Ex $F(x,y,z) = (xy,\, y^2 - z^2,\, yz)$

$$\operatorname{div} F = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(y^2 - z^2) + \frac{\partial}{\partial z}(yz) =$$

$$= y + 2y + y = 4y$$

$$\operatorname{Curl} F = \begin{vmatrix} \vec{e_1} & \vec{e_2} & \vec{e_3} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & y^2-z^2 & yz \end{vmatrix} =$$

$$= \left(\frac{\partial}{\partial y}(yz) - \frac{\partial}{\partial z}(y^2 - z^2),\, \frac{\partial}{\partial z}(xy) - \frac{\partial}{\partial x}(yz),\, \frac{\partial}{\partial x}(y^2 - z^2) - \frac{\partial}{\partial y}(xy) \right) =$$

$$= (z + 2z,\, 0,\, -x) = (3z, 0, -x)$$

Interpretation of the divergence

Let $\vec{F}$ be a smooth vector field and $\vec{N}$ be the unit outward normal vector field of S_ε, the sphere with radius ε centered at P. Then

$$\operatorname{div} \vec{F}(P) = \lim_{\varepsilon \to 0^+} \frac{3}{4\pi\varepsilon^3} \oiint_{S_\varepsilon} \vec{F} \cdot \vec{N}\, dS$$

$$\frac{1}{\text{volume of } S_\varepsilon}$$

"Proof": Let $P = \vec{0}$. Then $\vec{N} = \frac{1}{\varepsilon}(x,y,z)$

Taylor expansion of (F_1, F_2, F_3)

$$\vec{F}(x,y,z) = \underbrace{\vec{F}(0,0,0)}_{\vec{F_0}} + \underbrace{\left(\frac{\partial F_1}{\partial x}, \frac{\partial F_2}{\partial x}, \frac{\partial F_3}{\partial x}\right)}_{\vec{F}_{x0}} x +$$

$$+ \underbrace{\left(\frac{\partial F_1}{\partial y}, \frac{\partial F_2}{\partial y}, \frac{\partial F_3}{\partial y}\right)}_{\vec{F}_{y0}} y + \underbrace{\left(\frac{\partial F_1}{\partial z}, \frac{\partial F_2}{\partial z}, \frac{\partial F_3}{\partial z}\right)}_{\vec{F}_{z0}} z +$$

$+$ higher order terms

$$\vec{F} \cdot \vec{N} = \frac{1}{\varepsilon}\Big(\vec{F_0} \cdot (x,y,z) + \vec{F}_{x0}\, x^2 \cdot \vec{e_1} + \vec{F}_{x0}\, xy \cdot \vec{e_2}$$

$$+ \vec{F}_{x0}\, xz \cdot \vec{e_3} + \vec{F}_{y0}\, yx \cdot \vec{e_1} + \vec{F}_{y0}\, y^2 \cdot \vec{e_2} +$$

$$+ \vec{F}_{y0}\, yz \cdot \vec{e_3} + \vec{F}_{z0}\, xz \cdot \vec{e_1} + \vec{F}_{z0}\, yz \cdot \vec{e_2} +$$

$$+ \vec{F}_{z0}\, z^2 \cdot \vec{e_3} + \ldots$$

Now integrate termwise

$$\oiint_{S_\varepsilon} x\, dS = \oiint_{S_\varepsilon} y\, dS = \oiint_{S_\varepsilon} z\, dS = 0$$

$$\oiint_{S_\varepsilon} xy\, dS = \oiint_{S_\varepsilon} xz\, dS = \oiint_{S_\varepsilon} yz\, dS = 0$$

$$\oiint_{S_\varepsilon} x^2\, dS = \oiint_{S_\varepsilon} y^2 dS = \oiint_{S_\varepsilon} z^2 dS =$$

$$= \frac{1}{3} \oiint_{S_\varepsilon} x^2+y^2+z^2\, dS = \frac{1}{3}\varepsilon^2 \cdot 4\pi\varepsilon^2 = \frac{4\pi}{3}\varepsilon^4$$

Higher order terms involve ε^k, $k \geq 5$

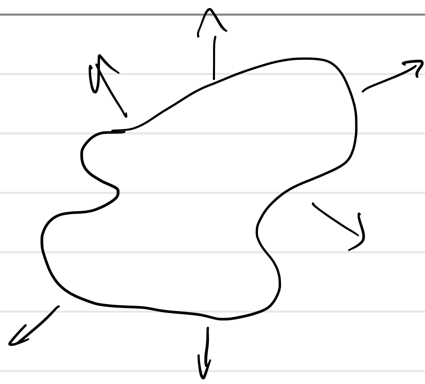
So $\frac{3}{4\pi\varepsilon^3} \oiint_{S_\varepsilon} \vec{F}\cdot\vec{N}\, dS =$

$$= \frac{3}{4\pi\varepsilon^3}\cdot\frac{1}{\varepsilon}\left(\oiint_{S_\varepsilon} (F_{x0}\cdot\vec{e_1})x^2 + (F_{y0}\cdot\vec{e_2})y^2 + (F_{z0}\cdot\vec{e_3})z^2\, dS + O(\varepsilon^5)\right)$$

and $\lim\limits_{\varepsilon\to 0+} \frac{3}{4\pi\varepsilon^3} \oiint_{S_\varepsilon} \vec{F}\cdot\vec{N}\, dS = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \operatorname{div} F.$

$\otimes$

That is div F measures "how much fluid is created or destroyed in each point".



Flow = "sum" of div F in the interior.

Interpretation of Curl

Ex Consider the vector field

$$\vec{V} = (-\omega y, \omega x, 0)$$

Calculate the circulation counterclockwise around the circle C_ε centered at (x_0, y_0) with radius ε in the xy-plane

$$C_\varepsilon(t) = (x_0 + \varepsilon \cos t, y_0 + \varepsilon \sin t, 0)$$

$$0 \le t \le 2\pi$$

$$\oint_{C_\varepsilon} \vec{V} \cdot d\vec{r} = \int_0^{2\pi} -\omega(y_0 + \varepsilon \sin t)(-\varepsilon \sin t) +$$

$$+ \omega(x_0 + \varepsilon \sin t)(\varepsilon \cos t)\, dt =$$

$$= \int_0^{2\pi} \omega\varepsilon(y_0 \sin t + x_0 \cos t) + \omega\varepsilon^2\, dt = 2\pi\omega\varepsilon^2$$

Also $\quad \text{Curl}\,\vec{v} = \nabla \times \vec{v} = \left(\frac{\partial}{\partial x}(\omega x) - \frac{\partial}{\partial y}(-\omega y) \right) \vec{e_3}$

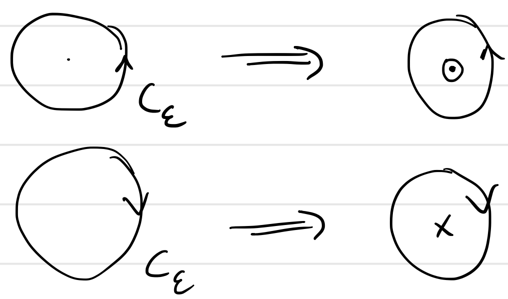
$$= 2\omega \vec{e_3}$$

Note that $\displaystyle\oint_{C_\varepsilon} \vec{v} \cdot d\vec{r} = \text{Area}(C_\varepsilon) \left(\text{Curl}\,\vec{v} \cdot \vec{e_3} \right)$

Theorem: If $\vec{F}$ is a smooth vector field in $\mathbb{R}^3$ and C_ε is a circle of radius ε centered at P and bounding a disk D_ε with unit normal $\vec{N}$ (orientation inherited from C_ε) then

$$\lim_{\varepsilon \to 0^+} \frac{1}{\pi \varepsilon^2} \oint_{C_\varepsilon} \vec{F} \cdot d\vec{r} = \vec{N} \cdot \text{Curl}\,\vec{F}(P)$$

Orientation inherited from C_ε?



$\vec{N}$ "comes out of paper"

$\vec{N}$ "going into paper"

Identities involving div, grad and Curl

ϕ, ψ functions , $\vec{F}, \vec{G}$ vector fields

a) $\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$

b) $\operatorname{div}(\phi\vec{F}) = \nabla\cdot(\phi\vec{F}) = (\nabla\phi)\cdot\vec{F} + \phi(\nabla\cdot\vec{F})$

c) $\nabla\times(\phi\vec{F}) = (\nabla\phi)\times\vec{F} + \phi(\nabla\times\vec{F})$

d) $\nabla\cdot(\vec{F}\times\vec{G}) = (\nabla\times\vec{F})\cdot\vec{G} - \vec{F}\cdot(\nabla\times\vec{G})$

e) $\nabla\times(\vec{F}\times\vec{G}) = (\nabla\cdot\vec{G})\vec{F} + (\vec{G}\cdot\nabla)\vec{F} - (\nabla\cdot\vec{F})\vec{G} - (\vec{F}\cdot\nabla)\vec{G}$

f) $\nabla(\vec{F}\cdot\vec{G}) = \vec{F}\times(\nabla\times\vec{G}) + \vec{G}\times(\nabla\times\vec{F}) + (\vec{F}\cdot\nabla)\vec{G} + (\vec{G}\cdot\nabla)\vec{F}$

g) $\nabla\cdot(\nabla\times\vec{F}) = 0$ div Curl $= 0$

h) $\nabla\times(\nabla\phi) = 0$ Curl grad $= 0$

i) $\nabla\times(\nabla\times\vec{F}) = \nabla(\nabla\cdot\vec{F}) - (\nabla\cdot\nabla)\vec{F}$

$\nabla^2 = \Delta =$ Laplace operator

Curl Curl = grad div − Laplace

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_n^2}$$

A function satisfying $\Delta f = 0$ is called harmonic.

Proof g) (Do the rest by yourself)

$$\nabla \times \vec{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

$$\nabla \cdot (\nabla \times \vec{F}) = \frac{\partial}{\partial x}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) + \frac{\partial}{\partial y}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) + \frac{\partial}{\partial z}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) = 0$$

$\otimes$

Green's Theorem in the plane

This can be seen as a higher-dimensional version of the Fundamental Theorem of Calculus.

- Classical version $\int_a^b \frac{d}{dx} f(x)\,dx = f(b) - f(a)$.