

Theorem If  $D$  is an open connected domain and  $F$  is a smooth vector field then the following are equivalent

1)  $F$  is conservative in  $D$ .

2)  $\oint_{\gamma} F \cdot d\vec{r} = 0$  for every piecewise smooth closed curve  $\gamma$  in  $D$ .

3)  $\int_{\gamma} F \cdot d\vec{r}$  depends only on the end-points of  $\gamma$ .

Proof: 1)  $\Rightarrow$  2) (in  $\mathbb{R}^2$ )

$$\nabla\phi = F$$

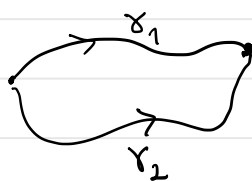
$$F \cdot d\vec{r} = \left( \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y} \right) \cdot \left( \frac{dx}{dt}, \frac{dy}{dt} \right) dt = \frac{d\phi(\gamma(t))}{dt} dt$$

$$\int_{\gamma} F \cdot d\vec{r} = \int_a^b \frac{d\phi(\gamma(t))}{dt} dt = \phi(\gamma(b)) - \phi(\gamma(a))$$

$$\gamma \text{ closed} \Rightarrow \gamma(b) = \gamma(a)$$

$$\Rightarrow \oint_{\gamma} F \cdot d\vec{r} = 0$$

2)  $\Rightarrow$  3)



$$\int_{\gamma_1} F \cdot d\vec{r} \neq \int_{\gamma_2} F \cdot d\vec{r}$$

$$\Rightarrow \int_{\gamma_1 - \gamma_2} F \cdot d\vec{r} \neq 0 \quad ( \neg 3 ) \Rightarrow \neg 2 )$$

3)  $\Rightarrow$  1) Fix  $(x_0, y_0) \in D$ . For  $(x, y) \in D$  choose  $\gamma$  from  $(x_0, y_0)$  to  $(x, y)$ .

$$\text{Define } \phi(x, y) = \int_{\gamma} \mathbf{F} \cdot d\vec{r}$$

$\phi(x, y)$  is well-defined (independent of  $\gamma$ )

$$\phi(x+h, y) - \phi(x, y) = \int_x^{x+h} F_1(\xi, y) d\xi$$

$$\frac{\partial \phi}{\partial x} = \lim_{h \rightarrow 0} \frac{\phi(x+h, y) - \phi(x, y)}{h} = F_1(x, y)$$

In the same way we get  $\frac{\partial \phi}{\partial y} = F_2(x, y)$   $\otimes$

This shows why conservative vector fields are so pleasant to work with.

Ex Find the work done when moving an object from  $(-1, 0, 1)$  to  $(0, -2, 3)$  along any smooth curve in the force field

$$\mathbf{F}(x, y, z) = (x+y)\vec{e}_1 + (x-z)\vec{e}_2 + (z-y)\vec{e}_3$$

Solutions: We try to construct a potential function

$$\frac{\partial \phi}{\partial x} = x+y \implies \phi(x, y, z) = \frac{x^2}{2} + xy + \alpha(y, z)$$

$$\frac{\partial \phi}{\partial y} = x + \frac{\partial \alpha}{\partial y} \implies \frac{\partial \alpha}{\partial y} = -z$$

$$\implies \alpha(y, z) = -yz + \beta(z)$$

$$\phi(x, y, z) = \frac{x^2}{2} + xy - yz + \beta(z)$$

$$\frac{\partial \phi}{\partial z} = -y + \beta'(z) \implies \beta'(z) = z$$

$$\implies \beta(z) = \frac{z^2}{2} + C$$

$$\phi(x, y, z) = \frac{x^2}{2} + xy - yz + \frac{z^2}{2} + C$$

Choose  $C = 0$ .

$$\int_{\gamma} F \cdot d\vec{r} = \phi(0, -2, 3) - \phi(-1, 0, 1) =$$

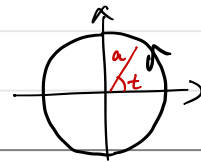
$$= -(-2)3 + \frac{3^2}{2} - \left(\frac{(-1)^2}{2} + \frac{1^2}{2}\right) = 6 + \frac{9}{2} - 1 = \frac{19}{2}$$

Ex Calculate  $\frac{1}{2\pi} \oint_{\gamma} \frac{-y dx + x dy}{x^2 + y^2}$

counterclockwise around the circle  $x^2 + y^2 = a^2$ .

Solution:

$$\begin{cases} x = a \cos t \\ y = a \sin t \end{cases}$$



$$\frac{1}{2\pi} \int_0^{2\pi} \frac{a^2 \sin^2 t dt + a^2 \cos^2 t dt}{a^2} = \frac{1}{2\pi} \int_0^{2\pi} dt = 1$$

So  $\oint \frac{-y dx + x dy}{x^2 + y^2} = 1$  and we see that

$F(x,y) = \left( \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$  is not conservative.

### Surfaces and surface integrals

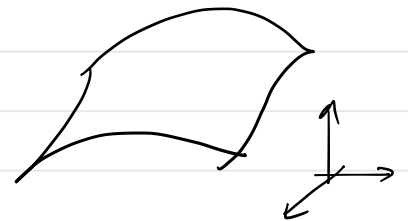
What do we mean by a surface?

Again we need a parametrization

$$\vec{r}: \underbrace{U}_{(u,v)} \subset \mathbb{R}^2 \rightarrow \mathbb{R}^n \quad (\text{usually } n=3 \text{ for us})$$

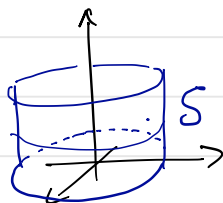
is a surface (under certain assumptions)

$$\vec{r}(u,v) = (x(u,v), y(u,v), z(u,v))$$



$$\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \neq \vec{0}$$

Ex  $r(u,v) = (\cos u, \sin u, v)$   
 $0 \leq u \leq 2\pi, 0 \leq v \leq 1.$



We can form surface integrals  $\iint_S f(x,y,z) dS$   
 as Riemann sums just  
 as for line integrals in  $\mathbb{R}^n$ .

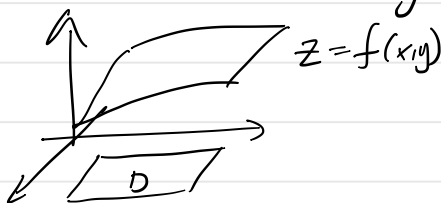
So how does one calculate surface integrals?  
 We use the parametrization and

$dS =$  "area scale factor"  $du dv$

$$dS = \left| \frac{\partial \vec{r}}{\partial u} du \times \frac{\partial \vec{r}}{\partial v} dv \right| = \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| du dv$$

$$\frac{\partial \vec{r}}{\partial u} = \left( \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right) \quad \& \quad \frac{\partial \vec{r}}{\partial v} = \left( \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right)$$

Ex What is the area of a graph  $z=f(x,y)$  over  $D$ ?



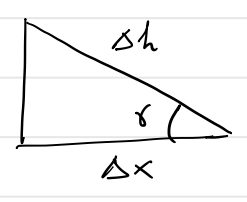
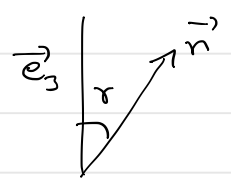
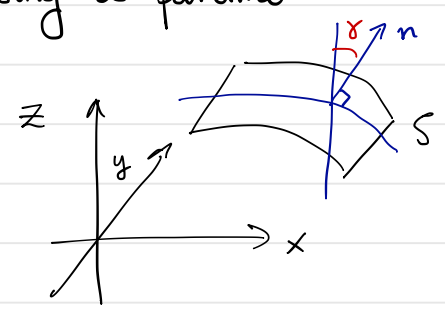
$$\vec{r}(u,v) = (u, v, f(u,v))$$

$$\frac{\partial \vec{r}}{\partial u} = (1, 0, \frac{\partial f}{\partial u}) \quad \frac{\partial \vec{r}}{\partial v} = (0, 1, \frac{\partial f}{\partial v})$$

$$\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 1 & 0 & \frac{\partial f}{\partial u} \\ 0 & 1 & \frac{\partial f}{\partial v} \end{vmatrix} = (-\frac{\partial f}{\partial u}, -\frac{\partial f}{\partial v}, 1)$$

$$\text{Surface area} = \iint_D \sqrt{1 + (\frac{\partial f}{\partial u})^2 + (\frac{\partial f}{\partial v})^2} \, du \, dv$$

There is a way of getting dS without using a parametrization



$$\cos \gamma = \frac{\Delta x}{\Delta h} \implies \Delta h = \frac{\Delta x}{\cos \gamma}$$

$$\implies dS = \frac{1}{\cos \gamma} \, dx \, dy$$

However, also  $\cos \gamma = \frac{\vec{n} \cdot \vec{e}_3}{|\vec{n}|}$  so

$$dS = \frac{|\vec{n}|}{\vec{n} \cdot \vec{e}_3} \, dx \, dy$$

This is useful if you have a surface given implicitly as  $\{G(x,y,z) = 0\} = S$   
 We know that  $\vec{n} = \nabla G$ .

So

$$dS = \frac{|\nabla G|}{|\partial G / \partial z|} dx dy$$

Surface area of a sphere with radius  $R > 0$ .

$$\{(x,y,z) \in \mathbb{R}^3; G(x,y,z) = x^2 + y^2 + z^2 - R^2 = 0\}$$

$$\nabla G = (2x, 2y, 2z) \quad \text{Take the part where } z > 0.$$

$$dS = \frac{|\nabla G|}{|G_z|} dx dy = \frac{\sqrt{4x^2 + 4y^2 + 4z^2}}{2z} dx dy =$$

$$= \frac{\sqrt{x^2 + y^2 + z^2}}{z} dx dy = \frac{R}{\sqrt{R^2 - x^2 - y^2}} dx dy$$

$$\text{Surface area} = 2 \iint_S 1 dS = 2 \iint_{x^2 + y^2 \leq R^2} \frac{R}{\sqrt{R^2 - x^2 - y^2}} dx dy =$$

$$= 2 \int_0^{2\pi} \int_0^R \frac{R}{\sqrt{R^2 - r^2}} r dr d\theta = 4\pi \int_0^R \frac{Rr}{\sqrt{R^2 - r^2}} dr =$$

$$= \int_{t=R^2-r^2}^{t=0} \frac{t}{\sqrt{t}} dt = 4\pi R \int_{R^2}^0 -\frac{1}{2} t^{-1/2} dt =$$

$$= 4\pi R \left[ -\frac{1}{2} \frac{t^{1/2}}{1/2} \right]_{R^2}^0 = 4\pi R^2$$

Ex Calculate  $\iint_S z \, dS$  over

$$S = \left\{ z = \sqrt{x^2 + y^2} \text{ and } 0 \leq z \leq 1 \right\}$$

$$\vec{r}(u,v) = (u, v, \sqrt{u^2 + v^2})$$

$$\frac{\partial \vec{r}}{\partial u} = \left( 1, 0, \frac{u}{\sqrt{u^2 + v^2}} \right) \quad \frac{\partial \vec{r}}{\partial v} = \left( 0, 1, \frac{v}{\sqrt{u^2 + v^2}} \right)$$

$$\iint_S z \, dS = \iint_{u^2 + v^2 \leq 1} \sqrt{u^2 + v^2} \cdot \sqrt{1 + \frac{u^2}{u^2 + v^2} + \frac{v^2}{u^2 + v^2}} \, du \, dv =$$

$$= \sqrt{2} \iint_{u^2 + v^2 \leq 1} \sqrt{u^2 + v^2} \, du \, dv = 2\pi \sqrt{2} \int_0^1 r^2 \, dr = \frac{2\pi\sqrt{2}}{3}$$

(Also note:



polar coord.

$$\cos 45^\circ = \frac{1}{\sqrt{2}}$$

$$\frac{1}{\cos 45^\circ} = \sqrt{2}$$

### Flux integrals

Say that we have a fluid flowing in  $\mathbb{R}^3$  and we want to calculate how much of the fluid that flows across a surface.