Lecture 6

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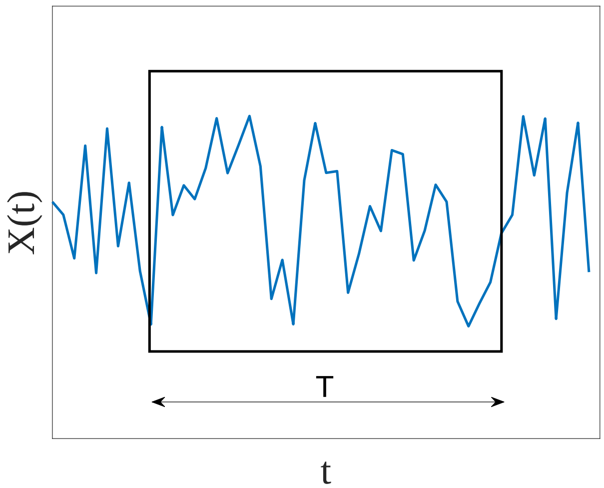
I. ELEMENTS OF NOISE THEORY FOR ELECTRICAL CIRCUITS

– Noise: determines the smallest signal that you can measure, so it limits the dynamical range from below.

$X(t)$ = random variable, time-dependent.



– We will work with ergodic, stationary processes.

Define:

- Average of x: $\overline{X} = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} X(t)dt$, where T = time window.
- Average power of x: $\overline{X^2} = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} X^2(t)dt$
  For example, suppose you have a resistor; then $\frac{\overline{V^2}}{R} = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} \frac{V^2(t)}{R}dt$. Here, $V^2(t)/R$ is the instantaneous power.
- Autocorrelation function: $R_{XX}(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} X(t)X(t+\tau)dt$
  – Obviously:

$$\begin{cases} \overline{X^2} = R_{XX}(\tau=0) \geq |R_{XX}(\tau)| \text{ for } |\tau| > 0 \\ R_{XX}(\tau) = R_{XX}(-\tau) \text{ — this is an even function!} \end{cases} \tag{1}$$

- Power spectral density (PSD):

$S_X(f) \rightarrow$ Collect data in a window T from your oscilloscope, then take the Fourier transform $X(f)$.

$$S_X(f) \equiv \lim_{T\rightarrow\infty} \frac{|X(f)|^2}{T} \,, \tag{2}$$

where $\frac{|X(f)|^2}{T}$ = periodogram (provides an estimate of $S_X(f)$.

Units: if X = Voltage, then PSD has units of $\mathrm{V}^2\mathrm{Hz}^{-1}$.

- What is the relation between $S_X(f)$ and the autocorrelation function? This is provided by the following theorem:

II. THE WIENER-KHINCHIN THEOREM

$$S_X(f) = \mathcal{F}[R_{XX}(\tau)] = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-2\pi i f\tau} d\tau \,. \tag{3}$$

So the PSD = the Fourier transform of the autocorrelation!

or:

$$R_{XX}(\tau) = \int_{-\infty}^{\infty} df e^{2\pi i f\tau} S_X(f) \tag{4}$$

– Note how interesting this is! If you measure any $S_X(f)$ you have access to any correlation (at any τ).

$$\text{For } \tau = 0,\ \overline{X^2} \equiv R_{XX}(0) = \int_{-\infty}^{\infty} df \cdot S_X(f) \tag{5}$$

We will not prove the Wiener-Khinchin theorem (it's not difficult ... if you are interested in you can find the proof in textbooks).

But let's give an alternative justification based on Parseval's theorem:

$\int_{-\infty}^{\infty} dt x^2(t) = \int_{-\infty}^{\infty} df |X(f)|^2$, where the integrands correspond to the power calculated in the time-domain and the frequency-domain, respectively.

So $\overline{X^2} = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} dt X^2(t) = \lim_{T\to\infty} \frac{1}{T}\int_{-\infty}^{\infty} df |X(f)|^2 = \int_{-\infty}^{\infty} df \left(\lim_{T\to\infty} \frac{|X(f)|^2}{T}\right) = \int_{-\infty}^{\infty} df \cdot S_X(f)$. Therefore,

$$\overline{X^2} = \int_{-\infty}^{\infty} df S_x(f) \; . \tag{6}$$

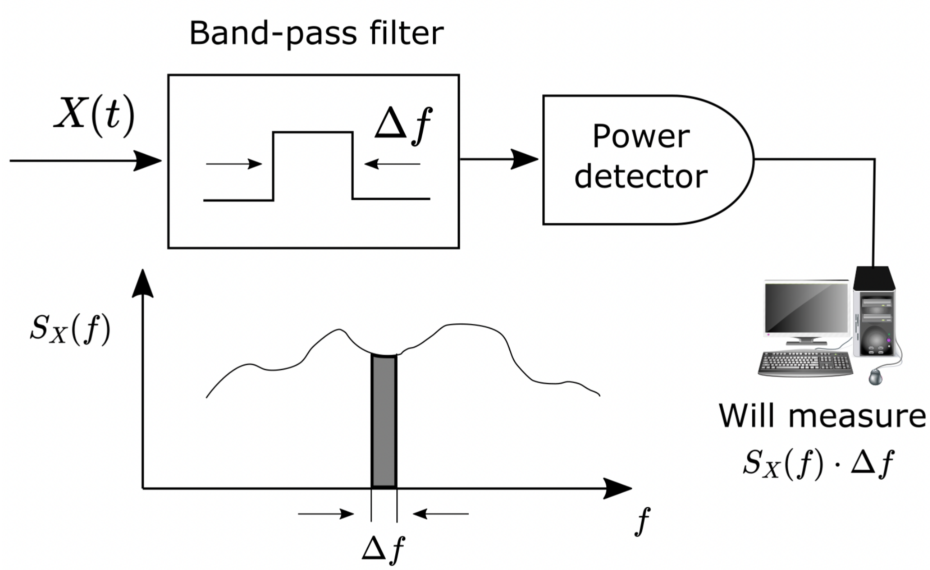
Therefore $S_X(f) \cdot df$ = power in the frequency interval df.

However, there is one little inconvenience. We can always have time taken at least approximately from $-\infty$ to $+\infty$. But what does it mean to have a negative frequency? All generators that we have in the lab produce a "positive" frequency. To bypass this issue it is convenient to restrict ourselves only to positive frequencies, which is what we will in the rest of the lecture.

How do we deal with this? Formally, note that for classical fields $X(t)$, $X(-\omega) = X^*(\omega)$ because $X(t)$ is real, therefore $S(f) = S(-f)$.
So $\overline{X^2} = \int_{-\infty}^{\infty} df S_X(f) = \int_0^{\infty} df S_X(f) + \int_0^{\infty} df S_X(-f) = \int_0^{\infty} df 2 S_X(f)$.

Let us introduce $\mathcal{S}_X(f) \equiv 2S_X(f)$ = single-sided spectral density (as opposed to S_X, which is called double-sided):

$$\overline{X^2} = \int_0^{\infty} df \cdot \mathcal{S}_X(f) \; . \tag{7}$$



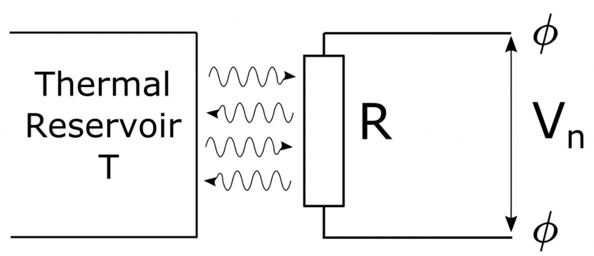
- What does this mean physically?

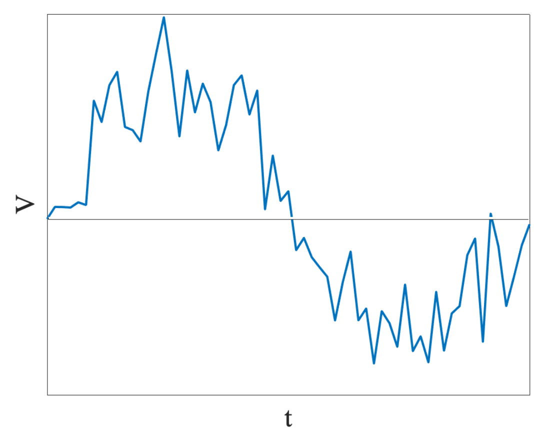
This means that the average power of X is dstributed over all frequencies (now we look at the positive ones only). Considering some small frequncy interval Δf, the single-sided spectral density $\mathcal{S}_X(f)$ represents the power obtained if a bandpass filter with bandwidth Δf is inserted before a power detector.

Units of $\mathcal{S}_X(f)$: if $X \equiv$ voltage , then V^2/Hz.

III. THERMAL NOISE IN ELECTRICAL CIRCUITS

Resistor at finite temperature - can be understood as being in thermal contact with a reservoir of temperature T, therefore exchanging energy back and forth. There will be voltage fluctuations across R, which we will call V_n (subscript n means noise).



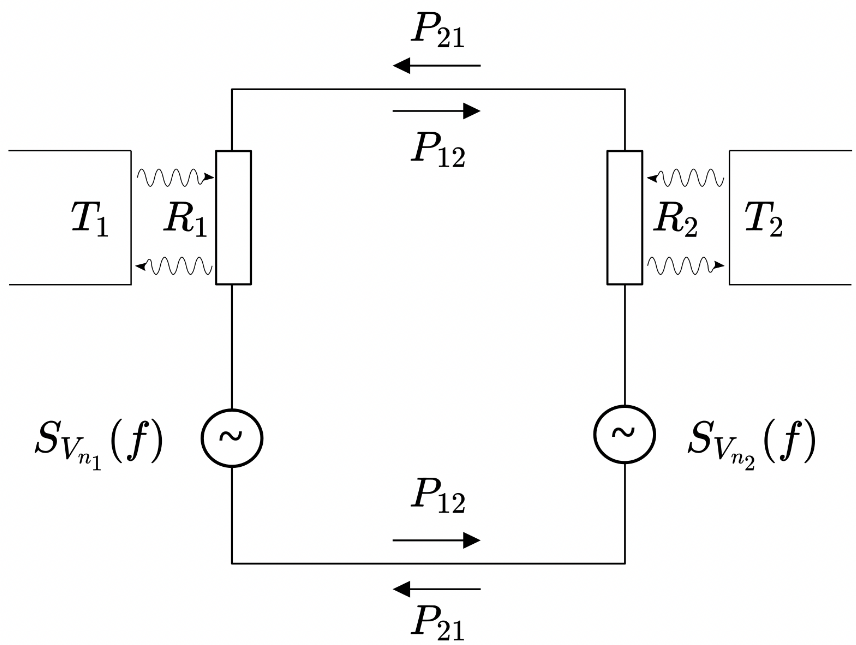


- Discovered by John Johnson (Bell Labs) in 1926.

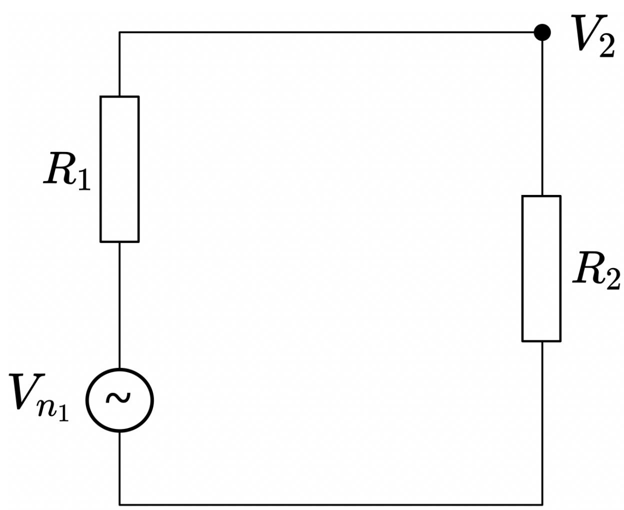
- Theory by Henry Nyquist (Bell Labs)

Derivation of Nyquist Formula cf. H. Nyquist, *Thermal agitation of electric charge in conductors, Phys. Rev. 32, 110* (1928).

1. Estimates and general considerations



Calculate P_{12}:



$V_2 = V_{n_1}\frac{R_2}{R_1+R_2}$, $\quad P_{12} = \frac{V_2^2}{R_2} = V_{n_1}^2 \cdot \frac{R_2}{(R_1+R_2)^2}$.
Similarly for $P_{21} = V_{n_2}^2 \cdot \frac{R_1}{(R_1+R_2)^2}$.

a.) Case $T_1 = T_2 = T$:

2^{nd} law of thermodynamics: $\overline{P_{12}} = \overline{P_{21}}$.

Otherwise by collecting the difference between them you would be able to extract <u>work</u> from a single reservoir at temperature T!

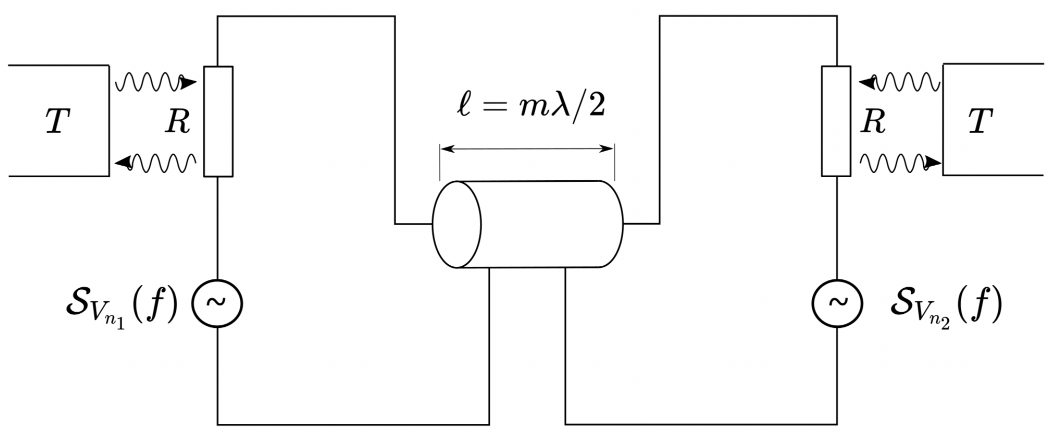
$\overline{P_{12}} = \overline{P_{21}} \implies \frac{\overline{V_{n_1}^2}}{R_1} = \frac{\overline{V_{n_2}^2}}{R_2} = \text{const.} \implies$ we expect $\underline{\overline{V_n^2} \sim R}$.

b.) <u>Case $T_1 \neq T_2$:</u>

We expect $(\overline{P_{12}} - \overline{P_{21}}) \sim (T_1 - T_2)$. This is just because usually energy $\sim k_B T$ in thermodynamics.

If we make $T_2 = 0K$, $P_{21} = 0$, then $\overline{P_{12}} \sim T_1 \implies \overline{V_n^2} \sim T$.

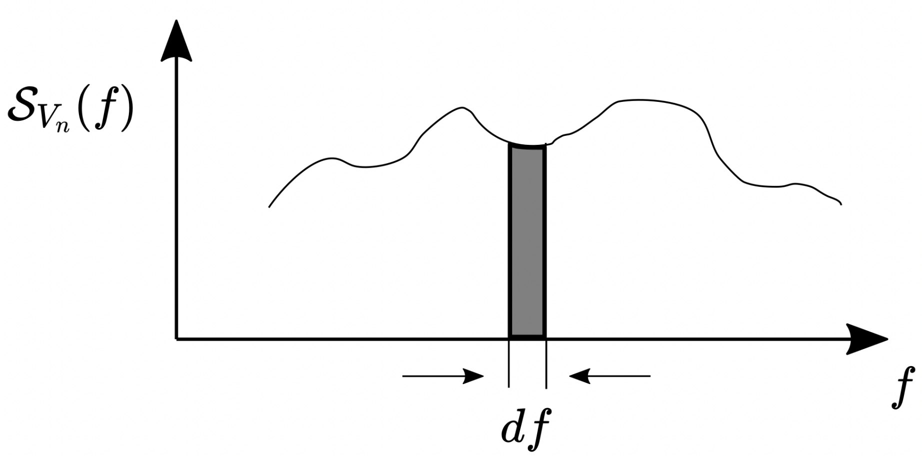
2. A more rigorous derivation



This time $R_1 = R_2 = R$ but the connection is done through a TL (transmission line).

Consider a frequency interval df. We want to find $\mathcal{S}_{V_n}(f)$ for a bandwidth df around f. How many modes dm of the TL are in df?

$\ell = m\frac{\lambda}{2} = m\frac{v}{2f} \implies f = m\frac{v}{2\ell} \implies \underline{dm = \frac{2\ell}{v}df}$, where $v =$ speed of light in the TL.



a.) $dE = dm \cdot \overbrace{\frac{hf}{e^{hf/k_BT}-1}}^{\text{Planck's energy per mode}} = \frac{2\ell}{v} \cdot \frac{hf}{e^{hf/k_BT}-1} df$ = energy contained in the TL in the frequency interval df.

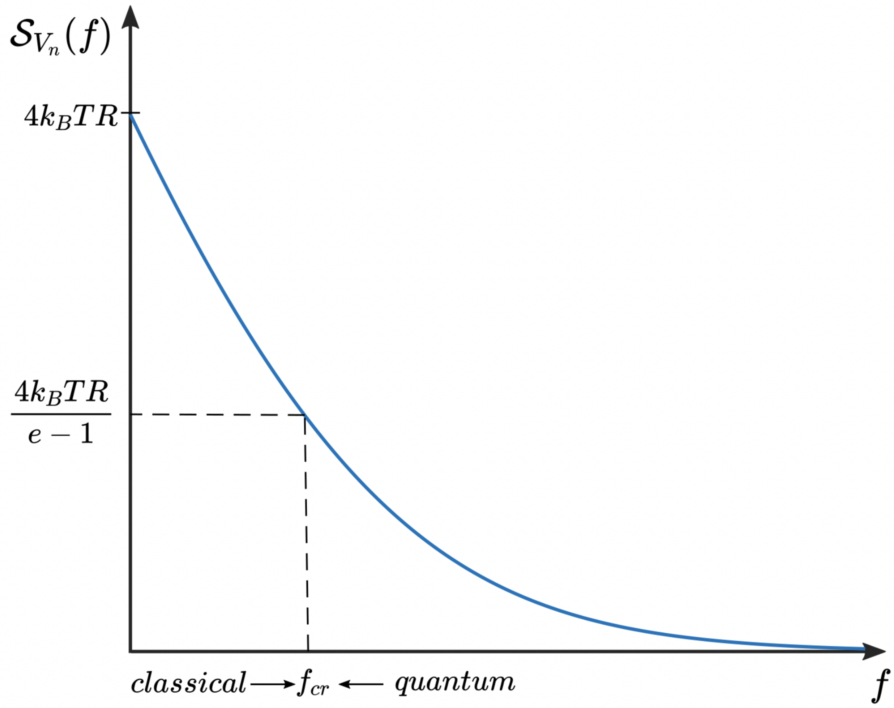
On the other hand, from $\overline{P_{12}} = \overline{V_{n_1}^2} \frac{R}{(R+R)^2} = \frac{\overline{V_{n_1}^2}}{4R} = \overline{P_{21}}$ and $\overline{P} = \frac{\overline{V_n^2}}{4R}$, we have

$$d\overline{P} = \frac{1}{4R} \mathcal{S}_{V_n}(f) \cdot df \tag{8}$$

b.) Therefore $dE = 2 \cdot \frac{\ell}{v} \cdot d\overline{P} = 2 \cdot \frac{\ell}{v} \cdot \frac{\mathcal{S}_{V_n}(f)}{4R} \cdot df$, where the pre-factor of 2 comes from the 2 sources of which inject this power in the TL, and where ℓ/v is the time propagation in the TL.

Now combining a.) and b.), we find

$$\mathcal{S}_{V_n}(f) = \frac{4hfR}{e^{hf/k_BT}-1} \ . \tag{9}$$



- Classical limit: If the temperature is high,

$$k_BT \gg hf \implies \quad \mathcal{S}_{V_n}(f) \simeq 4k_BTR\ . \tag{10}$$

  This could also have been obtained by applying the equipartition theorem:

  $dE = \frac{2\ell}{v} \cdot 2 \cdot \frac{k_BT}{2} \cdot df$ and $dE = \frac{2\ell}{v} \cdot \frac{\mathcal{S}_{V_n}(f)}{4R} df \implies \underline{\mathcal{S}_{V_n}(f) \simeq 4k_BTR}$.

  Note there are two degrees of freedom in the TL ($\vec{E}$ and $\vec{B}$) hence the multiplicative factor 2, and also note that $k_BT/2$ is the energy per degree of freedom.

- Crossover, quantum-classical: From the exponent we can define $f_{cr} = k_BT/h$.

  $\mathcal{S}_{V_n}(f) = \frac{4hfR}{e^{hf/k_BT}-1} \simeq 4hfR \cdot \frac{1}{\cancel{1}+\frac{hf}{k_BT}+\frac{1}{2}(\frac{hf}{k_BT})^2-\cancel{1}} \simeq 4k_BTR(1-\frac{hf}{2k_BT})$.

$$\begin{cases} f_{cr}\Big|_{T=300\mathrm{K}} = \frac{1.38\cdot 10^{-23}\mathrm{J/K}\cdot 300\mathrm{K}}{6.62\cdot 10^{-34}\mathrm{J\cdot s}} = 6.2\mathrm{THz}\ , \\ f_{cr}\Big|_{T=10\mathrm{mK}} = \frac{1.38\cdot 10^{-23}\mathrm{J/K}\cdot 10^{-2}\mathrm{K}}{6.62\cdot 10^{-34}\mathrm{J\cdot s}} = 0.2\mathrm{GHz} = 200\mathrm{MHz}\ . \end{cases} \tag{11}$$

– Thermal noise appears from the motion of electrons in the resistor. But where is the electron charge e in our final equations?!? It's in fact "hidden" in the microscopic models that we might use for R (for example, the Drude model, $\sigma = 1/\rho = ne^2\tau/m$).

<u>Discussion:</u>

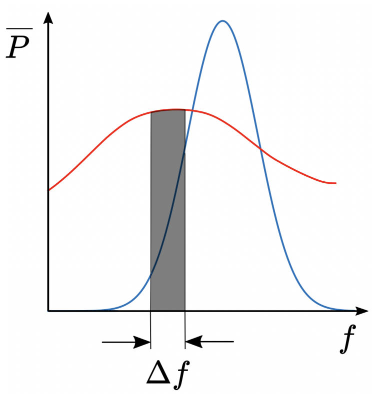
- How much power is emitted?

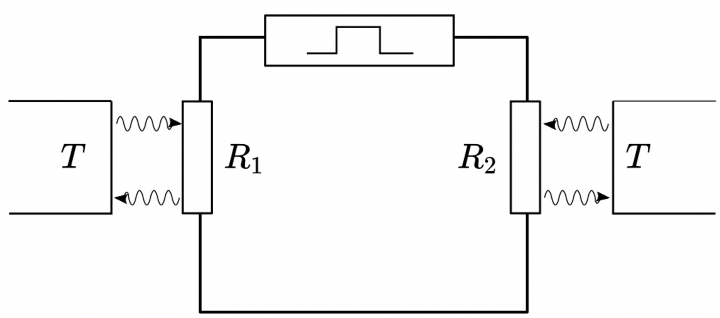
$\overline{P}_{tot} = \int_0^\infty df \frac{\mathcal{S}_{V_n}(f)}{R} = \frac{4(k_BT)^2}{h}\int_0^\infty \frac{udu}{e^u-1} = \frac{2\pi^2}{3}\cdot\frac{(k_BT)^2}{h}$, where $u = \frac{hf}{k_BT}$.

– Does not depend on R!

– Let's calculate it. $T = 300K$
$\overline{P_{tot}} = \frac{2\pi^2}{3}\cdot\frac{(1.38\cdot 10^{-23}\cdot 300)^2}{6.62\cdot 10^{-34}}\frac{J}{s} = 172\text{nW}$ — very small!

– Note that $\Delta\overline{P} = \frac{\mathcal{S}_V(f)}{R}\cdot\Delta f$ = independent of R!





Why is it so? It is a consequence of argument 1.a in the beginning. Had it been otherwise, it would be possible to create work from a single reservoir by using 2 resistors and a bandpass filter Δf.

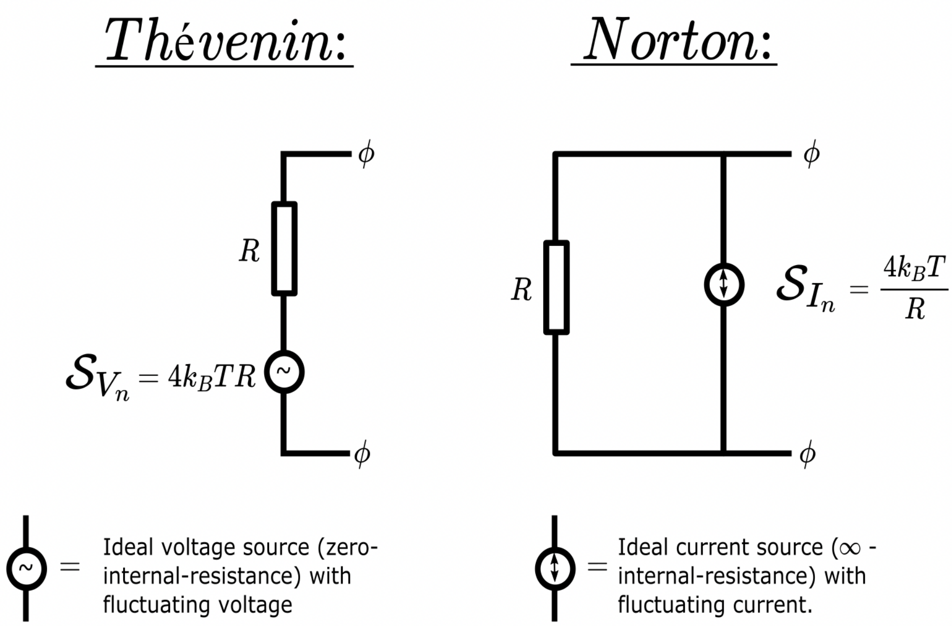
This would contradict the second law of thermodynamics (perpetual motion of the second kind).

- A more general statement: <u>The fluctuation-dissipation theorem:</u>

- In order for an object to emit radiation (due to fluctuations) it must also dissipate! Not restricted to electric circuits! E.g. in nanomechanics $\overline{S_{FF}}(\omega) = 2mk_BT\cdot\gamma$.
Example: A non-dissipative circuit element (ideal capacitor, inductance) will not emit radiation. There is no fluctuating voltage across an ideal capacitor! The ideal capacitor

only stores energy. Also if you have an I-V characteristic and define dV/dI it does not mean that there will be accompanying fluctuations!

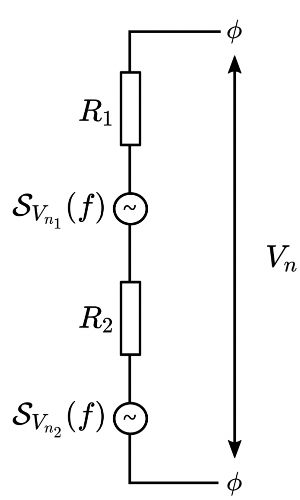
Circuit equivalent representation



Does it work?

Example:

Resistors in series:



$$\mathcal{S}_{V_{n_1}} = 4k_B T R_1 \ , \tag{12}$$

$$\mathcal{S}_{V_{n_2}} = 4k_B T R_2 \ , \tag{13}$$

But the fluctuations in the resistors are uncorrelated! Therefore we can add them (this is a known result from mathematical statistics)

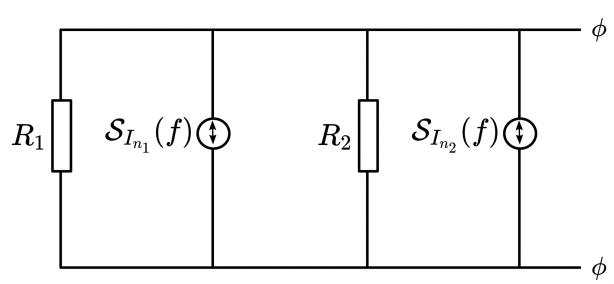
$$\mathcal{S}_{V_n} = \mathcal{S}_{V_{n_1}} + \mathcal{S}_{V_{n_2}} \tag{14}$$

Therefore,

$$\mathcal{S}_{V_n} = 4k_B T(R_1 + R_2) \ , \tag{15}$$

where $R = R_1 + R_2 =$ is the in-series resistance. So everything is consistent!

Resistors in parallel:



$$\begin{cases} \mathcal{S}_{I_{n_1}} = \frac{4k_B T}{R_1}, \\ \mathcal{S}_{I_{n_2}} = \frac{4k_B T}{R_2}, \end{cases} \tag{16}$$

$$\mathcal{S}_{I_n} = \mathcal{S}_{I_{n_1}} + \mathcal{S}_{I_{n_2}} \tag{17}$$

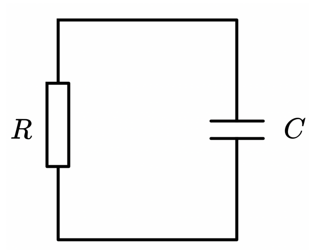
where these are uncorrelated fluctuations of current!

In-parallel resistance, $R_{||} = \frac{R_1 R_2}{R_1 + R_2}$. So again everything works well,

$$\mathcal{S}_{I_n} = \frac{4k_B T}{R_{||}} \ . \tag{18}$$

Coupling of a dissipative element to a non-dissipative one.

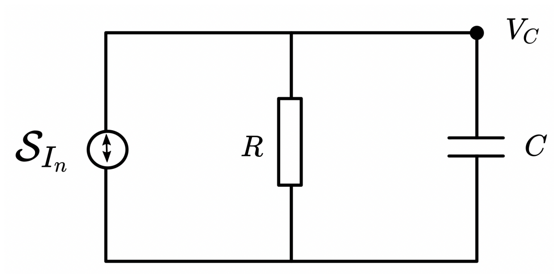
What happens for example when the fluctuations of the resistor charge and discharge a capacitor?



$$Z_C(f) = \frac{1}{iC \cdot 2\pi f} = \frac{1}{iC\omega}\ , \tag{19}$$

where $\omega = 2\pi f$.

<u>Norton:</u>



$\frac{V_C(f)}{Z_C(f)} + \frac{V_C)f)}{R} = I_n(f) \quad \therefore \quad V_C = \frac{I_n}{\frac{1}{R}+iC\cdot 2\pi f} = \frac{I_n R}{1+2\pi i f RC}.$

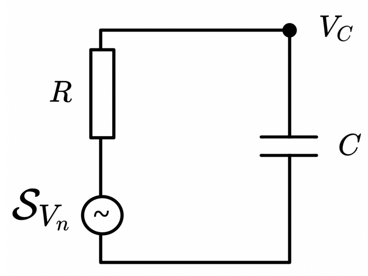
and

$$\mathcal{S}_{I_n}(f) = \frac{4k_B T}{R}. \tag{20}$$

So we obtain:

$$\mathcal{S}_{V_C}(f) = \lim_{T\to\infty} \frac{|V_C(f)|^2}{T} = \frac{R^2}{|1+2\pi i f RC|^2} \lim_{T\to\infty} \frac{|I_n(f)|^2}{T} = \frac{R^2}{|1+2\pi i f RC|^2} \mathcal{S}_{I_n}(f) = \frac{4k_B T R}{1+(2\pi RC)^2 f^2}\ . \tag{21}$$

<u>Therevin:</u>



$V_C(f) = V_n(f) \cdot \frac{Z_C(f)}{R+Z_C(f)} = V_n(f) \cdot \frac{1}{1+i2\pi RCf} \implies \mathcal{S}_{V_C}(f) = \mathcal{S}_{V_n} \cdot \frac{1}{1+(2\pi RC)^2 f^2} = \frac{4k_BTR}{1+(2\pi RC)^2 f^2}$.

This is the same result! So per bandwidth, the capacitors experience

$$\mathcal{S}_{V_C}(f) = \frac{4k_BTR}{1+(2\pi RC)^2 f^2} \ .$$

What is the <u>total</u> $\overline{V_C^2}$ (over all frequencies)?

$\overline{V^2_{C\,tot}} = \int_0^\infty S_{V_C}(f) \cdot df = \frac{4k_BT}{2\pi C} \cdot \int_0^\infty df \frac{2\pi RC}{1+(2\pi RCf)^2} = \frac{4k_BT}{2\pi C} \int_0^\infty dy \frac{1}{1+.y^2}\Big|_{y=2\pi RCf} = \frac{4k_BT}{2\pi C} \cdot \arctan y\Big|_0^\infty = \frac{4k_BT}{2\pi C} \cdot \frac{\pi}{2} = \frac{k_BT}{C}$.

Therefore,

$$\overline{V^2_{C(tot)}} = \frac{k_BT}{C} \ . \tag{22}$$

This is called K-T-over C-noise.

– Note that it does not depend on R! Why? – Noise increases with R but bandwidth decreases $\sim 1/R$.

– Suppose we want to increase the speed of a circuit (say a switch, AC converter, sampler, etc.) by making C smaller (smaller RC constant). Then $\overline{V^2_{C(tot)}}$ will increase! The circuit will be more noisy!

– We can derive the formula above based from the equipartition theorem: $\frac{1}{2}C\overline{V^2_{C(tot)}} = \frac{1}{2}k_BT \longleftarrow$ there is only one degree of freedom in a capacitor.

Take-home conclusions:

Dissipation (by resistive elements) produces noise. The noise tends to "bury" any quantum effects, making them useless. Two ways out of it:

1. Increase the frequency. This can be done in some systems (optics) but it is not always possible with microwaves and rf.
2. Use zero-resistance components. Is this possible? Yes, as we will see in the next lecture.

References

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- Ali Hajimiri — Analog Circuit Design (lectures at Caltech).
- A.M. Zagoskin — Quantum Engineering.

Highly Advanced:

- C. Gardiner — Quantum Noise.
- M. Cattaneo, GSP — *arXiv*:210316946.