cdot \vec{b} = 0 \quad \forall \vec{b} \quad \Leftrightarrow \quad \vec{a} = 0$
- ☐ $a, b \in C^0(\Omega)$: $\int_\Omega ab d\Omega = 0 \quad \forall b \quad \Leftrightarrow \quad a = 0 \text{ in } \Omega$
- ☐ $a, b \in C^2(\Omega)$: $\int_\Omega \nabla a \cdot \nabla b d\Omega = 0 \;\; \forall b \;\; \Leftrightarrow \;\; \nabla^2 a = 0 \text{ in } \Omega,\; a = \underline{a} \text{ or } \vec{n} \cdot \nabla a = 0 \text{ on } \partial\Omega$

In connection with principle of virtual work, b is taken to be kinematically admissible variation $\delta\vec{u}$ of displacement $\vec{u}$ (vanishes whenever $\vec{u}$ is known).

EXAMPLE Principle of virtual work for a Bernoulli beam problem is given by: find $w \in U$ such that $\forall \delta w \in U$

$$\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = \int_{\Omega} (-\frac{d^2 \delta w}{dx^2} EI \frac{d^2 w}{dx^2} + \delta w b) dx = 0$$

in which $\Omega = (0, L)$, $U = \{w \in C^4(\Omega) : w = dw / dx = 0 \text{ at } x = 0\}$ and the bending stiffness $EI(x)$ and $b(x)$ are given. Deduce the underlying boundary value problem by using integration by parts and the fundamental lemma of variation calculus.

Answer $-\frac{d^2}{dx^2}(EI \frac{d^2 w}{dx^2}) + b = 0$ in $(0, L)$, $\frac{d}{dx}(EI \frac{d^2 w}{dx^2}) = 0$ at $x = L$,

$-EI \frac{d^2 w}{dx^2} = 0$ at $x = L$, $\frac{dw}{dx} = 0$ at $x = 0$, and $w = 0$ at $x = 0$

Integration by parts twice in the first term gives an equivalent form (notice that $\delta w \in U$ and therefore $\delta w = d\delta w / dx = 0$ at $x = 0$)

$$\delta W = \int_\Omega (-\frac{d^2\delta w}{dx^2} EI \frac{d^2 w}{dx^2} + \delta w b) dx \quad \Leftrightarrow$$

$$\delta W = \int_\Omega [\frac{d\delta w}{dx}\frac{d}{dx}(EI \frac{d^2 w}{dx^2}) + \delta w b] dx - [\frac{d\delta w}{dx}(EI \frac{d^2 w}{dx^2})]_{x=L} \quad \Leftrightarrow$$

$$\delta W = \int_\Omega [-\frac{d^2}{dx^2}(EI \frac{d^2 w}{dx^2}) + b]\delta w dx - [\frac{d\delta w}{dx}(EI \frac{d^2 w}{dx^2}) - \delta w \frac{d}{dx}(EI \frac{d^2 w}{dx^2})]_{x=L} .$$

According to principle of virtual work $\delta W = 0 \ \forall \delta w \in U$. Let us first consider a subset $U_0 \subset U$ for which $\delta w = d\delta w / dx = 0$ at $x = L$ so that the boundary terms vanish. The equilibrium equation follows from the fundamental lemma of variation calculus:

$$\delta W = \int_\Omega [-\frac{d^2}{dx^2}(EI\frac{d^2 w}{dx^2}) + b]\delta w dx = 0 \quad \Rightarrow \quad -\frac{d^2}{dx^2}(EI\frac{d^2 w}{dx^2}) + b = 0 \quad \text{in } (0, L).$$

After that, let us consider U with restriction $d\delta w / dx = 0$ first and then with $\delta w = 0$ at $x = L$ and simplify the virtual work expression by using the equilibrium equation already obtained. The natural boundary conditions follow from the fundamental lemma of variation calculus

$$\delta W = [\delta w \frac{d}{dx}(EI\frac{d^2 w}{dx^2})]_{x=L} = 0 \quad \Rightarrow \quad \frac{d}{dx}(EI\frac{d^2 w}{dx^2}) = 0 \quad \text{at} \quad x = L,$$

$$\delta W = -[\frac{d\delta w}{dx}(EI\frac{d^2 w}{dx^2})]_{x=L} = 0 \quad \Rightarrow \quad -EI\frac{d^2 w}{dx^2} = 0 \quad \text{at} \quad x = L .$$

Boundary conditions $w = dw / dx = 0$ at $x = 0$ follow from assumption $w \in U$.

3.3 DERIVATION OF ENGINEERING MODELS

First, write the virtual work expression by using the virtual work densities of an engineering model. If not available, start with the generic virtual work expression, kinematical and kinetic assumptions of the model, and integrate over the small dimensions.

Second, use the principle of virtual work, integration by parts, and the fundamental lemma of variation calculus to deduce the field equation(s) and (natural) boundary conditions in terms of stress resultants. Consider suitable subset of function space U to deduce first the equilibrium equation and thereafter the conditions at the boundaries.

Third, use the definitions of the stress resultants to derive the constitutive equations corresponding to the material model required.

DENSITY EXPRESSIONS

Virtual work densities (virtual work per unit volume or area) of the internal forces, external volume forces, and external surface forces. In a Cartesian $(x, y, z)-$coordinate system

$$\delta w_V^{\text{int}} = \delta\vec{\vec{\varepsilon}}_{\text{c}} : \vec{\vec{\sigma}} = -\begin{Bmatrix} \delta\varepsilon_{xx} \\ \delta\varepsilon_{yy} \\ \delta\varepsilon_{zz} \end{Bmatrix}^{\text{T}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix} - \begin{Bmatrix} \delta\varepsilon_{xy} \\ \delta\varepsilon_{yz} \\ \delta\varepsilon_{zx} \end{Bmatrix}^{\text{T}} \begin{Bmatrix} \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix} - \begin{Bmatrix} \delta\varepsilon_{yx} \\ \delta\varepsilon_{zy} \\ \delta\varepsilon_{xz} \end{Bmatrix}^{\text{T}} \begin{Bmatrix} \sigma_{yx} \\ \sigma_{zy} \\ \sigma_{xz} \end{Bmatrix},$$

$$\delta w_V^{\text{ext}} = \delta\vec{u} \cdot \vec{f} = \begin{Bmatrix} \delta u_x \\ \delta u_y \\ \delta u_z \end{Bmatrix}^{\text{T}} \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix} \text{ and } \delta w_A^{\text{ext}} = \delta\vec{u} \cdot \vec{t} = \begin{Bmatrix} \delta u_x \\ \delta u_y \\ \delta u_z \end{Bmatrix}^{\text{T}} \begin{Bmatrix} t_x \\ t_y \\ t_z \end{Bmatrix}.$$

The terms of the expressions consist of work conjugate pairs of kinematic and kinetic quantities. As stress is symmetric $\vec{\vec{\sigma}} = \vec{\vec{\sigma}}_{\text{c}}$, one may write $(\delta\nabla\vec{u})_{\text{c}} : \vec{\vec{\sigma}} = \delta\vec{\vec{\varepsilon}}_{\text{c}} : \vec{\vec{\sigma}}$.

THIN BODY ASSUMPTIONS

Bar: $\vec{u}(x,y,z) = \vec{u}_0(x)$ and $\sigma_{yy} = \sigma_{zz} = \sigma_{xy} = \sigma_{yz} = \sigma_{zx} = 0$

String: $\vec{u}(s,n,b) = \vec{u}_0(s)$ and $\sigma_{nn} = \sigma_{bb} = \sigma_{sn} = \sigma_{nb} = \sigma_{bs} = 0$

Straight beam: $\vec{u}(x,y,z) = \vec{u}_0(x) + \vec{\theta}(x) \times \vec{\rho}(y,z)$ and $\sigma_{yy} = \sigma_{zz} = 0$

Curved beam: $\vec{u}(s,n,b) = \vec{u}_0(s) + \vec{\theta}(s) \times \vec{\rho}(n,b)$ and $\sigma_{nn} = \sigma_{bb} = 0$

Thin slab: $\vec{u}(x,y,z) = \vec{u}_0(x,y)$ and $\sigma_{zz} = \sigma_{yz} = \sigma_{zx} = 0$

Membrane: $\vec{u}(\alpha,\beta,n) = \vec{u}_0(\alpha,\beta)$ and $\sigma_{nn} = \sigma_{\beta n} = \sigma_{n\alpha} = 0$

Plate: $\vec{u}(x,y,z) = \vec{u}_0(x,y) + \vec{\theta}(x,y) \times \vec{\rho}(z)$ and $\sigma_{zz} = 0$

Shell: $\vec{u}(z,s,n) = \vec{u}_0(z,s) + \vec{\theta}(z,s) \times \vec{\rho}(n)$ and $\sigma_{nn} = 0$

BAR EQUATIONS

Bar is one of the loading modes of the beam model and it can be considered also as the elasticity problem in one dimension. The model assumes that displacement and stress have just axial components depending on the axial coordinate only. In a Cartesian $(x, y, z)-$ coordinate system, the bar boundary value problem is given by

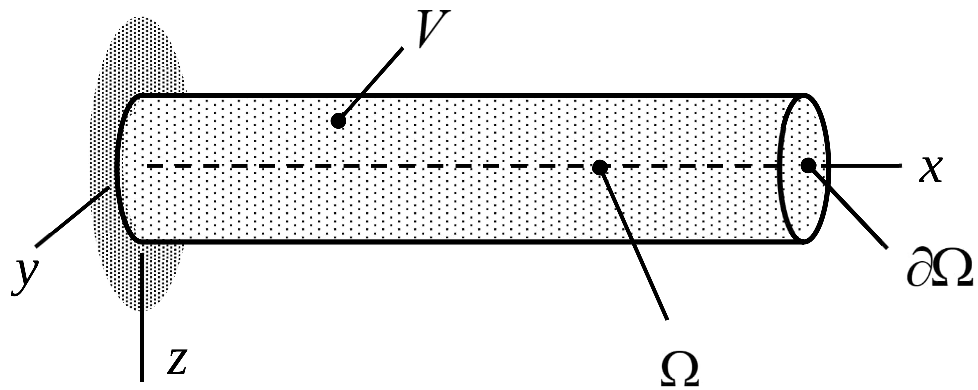
$$\frac{dN}{dx} + b = 0 \text{ in } \Omega \quad \text{and} \quad nN - \underline{F} = 0 \text{ or } u - \underline{u} = 0 \text{ on } \partial\Omega,$$

where

$$N = \int \sigma_{xx} dA, \; b = \int f_x dA, \text{ and } \underline{F} = \int t_x dA.$$

For a closed equation system (number of equations and unknown functions should match) a material model is also needed (Hooke's law).

The physical domain of the bar model is V occupied by a body althought the solution domain of the equations is the mid-line Ω. The starting point is the virtual work expression written for the physical domain.



Let us consider the steps in the Cartesian $(x, y, z)-$coordinate system for clarity. The bar model assumes that displacement and stress have just axial components depending on the axial coordinate only. Representations of stress, displacement and gradient operator are $\vec{\vec{\sigma}} = \sigma_{xx}\vec{i}\vec{i}$ and $\vec{u}(x) = u(x)\vec{i}$, $\nabla = \vec{i}\partial/\partial x + \vec{j}\partial/\partial y + \vec{k}\partial/\partial z$.

$$\delta W^{\text{int}} = -\int_V (\nabla \delta \vec{u})_{\text{c}} : \vec{\sigma} dV = -\int_\Omega \frac{d\delta u}{dx}(\int \sigma_{xx} dA)dx = -\int_\Omega \frac{d\delta u}{dx} N dx$$

$$\delta W^{\text{ext}} = \int_V \delta\vec{u} \cdot \vec{f} dV + \int_{\partial V} \delta\vec{u} \cdot \vec{t} dA = \int_\Omega \delta u b dx + \sum_{\partial\Omega} \delta u F$$

in which (integrals over the cross-sectional area)

$$N = \int \sigma_{xx} dA, \ \ b = \int f_x dA, \ \text{ and } \ F = \int t_x dA.$$

According to the principle of virtual work $\delta W = 0 \ \ \forall \delta u \in U$. Integration by parts is used first to obtain a more convenient form for deducing the bar equations.

$$\delta W = -\int_\Omega (N \frac{d\delta u}{dx}) dx + \int_\Omega (b \delta u) dx + \sum_{\partial\Omega} (F \delta u) = 0 \quad \Leftrightarrow$$

$$\delta W = \int_\Omega (\frac{dN}{dx} + b) \delta u dx + \sum_{\partial\Omega} (-nN + F) \delta u = 0 \ \text{ in which } n = \pm 1.$$

After that, by considering a subset of variations $\delta u \in U$ with restriction $\delta u = 0$ on $\partial\Omega$ and using the fundamental lemma of variational calculus

$$\delta W = \int_\Omega \left(\frac{dN}{dx} + b\right)\delta u dx = 0 \quad \forall\, \delta u \in U \quad \Leftrightarrow \quad \frac{dN}{dx} + b = 0 \text{ in } \Omega.$$

By considering next $\delta u \in U$ without restrictions on the boundary (and using the equilibrium equation to get rid of the first term of the virtual work expression)

$$\delta W = \sum_{\partial\Omega}(-nN + F)\delta u = 0 \quad \forall\, \delta u \in U \quad \Leftrightarrow \quad nN - F = 0 \text{ on } \partial\Omega.$$

The boundary term vanishes also if $\delta u = 0$ on $\partial\Omega$ which implies that u is given on $\partial\Omega$. Therefore, on the boundary either $u - \underline{u} = 0$ or $nN - F = 0$ but not both. In solid mechanics, one may specify the force or displacement, but not both. The constitutive equation for an elastic material follows from the generalized Hooke's law for the bar model $\sigma_{xx} = E du / dx$ and the definition of stress resultant

$$N = \int \sigma_{xx} dA = EA\frac{du}{dx}.$$

The bar model boundary value problem combines the equations

$$\left.\begin{array}{l} \dfrac{dN}{dx} + b = 0 \text{ and } N = EA\dfrac{du}{dx} \text{ in } \Omega, \\[2ex] nN - F = 0 \text{ or } u - \underline{u} = 0 \text{ on } \partial\Omega. \end{array}\right\} \text{Local form}$$

For an unique solution, the displacement boundary condition should be given at least on one boundary point.

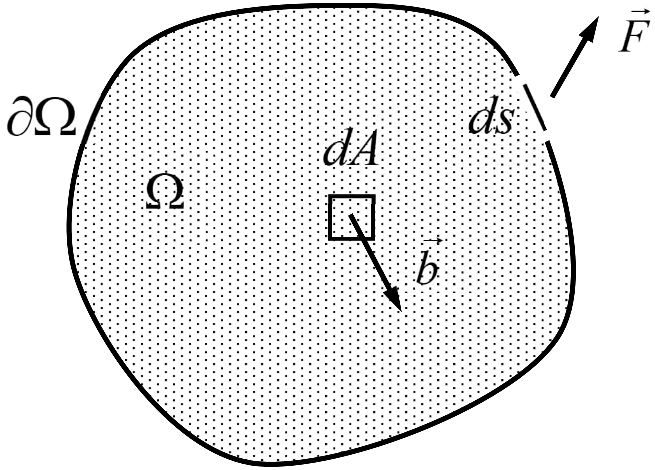
THIN SLAB EQUATIONS

Thin slab model assumes that the transverse displacement (perpendicular to the mid-plane) and stress components vanish and that the quantitities do not depend on the transverse coordinate. Principle of virtual work gives

$$\nabla \cdot \overleftrightarrow{N} + \vec{b} = 0 \quad \text{in} \quad \Omega,$$

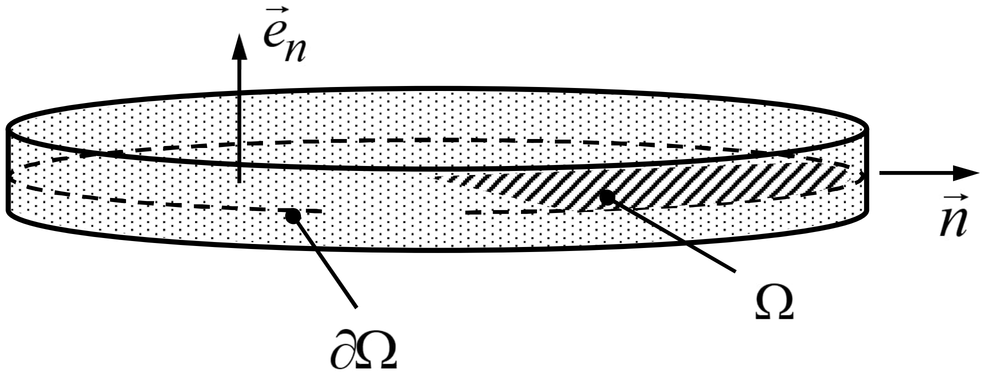
$$\vec{n} \cdot \overleftrightarrow{N} - \vec{F} = 0 \quad \text{or} \quad \vec{u} - \underline{\vec{u}} = 0 \quad \text{on} \quad \partial\Omega,$$

$$\overleftrightarrow{N} = \int \overleftrightarrow{\sigma} dn, \;\; \vec{b} = \int \vec{f} dn, \;\; \text{and} \;\; \vec{F} = \int \vec{t} dn.$$



Constitutive equation $f(\overleftrightarrow{N}, \vec{u}) = 0$, which is needed for a closed system of equations, follows form a material model and the stress resultant definition. Writing a boundary value problem in detail, requires specification of the coordinate system.

The physical domain of the thin-slab model is a prismatic body althought the solution domain of the equations is the mid plane. The starting point is virtual work expression written for the physical domain.



If the external forces on the top and bottom surfaces vanish and stress is symmetric ‘a priori’, virtual work expressions of the internal and external forces simplify to (volume element $dV = dndA$ and area element on the boundary $dA = dnds$)

$$\delta W^{\text{int}} = -\int \vec{\vec{\sigma}} : \delta(\nabla\vec{u})_{\text{c}} dV = -\int_{\Omega} (\int \vec{\vec{\sigma}} dn) : \delta(\nabla\vec{u})_{\text{c}} dA = -\int_{\Omega} \vec{\vec{N}} : \delta(\nabla\vec{u})_{\text{c}} dA,$$

$$\delta W_V^{\mathrm{ext}} = \int \vec{f} \cdot \delta\vec{u} dV = \int_\Omega (\int \vec{f} dn) \cdot \delta\vec{u} dA = \int_\Omega \vec{b} \cdot \delta\vec{u} dA,$$

$$\delta W_A^{\mathrm{ext}} = \int \vec{t} \cdot \delta\vec{u} dA = \int_{\partial\Omega} (\int \vec{t} dn) \cdot \delta\vec{u} ds = \int_{\partial\Omega} \vec{F} \cdot \delta\vec{u} ds$$

in which the stress resultants

$$\overset{\leftrightarrow}{N} = \int \overset{\leftrightarrow}{\sigma} dn, \quad \vec{b} = \int \vec{f} dn, \quad \text{and} \quad \vec{F} = \int \vec{t} dn.$$ integrals over the thickness!

Integration by parts with the vector identity $\overset{\leftrightarrow}{a} : (\nabla\vec{b})_{\mathrm{c}} = \nabla \cdot (\overset{\leftrightarrow}{a} \cdot \vec{b}) - (\nabla \cdot \overset{\leftrightarrow}{a}) \cdot \vec{b}$ in the virtual work expression gives an equivalent but more convenient form for the next step

$$\delta W = -\int_\Omega \overset{\leftrightarrow}{N} : \delta(\nabla\vec{u})_{\mathrm{c}} dA + \int_\Omega \vec{b} \cdot \delta\vec{u} dA + \int_{\partial\Omega} \vec{F} \cdot \delta\vec{u} ds \quad \Leftrightarrow$$

$$\delta W = \int_\Omega (\nabla \cdot \overset{\leftrightarrow}{N} + \vec{b}) \cdot \delta\vec{u} dA + \int_{\partial\Omega} (-\vec{n} \cdot \overset{\leftrightarrow}{N} + \vec{F}) \cdot \delta\vec{u} ds.$$

Principle of virtual work and the fundamental lemma of variation calculus imply the local forms. By considering first a subset of variations $\delta\vec{u} \in U$ with restriction $\delta\vec{u} = 0$ on $\partial\Omega$

$$\delta W = \int_{\Omega} (\nabla \cdot \overleftrightarrow{N} + \vec{b}) \cdot \delta\vec{u} dA = 0 \quad \forall \delta\vec{u} \in U \quad \Leftrightarrow \quad \nabla \cdot \overleftrightarrow{N} + \vec{b} = 0 \text{ in } \Omega.$$

According to the equilibrium equation, the first term of the virtual work expression vanishes. Next, by considering $\delta\vec{u} \in U$ without restrictions on the boundary

$$\delta W = \int_{\partial\Omega} (-\vec{n} \cdot \overleftrightarrow{N} + \vec{F}) \cdot \delta\vec{u} ds = 0 \quad \Rightarrow \; -\vec{n} \cdot \overleftrightarrow{N} + \vec{F} = 0 \text{ or } \delta\vec{u} = 0 \text{ on } \partial\Omega.$$

Vanishing of variation $\delta\vec{u} = 0$ on $\partial\Omega$ implies that displacement is given, i.e., $\vec{u} = \underline{\vec{u}}$. To be precise, one may specify a force component or the corresponding displacement component but not both. Constitutive equation $f(\overleftrightarrow{N}, \vec{u}) = 0$ follows from the definition

$$\overleftrightarrow{N} = \int \overleftrightarrow{\sigma} dn$$

when the stress-displacement relationship for plane-stress is subsitituted there. Altogether, the boundary value problem in its coordinate system invariant form

$$\left.\begin{aligned}&\nabla\cdot\overset{\leftrightarrow}{N}+\vec{b}=0 \ \text{ and } \ f(\overset{\leftrightarrow}{N},\vec{u})=0 \ \text{ in } \ \Omega,\\ &\vec{n}\cdot\overset{\leftrightarrow}{N}-\vec{F}=0 \ \text{ or } \ \vec{u}-\underline{\vec{u}}=0 \ \partial\Omega.\end{aligned}\right\}$$

Integration by parts step of derivation uses the Gauss theorem for a flat geometry which may exclude domains of non-vanishing curvature (it turns out later that the form is valid also in curved geometry).

THIN SLAB EQUATIONS IN (x, y)-COORDINATES

Component representation follows when the tensors of the equilibrium and constitutive equations are expressed in the Cartesian $(\vec{i}, \vec{j})-$basis. Assuming a linearly elastic isotropic material, equilibrium and constitutive equations take the forms,

$$\begin{Bmatrix} \dfrac{\partial N_{xx}}{\partial x} + \dfrac{\partial N_{xy}}{\partial y} + b_x \\ \dfrac{\partial N_{yy}}{\partial y} + \dfrac{\partial N_{xy}}{\partial x} + b_y \end{Bmatrix} = 0, \text{ where } \begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = t\left[E\right]_{\sigma} \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix}.$$

Boundary conditions define usually either displacement or traction in the normal and tangential directions to the boundary.

Representations in the Cartesian system (notice that the second form of the gradient is valid only when basis vectors are constants)

$$\nabla = \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \partial / \partial x \\ \partial / \partial y \end{Bmatrix} = \begin{Bmatrix} \partial / \partial x \\ \partial / \partial y \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix}, \; \overleftrightarrow{N} = \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} N_{xx} & N_{xy} \\ N_{yx} & N_{yy} \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix}, \; \vec{b} = \begin{Bmatrix} b_x \\ b_y \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix}$$

$$\nabla \cdot \overleftrightarrow{N} + \vec{b} = \begin{Bmatrix} \partial / \partial x \\ \partial / \partial y \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix} \cdot \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} N_{xx} & N_{xy} \\ N_{yx} & N_{yy} \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix} + \begin{Bmatrix} b_x \\ b_y \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix} = 0$$

$$\nabla \cdot \overleftrightarrow{N} + \vec{b} = \left(\begin{Bmatrix} \partial / \partial x \\ \partial / \partial y \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} N_{xx} & N_{xy} \\ N_{yx} & N_{yy} \end{bmatrix} + \begin{Bmatrix} b_x \\ b_y \end{Bmatrix}^{\mathrm{T}} \right) \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix} = 0. \quad \Leftarrow$$

A constitutive equation is needed for a closed system of equations (here the number of unknown stress components is 3, whereas the number of equations is 2. Assuming that the thin slab is made of isotropic homogeneous and linearly elastic material of thickness t (steel, aluminum etc.), stress-displacement relationship, kinematic assumption of the model, and elasticity tensor of the plane-stress case give $(N_{yx} = N_{xy})$

$$\overleftrightarrow{N} = \int \overleftrightarrow{\sigma} dn = \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{i}\vec{j} + \vec{j}\vec{i} \end{Bmatrix}^{\mathrm{T}} t\left[E\right]_{\sigma} \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} \text{ so } \begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = t\left[E\right]_{\sigma} \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix}. \quad \leftarrow$$

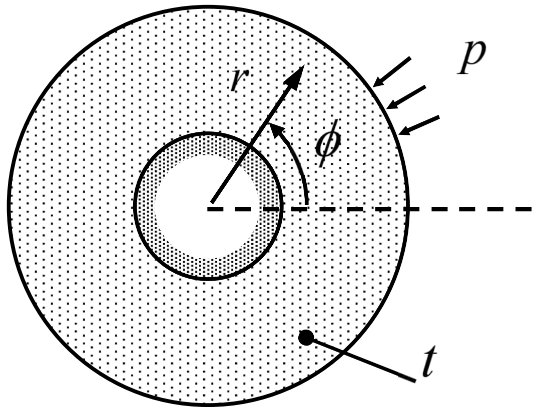
THIN SLAB EQUATIONS IN (r,ϕ)-COORDINATES

Component representation follows when the tensors of the equilibrium and constitutive equations are expressed in the polar $(\vec{e}_r, \vec{e}_\phi)-$basis. Assuming a linearly elastic isotropic material, equilibrium and constitutive equations take the forms,

$$\begin{Bmatrix} \frac{1}{r}[\frac{\partial(rN_{rr})}{\partial r}+\frac{\partial N_{r\phi}}{\partial\phi}-N_{\phi\phi}]+b_r \\ \frac{1}{r}[\frac{1}{r}\frac{\partial(r^2 N_{r\phi})}{\partial r}+\frac{\partial N_{\phi\phi}}{\partial\phi}]+b_\phi \end{Bmatrix}=0, \text{ where } \begin{Bmatrix} N_{rr} \\ N_{\phi\phi} \\ N_{r\phi} \end{Bmatrix}=t\left[E\right]_\sigma \begin{Bmatrix} \frac{\partial u_r}{\partial r} \\ \frac{1}{r}(u_r+\frac{\partial u_\phi}{\partial\phi}) \\ \frac{1}{r}\frac{\partial u_r}{\partial\phi}+r\frac{\partial}{\partial r}(\frac{u_\phi}{r}) \end{Bmatrix}.$$

Boundary conditions define usually either displacement or traction in the normal and tangential directions to the boundary.

EXAMPLE Consider a disk $r \in [\varepsilon R, R]$ which is loaded by traction $\vec{t} = -p\vec{e}_r$ on the outer edge $r = R$ (p is constant). Assuming rotation symmetry i.e. that all quantities depend only on the distance r from the center point, find the displacement components $u_r = u(r)$ and $u_\phi = v(r)$ for a linearly elastic material when Young's modulus E and Poisson's ratio ν are constants.



Answer $$u = \frac{(\varepsilon R)^2 - r^2}{r}\frac{p}{E}\frac{1-\nu^2}{1+\nu+\varepsilon^2(1-\nu)}$$

If the displacement and stress resultant components depend only on the radial coordinate, the equilibrium equations and the constitutive equations of the polar coordinate system simplify to (here $b_r = b_\phi = 0$)

$$\frac{dN_{rr}}{dr} + \frac{1}{r}(N_{rr} - N_{\phi\phi}) = \frac{1}{r}[\frac{d}{dr}(rN_{rr}) - N_{\phi\phi}] = 0, \quad \frac{\partial N_{r\phi}}{\partial r} + \frac{2}{r}N_{r\phi} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 N_{r\phi}) = 0$$

and

$$N_{rr} = \frac{tE}{1-\nu^2}(\frac{du}{dr} + \nu\frac{u}{r}), \quad N_{\phi\phi} = \frac{tE}{1-\nu^2}(\frac{u}{r} + \nu\frac{du}{dr}), \quad N_{r\phi} = tG(\frac{dv}{dr} - \frac{v}{r}) = tGr\frac{d}{dr}(\frac{v}{r}).$$

On the inner edge $r = \varepsilon R$ displacement vanishes, i.e., $u_r \equiv u = 0$. On the outer edge $r = R$, $\vec{n} = \vec{e}_r$, $\vec{n} \cdot \vec{\vec{N}} - \vec{F} = 0$, and $\vec{F} = -pt\vec{e}_r$. These conditions give the boundary value problem,

$$\frac{1}{r}[\frac{d}{dr}(rN_{rr}) - N_{\phi\phi}] = 0, \; N_{rr} = \frac{tE}{1-\nu^2}(\frac{du}{dr} + \nu\frac{u}{r}), \quad N_{\phi\phi} = \frac{tE}{1-\nu^2}(\frac{u}{r} + \nu\frac{du}{dr}) \;\text{ in }\; (\varepsilon R, R),$$

$$u = 0 \text{ at } r = \varepsilon R \quad \text{and} \quad N_{rr} = -pt \text{ at } r = R.$$

Elimination of the stress resultants from the equilibrium equation and boundary conditions gives the boundary value problem for the radial displacement component

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d(ru)}{dr}\right] = 0 \quad \text{in} \quad (\varepsilon R, R),$$

$$u = 0 \text{ at } r = \varepsilon R \quad \text{and} \quad \frac{tE}{1-\nu^2}\left(\frac{du}{dr} + \nu\frac{u}{r}\right) = -pt \text{ at } r = R.$$

The generic solution to the differential equation is $u = a/r + br$. Thereafter, the boundary conditions give the values of the integration constants and solution,

$$u = \frac{(\varepsilon R)^2 - r^2}{r}\frac{p}{E}\frac{1-\nu^2}{1+\nu+\varepsilon^2(1-\nu)}. \quad \Leftarrow$$

The boundary value problem for the displacement component in the angular direction (in terms of displacement component and stress resultant) is given by

$$\frac{1}{r^2}\frac{d}{dr}(r^2 N_{r\phi}) = 0 \quad \text{and} \quad N_{r\phi} = tGr\frac{d}{dr}\left(\frac{v}{r}\right) \quad \text{in} \quad (\varepsilon R, R),$$

$$v = 0 \quad \text{at} \quad r = \varepsilon R \quad \text{and} \quad N_{r\phi} = 0 \quad \text{at} \quad r = R.$$

Equilibrium equation and the condition on the outer edge imply first $N_{r\phi}(r) = 0$. After that, the constitutive equation, and the displacement boundary condition result into

$$v = 0. \quad \Leftarrow$$