Ex Calculate $\iint_S z \, dS$ over

$$S = \{ z = \sqrt{x^2+y^2} \text{ and } 0 \leq z \leq 1 \}$$

$$\vec{r}(u,v) = (u, v, \sqrt{u^2+v^2})$$

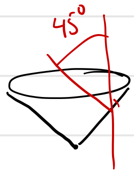
$$\frac{\partial \vec{r}}{\partial u} = \left(1, 0, \frac{u}{\sqrt{u^2+v^2}}\right) \qquad \frac{\partial \vec{r}}{\partial v} = \left(0, 1, \frac{v}{\sqrt{u^2+v^2}}\right)$$

$$\iint_S z \, dS = \iint_{u^2+v^2 \leq 1} \sqrt{u^2+v^2} \cdot \sqrt{1 + \frac{u^2}{u^2+v^2} + \frac{v^2}{u^2+v^2}} \, du dv =$$

$$= \sqrt{2} \iint_{u^2+v^2 \leq 1} \sqrt{u^2+v^2} \, du dv = 2\pi\sqrt{2} \int_0^1 r^2 \, dr = \frac{2\pi\sqrt{2}}{3}$$

polar coord.

(Also note :

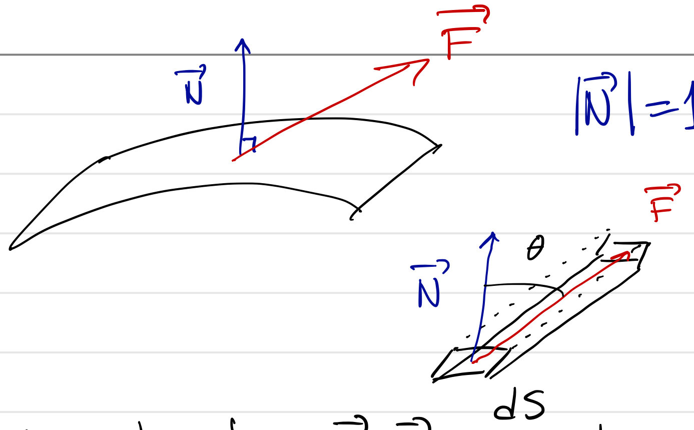


$$\cos 45° = \frac{1}{\sqrt{2}}$$

$$\frac{1}{\cos 45°} = \sqrt{2}$$

Flux integrals

Say that we have a fluid flowing in $\mathbb{R}^3$ and we want to calculate how much of the fluid that flows across a surface.



$|\vec{N}| = 1$

We integrate $\vec{F} \cdot \vec{N}$ over the surface

$$\text{Flux integral} = \iint_S \vec{F} \cdot \vec{N} \, dS$$

How do we find a normal field?

Given a parametrization we have a candidate.

$$\hat{N} = \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}$$

$$\Longrightarrow \vec{N} = \frac{\hat{N}}{|\hat{N}|}$$

$$\iint_S \vec{F} \cdot \vec{N} = \iint_S \vec{F} \cdot \frac{\hat{N}}{|\hat{N}|} \, |\hat{N}| \, du\,dv =$$

$$= \iint_S \vec{F} \cdot \hat{N} \, du\,dv$$

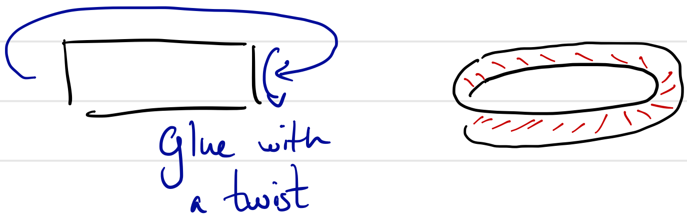
We can also use $S = \{G(x,y,z) = 0\}$

$$dS = \frac{|\nabla G|}{|G_z|}\,dxdy$$

$$\vec{N} = \frac{\nabla G}{|\nabla G|} \quad \left(\text{or } -\frac{\nabla G}{|\nabla G|}\right)$$

$$\Longrightarrow \quad \vec{N}\,dS = \pm \frac{\nabla G}{G_z}\,dxdy$$

The sign depends on which normal points in the correct direction. Note that some surfaces are "one-sided". That is not every surface is orientable. The Möbius strip is an example of a surface that is non-orientable



Ex Calculate the flux of
$F(x,y,z) = (z, 0, x^2)$ upwards through
$z = x^2 + y^2$ over $-1 \leq x \leq 1, -1 \leq y \leq 1$.

Method 1 $\quad \vec{r}(u,v) = (u, v, u^2+v^2)$

$$\frac{\partial \vec{r}}{\partial u} = (1,0,2u) \qquad \frac{\partial \vec{r}}{\partial v} = (0,1,2v)$$

$$\vec{N} = \begin{vmatrix} \vec{e_1} & \vec{e_2} & \vec{e_3} \\ 1 & 0 & 2u \\ 0 & 1 & 2v \end{vmatrix} = (-2u, -2v, 1)$$

points in correct direction

$$\int_{-1}^{1}\int_{-1}^{1} (z, 0, x^2) \cdot \vec{N}\, du\,dv =$$

$$= \int_{-1}^{1}\int_{-1}^{1} (u^2+v^2, 0, u^2) \cdot (-2u, -2v, 1)\, du\,dv =$$

$$= \int_{-1}^{1}\int_{-1}^{1} -2u(u^2+v^2) + u^2\, du\,dv =$$

$$= \int_{-1}^{1}\int_{-1}^{1} -2u^3 - 2uv^2 + u^2\, du\,dv =$$

$$= \int_{-1}^{1} \left[-\frac{2u^4}{4} - u^2v^2 + \frac{u^3}{3} \right]_{u=-1}^{u=1} dv = \frac{2}{3}\int_{-1}^{1} dv = \frac{4}{3}$$

Method 2 $\quad G(x,y,z) = x^2+y^2-z$

$$\nabla G = (2x, 2y, -1)$$

$$\frac{\nabla G}{G_z} = (-2x, -2y, 1)$$ points in the correct directions

$$\int_{-1}^{1}\int_{-1}^{1} F\cdot\frac{\nabla G}{G_z}\,dx\,dy = \int_{-1}^{1}\int_{-1}^{1} -2x(x^2+y^2)+x^2\,dx\,dy =$$

$$= \dots = 4/3.$$

Gradient, Divergence and Curl.

We know that the gradient of $f:\mathbb{R}^n\to\mathbb{R}$ is
$$\nabla f = \left(\frac{\partial f}{\partial x_1},\dots,\frac{\partial f}{\partial x_n}\right)$$

It gives the direction in which f is growing fastest.
We introduce a formal ~~vector~~ of differential ~~vector~~ operators.

The Nabla operator $\nabla = \left(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}\right)$

Definition $F:\mathbb{R}^n\to\mathbb{R}^n$ vector field

(a function) 1) $\operatorname{div} F = \nabla\cdot F = \frac{\partial F_1}{\partial x_1}+\frac{\partial F_2}{\partial x_2}+\dots+\frac{\partial F_n}{\partial x_n}$

(a vector field) 2) $\boxed{n=3}$

$$\operatorname{Curl} F = \nabla\times F = \begin{vmatrix}\vec{e_1} & \vec{e_2} & \vec{e_3}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ F_1 & F_2 & F_3\end{vmatrix} =$$

$$= \left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}\right)\vec{e_1}+\left(\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x}\right)\vec{e_2}+\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)\vec{e_3}$$

Curl works in $\mathbb{R}^3$ (and $\mathbb{R}^2$ in a special way)

Ex $F(x,y,z) = (xy,\ y^2 - z^2,\ yz)$

$$\operatorname{div} F = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(y^2 - z^2) + \frac{\partial}{\partial z}(yz) =$$

$$= y + 2y + y = 4y$$

$$\operatorname{Curl} F = \begin{vmatrix} \vec{e_1} & \vec{e_2} & \vec{e_3} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & y^2 - z^2 & yz \end{vmatrix} =$$

$$= \left(\frac{\partial}{\partial y}(yz) - \frac{\partial}{\partial z}(y^2 - z^2),\ \frac{\partial}{\partial z}(xy) - \frac{\partial}{\partial x}(yz),\ \frac{\partial}{\partial x}(y^2 - z^2) - \frac{\partial}{\partial y}(xy) \right) =$$

$$= (z + 2z,\ 0,\ -x) = (3z, 0, -x)$$

Interpretation of the divergence

Let $\vec{F}$ be a smooth vector field and $\vec{N}$ be the unit outward normal vector field of S_ε, the sphere with radius ε centered at P. Then

$$\operatorname{div} \vec{F}(P) = \lim_{\varepsilon \to 0^+} \frac{3}{4\pi\varepsilon^3} \oint\!\!\oint_{S_\varepsilon} \vec{F} \cdot \vec{N}\, dS$$

$$\frac{1}{\text{volume of } S_\varepsilon}$$