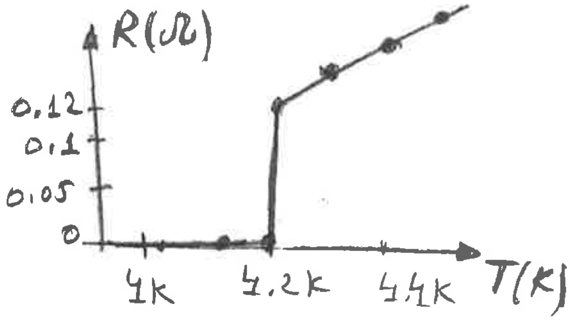
Superconductivity

1911 – Heike Kamerlingh Onnes

Electrical resistance of Hg (metal!) dropped to $< 10^{-5}\Omega$ at $T_c = 4.2K$



Other typical metals become superconductors:

$T_c = 1.2K$ for Al
$T_c = 7.2K$ for Pb
$T_c = 9.2K$ for Nb

1986 – discovery of high-T_c compounds by J.G. Bednorz and K.A. Müller

$T_c = 95K$ for $YBa_2Cu_3O_{7-\delta}$
$T_c = 125K$ for $Tl_2Ba_2Ca_2Cu_3O_{10}$
$T_c = 133K$ for $HgBa_2Ca_2Cu_3O_{8+\delta}$

These are not metals, they are ceramic materials at room temperature!

Meissner effect

- In the beginnings of superconductivity research it was hoped that the electromagnetic properties could be derived from the property of infinite conductivity.

$\sigma = \infty$ $\quad$ $\left.\begin{array}{l}\vec{J} = \sigma \cdot \vec{E} \\ \vec{J} = \text{finite}\end{array}\right| \Rightarrow \vec{E} = 0 \Rightarrow \vec{\nabla} \times \vec{E} = 0$

Maxwell: $\vec{\nabla} \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$ $\Big\} \Rightarrow \dfrac{\partial \vec{B}}{\partial t} = 0$

So $\vec{B}$ = constant inside a superconductor and also we expect it to be dependent on the way it was cooled down (e.g. either in the presence or absence of the magnetic field)

BUT in 1933 Meissner and Ochsenfeld discovered that $\vec{B} = 0$.
The magnetic field inside the superconductor is not just constant, but it is exactly zero. Magnetic field lines are expelled.
A superconductor is a perfect diamagnet.

Theory development

- 1935 – phenomenological theory developed by F. & H. London (two brothers!)
- 1957 – BCS (Bardeen – Cooper – Schrieffer) theory
- high-T_c superconductivity – maybe you?

Elements of London theory

Consider a particle of mass m^* and charge q^*. It will turn out that $m^* = 2m_e$ and $q^* = -2e$; these particles are Cooper pairs, and a complete understanding of what they are is provided by the BCS theory.

$\vec{B} = \vec{\nabla} \times \vec{A}$ $\quad$ $\vec{A}$ = magnetic vector potential $\quad$ V = electric potential

Schrödinger equation: $$i\hbar \frac{\partial}{\partial t}\psi(\vec{r},t) = \frac{1}{2m^*}\left(-i\hbar\vec{\nabla} - q^*\vec{A}(\vec{r})\right)^2 \psi(\vec{r},t) + q^* V(\vec{r},t)\psi(\vec{r},t)$$

- Recall also that $\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}V$

Note: the Hamiltonian of a free particle in a magnetic field is

$$H = \frac{\Pi^2}{2m^*} \quad \text{where } \Pi(\vec{r}) = -i\hbar\vec{\nabla} - q^*\vec{A}(\vec{r})$$

canonical momentum

Probability density

$$P(\vec{r},t) = |\psi(\vec{r},t)|^2$$

$$\frac{\partial P(\vec{r},t)}{\partial t} = \frac{\partial \psi^*(\vec{r},t)}{\partial t}\psi(\vec{r},t) + \psi^*(\vec{r},t)\frac{\partial \psi(\vec{r},t)}{\partial t}$$

$$= \frac{i}{\hbar}\Big\{\left[\frac{1}{2m^*}\left(i\hbar\vec{\nabla} - q^*\vec{A}(\vec{r})\right)^2\psi^*(\vec{r},t)\right]\psi(\vec{r},t) - \psi^*(\vec{r},t)\left[\frac{1}{2m^*}\left(-i\hbar\vec{\nabla} - q^*\vec{A}(\vec{r})\right)^2\psi(\vec{r},t)\right]\Big\}$$

$$= -\vec{\nabla}\cdot\vec{J}(\vec{r},t)$$

So $$\frac{\partial P(\vec{r},t)}{\partial t} = -\vec{\nabla}\cdot\vec{J}(\vec{r},t)$$

where $$\vec{J}(\vec{r},t) = \frac{1}{2m^*}\left[\left(-i\hbar\vec{\nabla} - q^*\vec{A}(\vec{r})\right)\psi(\vec{r},t)\right]^*\psi(\vec{r},t) + \frac{1}{2m^*}\psi^*(\vec{r},t)\cdot\left[\left(-i\hbar\vec{\nabla} - q^*\vec{A}(\vec{r})\right)\psi(\vec{r},t)\right]$$

Key point: the wavefunction $\psi(\vec{r},t)$ for a superconductor can be regarded as an order parameter (a macroscopic wavefunction!) Let us call this "solution" ψ_s.

Ginzburg-Landau order parameter

$$\psi_s(\vec{r},t) = \sqrt{n_s(\vec{r},t)}\, e^{i\theta(\vec{r},t)}$$

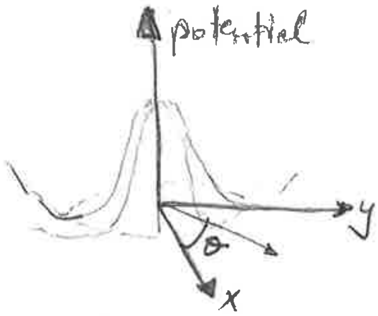
$$\int \psi_s^*(\vec{r},t)\,\psi(\vec{r},t) = \text{total number of superconducting particles ("superelectrons")}$$

$n_s(\vec{r},t)$ = density of superconducting particles

$\theta(\vec{r},t)$ = superconducting phase

Appears as a result of a broken symmetry.

• From now on we will assume $n_s(\vec{r},t) \equiv n_s = \text{const.}$



So $\vec{j}_s(\vec{r},t) = \frac{\hbar n_s}{m^*}\left[\vec{\nabla}\theta(\vec{r},t) - \frac{e^*}{\hbar}\vec{A}(\vec{r},t)\right]$

Electrical current: $\vec{J} = e^*\vec{j}$

$$\vec{J}_s(\vec{r},t) = \frac{\hbar e^* n_s}{m^*}\left[\vec{\nabla}\theta(\vec{r},t) - \frac{e^*}{\hbar}\vec{A}(\vec{r},t)\right] = \text{superconducting current density}$$

supercurrent

→ Gauge-invariant phase gradient

$\theta \to \theta + \frac{e^*}{\hbar}\chi$

$\vec{A} \to \vec{A} + \vec{\nabla}\chi$

CONSEQUENCES:

• PERFECT CONDUCTIVITY

• Here we consider θ = constant in $\vec{r}$; $\vec{\nabla}\theta = 0$

$$\vec{J}_s = -\frac{e^{*2}}{m^*} n_s \vec{A} \Rightarrow$$

$$\frac{d\vec{J}_s(\vec{r},t)}{dt} = -\frac{e^{*2}}{m^*} n_s \frac{d\vec{A}(\vec{r},t)}{dt}$$

or

$$\boxed{\frac{d\vec{J}_s(\vec{r},t)}{dt} = +\frac{e^{*2} n_s}{m^*}\vec{E}(\vec{r},t)}$$

Recall Maxwell: $\begin{cases} \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \\ \vec{B} = \vec{\nabla}\times\vec{A} \end{cases}$

What does it mean?

Take a ballistic superelectron (no collisions with atoms, impurities etc.)

$$\left.\begin{array}{l} m^*\frac{d\vec{v}_s}{dt} = e^*\vec{E} \\ \vec{J}_s = n_s \cdot e^* \cdot \vec{v}_s \end{array}\right\} \Rightarrow \boxed{\frac{d\vec{J}_s}{dt} = \frac{e^{*2} n_s}{m^*}\vec{E}}$$

Note the difference w.r.t. $\vec{J} = \sigma\vec{E}$ (Ohm's law)!

MEISSNER EFFECT

Let us look at Maxwell's equations: $\begin{cases} \vec{\nabla}\cdot\vec{B} = 0 \\ \vec{\nabla}\times\vec{B} = \mu_0 \vec{J}_s \end{cases}$

Now $\vec{B} = \vec{\nabla}\times\vec{A}$ so

$$\vec{\nabla}\times\vec{B} = \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}\underbrace{(\vec{\nabla}\cdot\vec{A})}_{\text{we can use Coulomb gauge } \vec{\nabla}\cdot\vec{A}=0} - \vec{\nabla}^2\vec{A} = -\vec{\nabla}^2\vec{A}$$

So $\left.\begin{aligned} \vec{\nabla}^2\vec{A} &= -\mu_0 \vec{J}_s \\ \vec{J}_s &= -\frac{e^{*2}}{m^*} n_s \vec{A} \end{aligned}\right\} \Rightarrow \boxed{\vec{\nabla}^2\vec{A} = \frac{\mu_0 e^{*2} n_s}{m^*}\vec{A}}$

<u>Notation</u>: $\lambda_L = \sqrt{\dfrac{m^*}{\mu_0 n_s e^{*2}}}$ = London penetration length.

Since $\vec{J}_s = -\dfrac{e^{*2} n_s}{m^*}\vec{A}$ we have $\boxed{\vec{J}_s = -\dfrac{1}{\mu_0 \lambda_L^2}\vec{A}}$

and $\boxed{\vec{\nabla}^2\vec{A} = \dfrac{1}{\lambda_L^2}\vec{A}}$

So $\mu_0 \lambda_L^2 \vec{J}_s = -\vec{A} \Rightarrow \mu_0\lambda_L^2\, \vec{\nabla}\times\vec{J}_s = -\vec{\nabla}\times\vec{A} = -\vec{B}$

$\Downarrow$

$$\mu_0 \lambda_L^2 \frac{\partial}{\partial t}(\underbrace{\nabla\times\vec{J}_s}_{\text{but } \vec{J}_s = \frac{1}{\mu_0}\vec{\nabla}\times\vec{B}}) = -\vec{\nabla}\times \underbrace{\frac{\partial\vec{A}}{\partial t}}_{= -\vec{E}}$$

because the voltage is zero

see $\vec{E} = -\frac{\partial\vec{A}}{\partial t} - \nabla V$

$\Downarrow$

$$\lambda_L^2 \frac{\partial}{\partial t}\cdot \underbrace{\left(\vec{\nabla}\times(\vec{\nabla}\times\vec{B})\right)}_{= \vec{\nabla}\cdot(\vec{\nabla}\cdot\vec{B}) - \vec{\nabla}^2 B} = \underbrace{\vec{\nabla}\times\vec{E}}_{= -\frac{\partial\vec{B}}{\partial t}}$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

$\Rightarrow \lambda_L^2\, \vec{\nabla}^2\vec{B} = +\vec{B}$

or $\left[\dfrac{1}{\lambda_L^2} - \vec{\nabla}^2\right]\vec{B}(\vec{r}) = 0$

Take $\vec{B}(\vec{r}) = (0, 0, B(z)) \Rightarrow B(z) = B_0 \exp\left(-z/\lambda_L\right)$

This is the Meissner effect. The field decays exponentially in the superconductor.

To review : we found $\boxed{\vec{J}_s = -\frac{1}{\mu_0 \lambda_L^2}\vec{A}}$

and $\boxed{\nabla^2 \vec{A} = \frac{1}{\lambda_L^2}\vec{A}}$

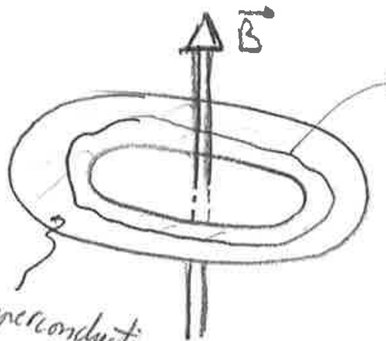
or $\boxed{\frac{d\vec{J}_s}{dt} = \frac{1}{\mu_0 \lambda_L^2}\vec{E}}$ – called 1^{st} London equation

$\boxed{\vec{B} = -\mu_0 \lambda_L^2 \vec{\nabla} \times \vec{J}_s}$ – called 2^{nd} London equation

So the magnetic field can penetrate at most to depths $\simeq \lambda_L$. Currents can flow in this region, but deep in the bulk they will be zero.

• QUANTIZATION OF FLUX

So far we did not discuss the phase θ from the general expression of the current. Now it's the time .... with a spectacular example!



Superconducting ring

contour of integration deep in the bulk, where $\vec{J}_s(\vec{r},t) = 0$. From the expression of $\vec{J}_s$,

$$\Rightarrow \hbar \vec{\nabla}\theta(\vec{r}) = e^* \vec{A}(\vec{r})$$

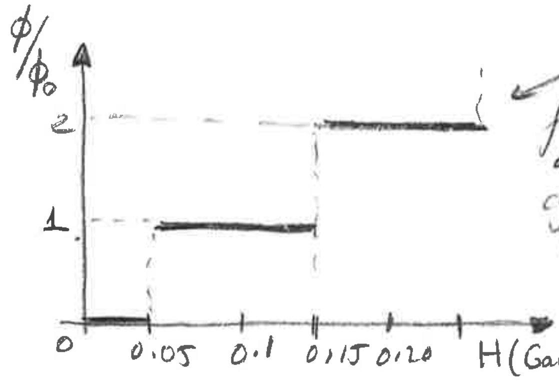
$$\Rightarrow \hbar \underbrace{\oint \vec{\nabla}\cdot\theta(\vec{r})\, d\vec{\ell}}_{=2\pi n} = e^* \Phi$$

Φ = magnetic flux

$$\phi = \iint \vec{B}\, d\vec{S}$$

$$\Rightarrow \quad \phi = \frac{2\pi n \hbar}{e^*} = \frac{h}{e^*}\cdot n$$

n = integer no.



measured flux in a superconducting cylinder cooled below T_c in various magnetic fields

$$\Phi_0 = \frac{h}{2e} = \text{flux quantum}$$

$$= 2.067 \times 10^{-15}\,\text{Wb}$$

Here I already have to disclose that : $e^* = -2e$.

Another useful relation: the energy-phase relationship

$$\underbrace{-\hbar\frac{\partial\theta}{\partial t}}_{\text{change of phase}} = \underbrace{\frac{1}{2}\frac{\mu_0\lambda_L^2}{n_s}\cdot J_s^2}_{\text{"Kinetic" energy}} + \underbrace{q^* V}_{\text{potential energy}}$$

Proof:

From the Schrödinger equation $i\hbar\frac{\partial}{\partial t}\psi = \frac{1}{2m^*}\left(-i\hbar\vec{\nabla} - q^*\vec{A}\right)^2\psi + q^* V\psi$

we replace $\psi = \sqrt{n_s}\, e^{i\theta}$, where $n_s = \text{const.}$

$$\left.\begin{aligned} &\Rightarrow -\hbar\frac{\partial\theta}{\partial t}\cancel{\sqrt{n_s}} = \frac{1}{2m^*}\left(+\hbar\vec{\nabla}\theta - q^*\vec{A}\right)^2\cancel{\sqrt{n_s}} + q^* V\cancel{\sqrt{n_s}} \\ &\quad\text{But } \vec{J_s}^2 = \frac{q^{*2} n_s^2}{m^{*2}}\left(\hbar\vec{\nabla}\theta - q^*\vec{A}\right)^2 \end{aligned}\right\} \Rightarrow -\frac{\hbar\partial\theta}{\partial t} = \frac{1}{2}\underbrace{\frac{m^*}{n_s^2 q^{*2}}}_{\equiv \frac{\mu_0\lambda_L^2}{n_s}} J_s^2 + q^* V$$

∘ Let us recap a bit:

Electrodynamics of superconductors is described by

1st London equation $$\boxed{\frac{d\vec{J}_s}{dt} = \frac{1}{\mu_0 \lambda_L^2}\vec{E}}$$

2nd London equation $$\boxed{\vec{B} = -\mu_0 \lambda_L^2 \vec{\nabla} \times \vec{J}_s}$$

Here $$\vec{J}_s = \frac{\hbar q^* n_s}{m^*}\left[\vec{\nabla}\theta - \frac{q^*}{\hbar}\vec{A}\right]$$

× London penetration length: $$\lambda_L^2 = \frac{m^*}{\mu_0 n_s q^{*2}}$$

or

$$\vec{J}_s = \frac{-\phi_0}{2\pi\mu_0\lambda_L^2}\left[\vec{\nabla}\theta + \frac{2\pi}{\Phi_0}\vec{A}\right]$$

$\Phi_0 = \frac{h}{2e}$ = flux quantum

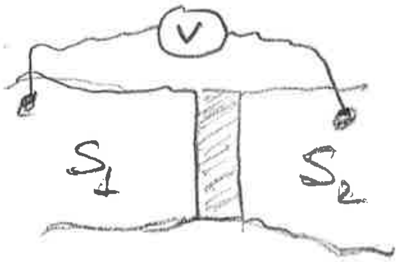
$q^* = -2e$

The quantity: $\vec{\nabla}\theta + \frac{2\pi}{\Phi_0}\vec{A}$ = gauge-invariant phase gradient

JOSEPHSON EFFECT

• What happens when we put a voltage across a weak link between two superconductors?

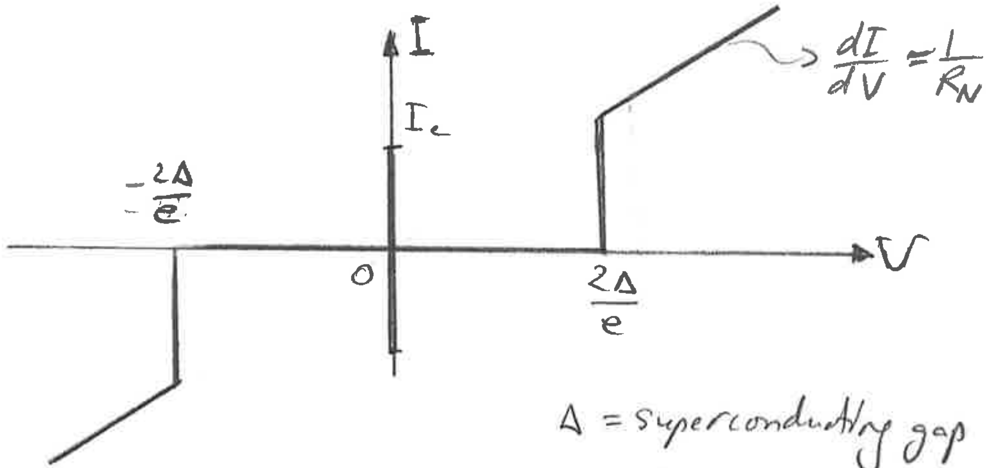
Weak link = can be a S-I-S (insulator between two superconductors)



S-N-S (a metal in-between)

S-s-S (a constriction)

• What we measure:



Δ = superconducting gap

$\Delta = 1.764\, k_B T_c$

T_c = critical temperature

(from BCS theory)

R_N = normal-state resistance

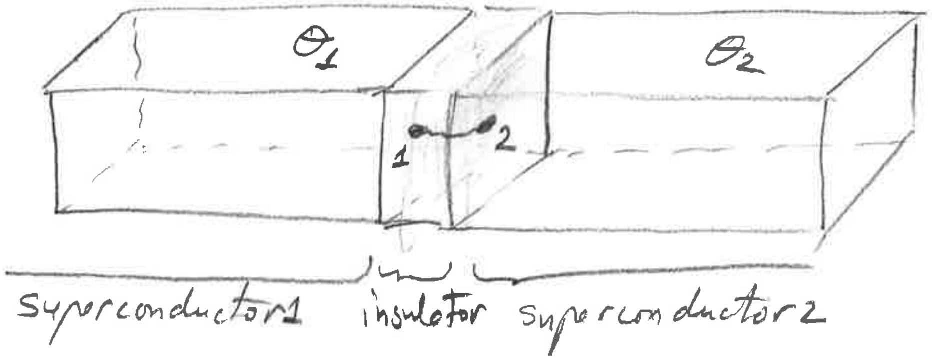
– Currents flowing for $|V| \geq \frac{2\Delta}{e}$ are no surprise – they are associated to breaking the Cooper pairs by the voltage.

But, at $V = 0$ there is a current flowing, with max. value = I_c (critical current of the junction).

This is the Josephson effect.

– Circuit symbol

The Josephson current-phase and phase-voltage relations



1] Current-phase.

Consider the gauge-invariant phase difference, obtained by integrating the gauge-invariant phase gradient,

$$\varphi = \int_1^2 d\vec{r}\left(\vec{\nabla}\theta + \frac{2\pi}{\phi_0}\vec{A}\right) = \theta_2 - \theta_1 + \frac{2\pi}{\phi_0}\int_1^2 d\vec{r}\,\vec{A}(\vec{r},t)$$

This is the only quantity which is gauge-invariant and includes the difference in phases $\theta_2 - \theta_1$ as we cross the insulator.

So perhaps $J_s = J_s(\varphi)$, that is, a function of φ.

Which one?

well, we should have also: 1) periodicity $J_s(\varphi) = J_s(\varphi + 2\pi n)$

2) $J_s(0) = 0$ (no current when there is no phase difference)

$$\Rightarrow \quad J_s(\varphi) = \underbrace{J_c}_{\substack{\text{constant,}\\ = \text{critical}\\ \text{Josephson current density.}}} \sin\varphi + \underbrace{\sum_{m=2}^{\infty} J_m \sin(m\varphi)}_{\text{this can be neglected}}$$

Therefore, for a given device we will have

$$\boxed{I = I_c \sin\varphi}$$

I_c = critical Josephson current

2] Voltage-phase

Consider again the gauge-invariant phase difference

$$\varphi = \theta_2 - \theta_1 + \frac{2\pi}{\phi_0}\int_1^2 d\vec{r}\,\vec{A}(\vec{r},t)$$

Recall now the energy-phase relationship

$$-\hbar\frac{\partial\theta}{\partial t} = \frac{1}{2}\frac{\mu_0\lambda_L^2}{n_s}J_s^2 + q^* V$$ valid for the phases θ_1 and θ_2 inside the superconductors 1 and 2

$$\Rightarrow \frac{\partial\varphi}{\partial t} = -\frac{1}{\hbar}\frac{\mu_0\lambda_L^2}{2n_s}\underbrace{\left(J_s^2(2) - J_s^2(1)\right)}_{} - \frac{q^*}{\hbar}\left(V(2) - V(1)\right) + \frac{2\pi}{\phi_0}\int_1^2 d\vec{r}\,\frac{\partial\vec{A}}{\partial t}$$

but $J_s(1) = J_s(2)$
(conservation of charge or Kirchhoff's current law)

$$\Rightarrow \frac{\partial\varphi}{\partial t} = \frac{2\pi}{\phi_0}\int_1^2 d\vec{r}\left(\vec{\nabla}V + \frac{\partial\vec{A}}{\partial t}\right)$$ But $\vec{E} = -\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t}$

$$\Rightarrow \frac{\partial\varphi}{\partial t} = -\frac{2\pi}{\phi_0}\underbrace{\int_1^2 d\vec{r}\cdot\vec{E}}_{\equiv -(V_2 - V_1) = -V}$$ where $V \equiv V_2 - V_1$

$$\Rightarrow \boxed{\frac{\partial\varphi}{\partial t} = \frac{2\pi}{\phi_0}V}$$

$$\Rightarrow \quad \boxed{I = I_c \sin\varphi}$$ current-phase relation

$$\boxed{\frac{\partial \varphi}{\partial t} = \frac{2e}{\hbar} V}$$ phase-voltage relation

flux

or: $V = \frac{\partial}{\partial t}\left(\frac{\phi_0}{2\pi}\cdot\varphi\right)$

very similar to Faraday's law

- Consequences:

DC JOSEPHSON EFFECT

$V = 0 \Rightarrow \frac{\partial \varphi}{\partial t} = 0 \Rightarrow \varphi = \text{const.}$

$I = I_c \sin\varphi$ — The current can reach a max. value of I_c

AC JOSEPHSON EFFECT

$V = \text{const} \neq 0 \Rightarrow \varphi = \frac{2e}{\hbar} V \cdot t$

$\Rightarrow I = I_c \sin\left(\frac{2e}{\hbar} V \cdot t\right) = I_c \sin\left(2\pi \frac{V}{\phi_0} t\right)$

$f_J = \frac{V}{\phi_0}$ = Josephson frequency $= 483 \times 10^{12} V_0$ (Hz)

JOSEPHSON INDUCTANCE

$$\frac{\partial I}{\partial t} = \frac{\partial I}{\partial \varphi} \cdot \frac{\partial \varphi}{\partial t} = I_c \cos\varphi \cdot \frac{2\pi}{\phi_0} V$$

or $V = L_J(\varphi) \frac{\partial I}{\partial t}$ $\quad L_J(\varphi)$ = Josephson inductance

$$L_J(\varphi) = \frac{\phi_0}{2\pi I_c \cos\varphi}$$

— depends on phase!

— can be ∞ if $\varphi = \frac{\pi}{2} + n\pi$

JOSEPHSON ENERGY

$$E_J = \int dt\, I \cdot V = \int d\varphi\, I_c \sin\varphi \cdot \frac{\phi_0}{2\pi}$$

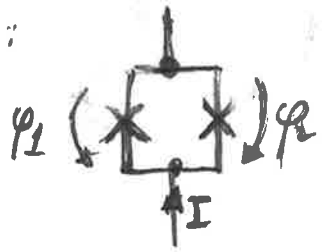
$$E_J(\varphi) = -\frac{I_c \phi_0}{2\pi} \cos\varphi = -E_J \cos\varphi$$

$$E_J = \frac{\phi_0 I_c}{2\pi} = \text{Josephson energy}$$

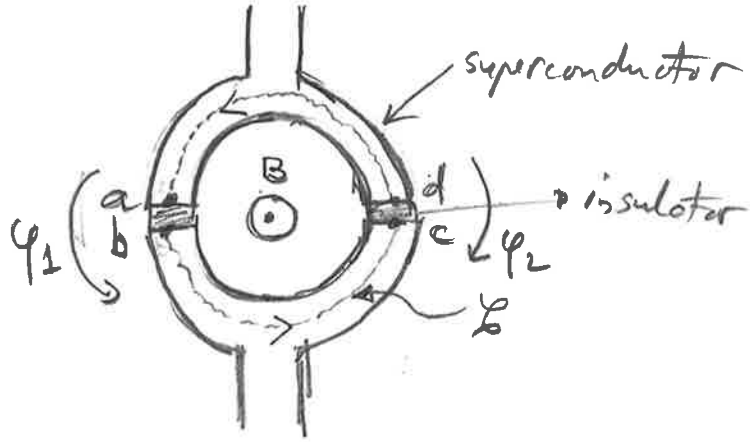
Application: the dc-SQUID

(superconducting quantum interference device)

⇒ Schematic:



How it is fabricated:



$$\oint_{\mathcal{C}} \vec{\nabla}\theta\cdot d\vec{r} = 2\pi n$$

$$= \underbrace{(\theta_b-\theta_a)}_{\varphi_1 - \frac{2\pi}{\phi_0}\int_a^b \vec{A}\,d\vec{r}} + (\theta_c-\theta_b) + \underbrace{(\theta_d-\theta_c)}_{-\varphi_2 - \frac{2\pi}{\phi_0}\int_c^d \vec{A}\,d\vec{l}} + (\theta_a-\theta_d)$$

$$\int_b^c d\vec{r}\cdot\vec{\nabla}\theta = -\frac{2\pi}{\phi_0}\mu_0\lambda_L^2\int_b^c d\vec{r}\,\vec{J}_s - \frac{2\pi}{\phi_0}\int_b^c d\vec{r}\,\vec{A}$$

zero, $J_s = 0$ inside the superconductor

$$\int_d^a d\vec{r}\cdot\vec{\nabla}\theta = -\frac{2\pi}{\phi_0}\mu_0\lambda_L^2\int_d^a \vec{J}_s\,d\vec{r} - \frac{2\pi}{\phi_0}\int_d^a d\vec{r}\,\vec{A}$$

zero

$$= \varphi_1 - \varphi_2 - \frac{2\pi}{\phi_0}\underbrace{\oint_{\mathcal{C}} d\vec{r}\cdot\vec{A}}_{=\phi}$$

$= \phi$ (magnetic flux piercing the SQUID)

$$\Rightarrow \boxed{\varphi_1 - \varphi_2 = 2\pi n + \frac{2\pi\phi}{\phi_0}}$$

So $I = I_1 + I_2 = I_c \sin\varphi_1 + I_c \sin\varphi_2$

(we assume that the junctions are identical)

$$= 2I_c \sin\frac{\varphi_1+\varphi_2}{2}\cos\frac{\varphi_1-\varphi_2}{2}$$

Let $\varphi = \frac{\varphi_1+\varphi_2}{2} \Rightarrow$

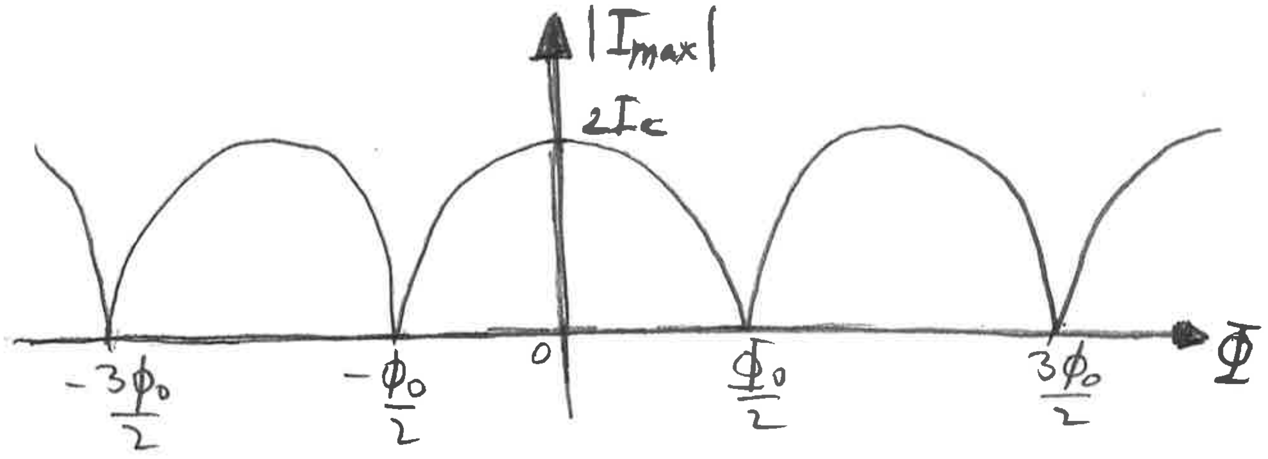
$$I = 2I_c \sin\varphi \cdot \cos\left(\frac{\pi\Phi}{\Phi_0} + \pi n\right)$$

$$I \equiv I_{max}(\Phi)\cdot\sin\varphi$$

where

$$I_{max}(\Phi) = 2I_c\cos\left(\frac{\pi\Phi}{\Phi_0} + \pi n\right)$$

The SQUID behaves as a single Josephson junction with critical current controlled by the magnetic flux.



- I_{max} never exceeds $2I_c$
- I_{max} can be zero! This can be understood as destructive interference of the currents in the two branches of the SQUID.

References

- Terry P. Orlando and Kevin A. Delin — Foundations of Applied Superconductivity
- D. R. Tilley and J. Tilley — Superfluidity and Superconductivity
- Antonio Barone and Gianfranco Paternò — Physics and Applications of the Josephson Effect