Introduction to quantization of electrical circuits and superconducting qubits

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IMPORTANT NOTE: Owing to lack of time, we will only go trhough about 35% of this material during the lectures: Briefly the Lagrangian formalism and quantization of the harmonic oscillator, Jospehson energy, and quantization of the qubit.

Material credit: This material is based on the courese *COM2: Superconducting Quantum Computer* that was held as a part of The 26th Jyväskylä Summer School in 2016. The material has been produced by Mikko Möttönen and **Juha Jeronen** while they were working for University of Jyväskylä and revised by Mikko Möttönen for the QuICC summer school in 2018 and for the Qauntum Connections in Sweden 6 and EQS summer schools in 2019.

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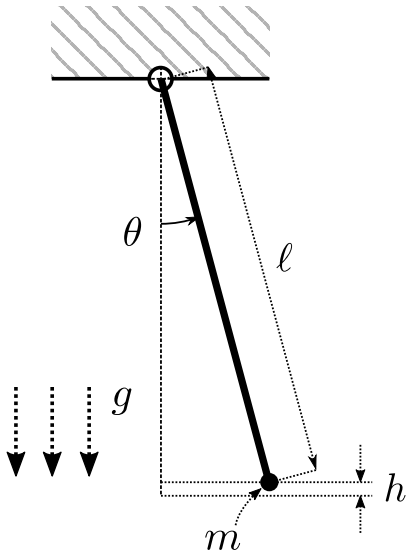

Figure 1: Classical pendulum.

1 Introduction: the classical pendulum

During this course, Lagrangian and Hamiltonian mechanics are used for analyzing quantum computing circuits. As a refresher on Lagrangian mechanics, let us first consider a classical pendulum, with a point mass m suspended by a rigid wire of length ℓ, subjected to a uniform gravitational field having acceleration g. Refer to Figure 1.

The magnitude of the linear velocity of the point mass is

$$v = \ell\,\dot{\theta}\,, \tag{1}$$

where θ is the angle of the pendulum as measured from the vertical.

The gravitational potential energy is, by geometry,

$$V = mgh = mg\ell\,(1 - \cos\theta) \approx \frac{1}{2} mg\ell\theta^2\,, \tag{2}$$

where the zero level for the height has been taken as the stable equilibrium position of the pendulum at $\theta = 0$, and the last form is an $O(\theta^4)$ Taylor approximation valid near $\theta = 0$.

The kinetic energy is

$$T = \frac{1}{2} mv^2 = \frac{1}{2} m\ell^2\dot{\theta}^2\,, \tag{3}$$

where we have used (1).

Recall that the Lagrangian is defined as the kinetic energy minus the potential energy,

$$L \equiv T - V\,. \tag{4}$$

We choose the generalized position and momentum coordinates as

$$q = \ell\,\theta\,, \tag{5}$$

$$p \equiv \frac{\partial L}{\partial \dot{q}} \approx \frac{\partial}{\partial \dot{q}} \left(\frac{1}{2} m\dot{q}^2 - \frac{mg}{2\ell} q^2 \right) = m\dot{q} = m\ell\,\dot{\theta}\,. \tag{6}$$

The Euler–Lagrange equation in Lagrangian mechanics states that

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}} \right) = \frac{\partial L}{\partial q}\,. \tag{7}$$

Observe that $T = T(\dot{q})$ only (no dependence on q in the kinetic energy), so actually

$$\frac{\partial L}{\partial q} = -\frac{\partial V}{\partial q}\,.$$

Since $p = \partial L/\partial \dot{q}$, the left-hand side of (7) can be written as $\mathrm{d}p/\mathrm{d}t$. Hence (7) coincides with Newton's equation of motion

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{\partial V}{\partial q}$$

for a particle subjected to a potential force (in one space dimension).

Inserting (5) and (6) into (7), we have

$$\dot{p} = -mg\,\theta\ . \tag{8}$$

On the other hand, by differentiating (6) with respect to time, we have

$$\dot{p} = m\ell\,\ddot{\theta}\ . \tag{9}$$

Combining (8) and (9), we obtain the linearized equation of motion, describing small vibrations of the pendulum, as

$$m\ell\,\ddot{\theta} + mg\theta = 0\ ,$$

i.e.

$$\ddot{\theta} + \frac{g}{\ell}\theta = 0\ . \tag{10}$$

Inserting the standard trial function

$$\theta = C\exp(i\omega t)\ , \tag{11}$$

where $C \neq 0$ is a constant, i is the imaginary unit and ω the angular frequency, we obtain

$$i^2\omega^2 C\exp(i\omega t) + \frac{g}{\ell}C\exp(i\omega t) = 0\ ,$$

or after regrouping,

$$\left(-\omega^2 + \frac{g}{\ell}\right)C\exp(i\omega t) = 0\ . \tag{12}$$

Equation (12) holds at all t if and only if the expression in the parentheses vanishes:

$$-\omega^2 + \frac{g}{\ell} = 0\ . \tag{13}$$

Equation (13) is the characteristic polynomial of the linearized equation of motion (10). Solving (13) for ω, we obtain the natural angular frequency of small vibrations of the pendulum as

$$\omega = \sqrt{g/\ell}\ . \tag{14}$$

Physically, the angle θ is of course real-valued; the complex-valued trial function is only used because it is mathematically convenient. For a second-order linear differential equation of the form (10) in particular, physicists may prefer to use $A_1\sin(\omega t)$ and $A_2\cos(\omega t)$ as the trial functions. The trial (11) has the advantage that it is more general; it works also for problems where a first-order term (such as a damping term) is present.

Because equation (10) is linear in θ and has only real coefficients, the real and imaginary parts of a complex-valued solution are real-valued solutions. This can be seen as follows. Let $\theta = \theta(t)$ be a complex-valued function of time, and $\mathcal{L}$ a linear differential operator with real-valued coefficients. From the differential equation

$$\mathcal{L}(\theta) = 0\ ,$$

it follows by taking the real part that

$$\begin{aligned}
\mathrm{Re}[\mathcal{L}(\theta)] &= \mathrm{Re}\{\mathcal{L}[\,\mathrm{Re}(\theta) + i\mathrm{Im}(\theta)\,]\} \\
&= \mathrm{Re}\{\mathcal{L}[\mathrm{Re}(\theta)] + i\mathcal{L}[\mathrm{Im}(\theta)]\,]\} \\
&= \mathrm{Re}\{\mathcal{L}[\mathrm{Re}(\theta)]\} + \mathrm{Re}\{i\mathcal{L}[\mathrm{Im}(\theta)]\,]\} \\
&= \mathrm{Re}\{\mathcal{L}[\mathrm{Re}(\theta)]\} \\
&= \mathcal{L}[\mathrm{Re}(\theta)]\ .
\end{aligned}$$

Similarly we have $\mathrm{Im}[\mathcal{L}(\theta)] = \mathcal{L}[\mathrm{Im}(\theta)]$. Now because $\mathcal{L}[\mathrm{Re}(\theta)] = 0$ and $\mathcal{L}[\mathrm{Im}(\theta)] = 0$, it is seen that the functions $\mathrm{Re}(\theta)$ and $\mathrm{Im}(\theta)$ are real-valued solutions of $\mathcal{L}(\dots) = 0$.

Now recall Euler's formula, which states that

$$e^{i\alpha} = \cos\alpha + i\sin\alpha \ . \tag{15}$$

The trial $\theta = C\exp(i\omega t)$ thus leads to the linearly independent real-valued solutions

$$\theta_1(t) = A_1\sin(\omega t) \ ,$$
$$\theta_2(t) = A_2\cos(\omega t) \ .$$

Their linear combination is equivalent to

$$\theta = A\sin(\theta_0 + \omega t) \ , \tag{16}$$

where the constants A and θ_0 are determined by the initial conditions of the canonical variables. This follows from the trigonometric identity

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) \ .$$

Applying the identity to (16), we obtain

$$A\sin(\omega_0 + \omega t) = A\sin(\theta_0)\cos(\omega t) + A\cos(\theta_0)\sin(\omega t) \ ,$$

whence we read

$$A_1 = A\cos(\theta_0) \ , \quad A_2 = A\sin(\theta_0) \ .$$

We recognize this as a conversion from polar (A, θ_0) to Cartesian (A_1, A_2) coordinates. To show that the representation (16) is applicable to any choice of coefficients (A_1, A_2), we need only note that the inverse conversion always exists. Explicitly, it is

$$A = \sqrt{A_1^2 + A_2^2} \ , \quad \theta_0 = \mathrm{atan2}(A_2, A_1) \ ,$$

where $\mathrm{atan2}(y, x)$ is the quadrant-preserving $\arctan(\dots)$ function. (If $A_1 = A_2 = 0$, then θ_0 is undefined, but this does not matter.)

2 The classical superconducting LC oscillator

Let us next consider a classical (continuous) LC oscillator. Strictly speaking, the name "LC resonator" would be technically more appropriate: this circuit starts to resonate when power is applied at the resonant angular frequency ω. However, "LC oscillator" has stuck as the standard term, and we will use it here.

In a classical superconducting LC circuit, electrical energy oscillates between potential energy stored in the capacitor, and kinetic energy related to the magnetic flux in the coil. Refer to Figure 2.

The instantaneous power fed into any electrical circuit can be obtained by the product of the voltage across the circuit times the current flowing into the node of the positive voltage from outside the circuit. For a capacitor, this principle yields

$$P = U\dot{Q} \ , \tag{17}$$

where U is the voltage across the capacitor induced by the electrical charge Q stored in the capacitor. The capacitance C is defined by the relation

$$Q = CU \ . \tag{18}$$

We define the direction of the current according to Figure 2 as

$$I = -\dot{Q} \ . \tag{19}$$

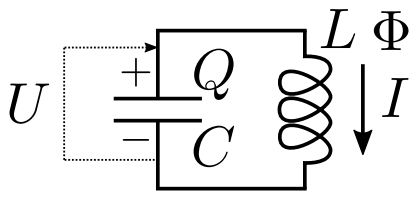

Figure 2: Superconducting LC oscillator.

Thus a positive current discharges the capacitor in this notation.

Since power is simply the temporal derivative of energy, by (17) and (18) we obtain the following relation for the electrical potential energy stored in the capacitor:

$$V = \int_{t_0}^{t_1} P \,\mathrm{d}t = \int_{t_0}^{t_1} U\dot{Q} \,\mathrm{d}t = \int_{t_0}^{t_1} U \left(\frac{\mathrm{d}Q}{\mathrm{d}t}\mathrm{d}t\right) = \int_0^Q U \,\mathrm{d}Q' = \int_0^Q \frac{Q'}{C} \,\mathrm{d}Q' = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}QU = \frac{1}{2}CU^2 \,, \tag{20}$$

where, in the change of variables, $Q' = Q'(t)$ is a dummy integration variable for the charge Q, and we have taken $Q'|_{t=t_0} = 0$ and $Q'|_{t=t_1} = Q$.

For the magnetic flux Φ in a coil, it holds that

$$\Phi = LI \,, \tag{21}$$

where L is the inductance of the coil, and I is the current flowing through it. Lenz's law states that

$$\dot{\Phi} = U \,. \tag{22}$$

In textbooks, Lenz's law is usually given as $\dot{\Phi} = -U$, but in that case U is defined as the change of electrical potential across the inductor in the direction of the current. Compared to this convention, in Figure 2, the voltage U is defined in the opposite sense (from the viewpoint of the current flowing in the inductor), which gives the plus sign in (22).

[To illustrate, let us temporarily replace the capacitor in Figure 2 by a voltage source grounded on its minus side. At time $t = 0$, let the current $I = 0$. As t increases, the current starts flowing through the inductor in the direction indicated by the arrow in the Figure, and the voltage drop across the inductor is U (i.e. the electrical potential changes by $-U$), matching the voltage U of the source.]

By differentiating both sides of (21) with respect to time and inserting (22), we have

$$U = L\dot{I} \,. \tag{23}$$

Similarly as above, by (21) and (23) we obtain the electrical kinetic energy in the coil as

$$T = \int_{t_0}^{t_1} P \,\mathrm{d}t = \int_{t_0}^{t_1} UI \,\mathrm{d}t = \int_{t_0}^{t_1} (L\dot{I})I \,\mathrm{d}t = \int_{t_0}^{t_1} LI \left(\frac{\mathrm{d}I}{\mathrm{d}t}\mathrm{d}t\right) = \int_0^I LI' \,\mathrm{d}I' = \frac{1}{2}LI^2 = \frac{\Phi^2}{2L} \,. \tag{24}$$

Now we apply Lagrangian mechanics. We have

$$V = \frac{1}{2}CU^2 = \frac{Q^2}{2C} \,, \tag{25}$$

$$T = \frac{1}{2}LI^2 = \frac{1}{2}L\dot{Q}^2 \,, \tag{26}$$

allowing us to write the Lagrangian as

$$L = T - V = \frac{1}{2}L\dot{Q}^2 - \frac{Q^2}{2C} \,. \tag{27}$$

We choose the generalized coordiantes as

$$q = Q \,, \tag{28}$$

$$p \equiv \frac{\partial L}{\partial \dot{q}} = L\dot{Q} = -LI = -\Phi \,. \tag{29}$$

The Euler–Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q}$$

with the Lagrangian (27) gives the equation of motion for the electrical charge Q:

$$\dot{p} = L\ddot{Q} = -\frac{Q}{C} \,,$$

i.e.

$$\ddot{Q} + \frac{1}{LC}Q = 0 \,, \tag{30}$$

whence the analogy with the pendulum is obvious. As above, the natural angular frequency of oscillations is obtained as

$$\omega = \frac{1}{\sqrt{LC}} \,. \tag{31}$$

In this course, we mostly use Hamiltonian mechanics. The Hamiltonian is defined as

$$H \equiv \dot{q}p - L \,. \tag{32}$$

Taking the (total) time derivative of H, we find [keep in mind that in general, $L = L(q, \dot{q}, t)$]

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \ddot{q}p + \dot{q}\dot{p} - \frac{\partial L}{\partial q}\dot{q} - \frac{\partial L}{\partial \dot{q}}\ddot{q} - \dot{L} \,.$$

Using $p \equiv \partial L/\partial \dot{q}$ to rewrite some terms [note $p = p(t)$ only, so $\mathrm{d}p/\mathrm{d}t = \dot{p}$],

$$\begin{aligned}
\frac{\mathrm{d}H}{\mathrm{d}t} &= \ddot{q}p + \dot{q}\dot{p} - \frac{\partial L}{\partial q}\dot{q} - p\ddot{q} - \dot{L} \\
&= \dot{q}\dot{p} - \frac{\partial L}{\partial q}\dot{q} - \dot{L} \\
&= \dot{q}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q}\dot{q} - \dot{L} \\
&= \dot{q}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q}\right] - \dot{L} \,.
\end{aligned}$$

By the Euler–Lagrange equation, the parenthetical expression is zero. If the Lagrangian does not depend explicitly on time, $L = L(q, \dot{q})$, then the last term vanishes. In this case,

$$\frac{\mathrm{d}H}{\mathrm{d}t} = 0 \,,$$

or in other words, the Hamiltonian is then a constant of motion.

It often (although not necessarily always) happens that the Hamiltonian is the total energy. Indeed, for our LC circuit, inserting (27)–(29) into (32) gives

$$H = \dot{Q}(L\dot{Q}) - \left(\frac{1}{2}L\dot{Q}^2 - \frac{Q^2}{2C}\right) = \frac{1}{2}L\dot{Q}^2 + \frac{Q^2}{2C} \,, \tag{33}$$

so for this system, $H = T + V$. By the definition (19), and equation (21), we may equivalently write

$$H = \frac{1}{2}LI^2 + \frac{Q^2}{2C} = \frac{\Phi^2}{2L} + \frac{Q^2}{2C} \,. \tag{34}$$

By the last form of (20), and equation (22), a third possible representation is

$$H = \frac{\Phi^2}{2L} + \frac{1}{2}CU^2 = \frac{\Phi^2}{2L} + \frac{1}{2}C\dot{\Phi}^2 \,.$$

In practice the form (34) turns out to be the most convenient. In terms of the generalized variables (28) and (29), we have the result

$$H = \frac{p^2}{2L} + \frac{q^2}{2C} \,. \tag{35}$$

We will next look at the modifications one must make to (35) in order to proceed from classical to quantum mechanics.

Remark If we instead wish to formulate the problem in terms of the magnetic flux Φ, we may swap the roles of kinetic and potential energy, writing

$$V = \frac{\Phi^2}{2L} \,, \tag{36}$$

$$T = \frac{1}{2}C\dot{\Phi}^2 \,, \tag{37}$$

where in (37) we have used (22). Here (36) is the previous T and (37) is the previous V. In this case the generalized variables are

$$q = \Phi \,, \tag{38}$$

$$p \equiv \frac{\partial L}{\partial \dot{q}} = \frac{\partial}{\partial \dot{q}} (T - V) = \frac{\partial}{\partial \dot{\Phi}} \left(\frac{1}{2} C \dot{\Phi}^2 - \frac{\Phi^2}{2L} \right) = C\dot{\Phi} = CU = Q \,, \tag{39}$$

where in the last step we have used (18).

3 Quantization of the LC oscillator

In quantum mechanics, the classical generalized variables q and p are replaced by operators,

$$q \to \widehat{q} : \mathscr{H} \to \mathscr{H} \,, \tag{40}$$

$$p \to \widehat{p} : \mathscr{H} \to \mathscr{H} \,, \tag{41}$$

which are linear mappings operating on a Hilbert space $\mathscr{H}$. The Hilbert space of kets, $\mathscr{H}$, is the space of all physical quantum states of the system: $\mathscr{H} = \{|\psi\rangle\}$.

In quantum mechanics, operators and matrices are often identified. However, strictly speaking, it is important to keep in mind that (for example) the operator $\widehat{q}$ operates on $\mathscr{H}$, while its matrix representation q, which has matrix elements given by

$$q_{k\ell} = \langle e_k | \, \widehat{q} \, | e_\ell \rangle \,,$$

operates in a fixed basis, i.e. in a setting where we have already chosen to use some particular set of basis vectors $\{|e_j\rangle\}$ spanning the space $\mathscr{H}$.

The operators $\widehat{p}$ and $\widehat{q}$ satisfy the commutation relation

$$[\widehat{p}, \widehat{q}] \equiv \widehat{p}\widehat{q} - \widehat{q}\widehat{p} = -i\hbar \,, \tag{42}$$

where i is the imaginary unit and $\hbar$ is the reduced Planck constant, $\hbar \equiv h/2\pi$, where the Planck constant $h = 6.626070040(81) \times 10^{-34}$ J s. Recall that in general, operators do not commute; one must thus be careful to retain the ordering of operations when performing algebra with them.

The operators $\widehat{p}$ and $\widehat{q}$ are Hermitian, i.e. they satisfy

$$\widehat{p}^\dagger = \widehat{p} \,, \quad \widehat{q}^\dagger = \widehat{q} \,, \tag{43}$$

where $(\dots)^\dagger$ denotes the Hermitian conjugate.

Similarly, the Hamiltonian H is replaced by its operator counterpart,

$$H \to \widehat{H} \,. \tag{44}$$

The most important equation in quantum mechanics is the Schrödinger equation, which governs the quantum state $|\psi\rangle$. The time-dependent Schrödinger equation reads

$$i\hbar \, \partial_t \, |\psi\rangle = \widehat{H} \, |\psi\rangle \ . \tag{45}$$

3.1 Diagonalization of the Hamiltonian

Let us quantize the superconducting LC oscillator. We use the quantum counterpart of the generalized variables (28) and (29), i.e. $\widehat{q} = \widehat{Q}$ and $\widehat{p} = \widehat{\Phi}$.

Analogously to (35), for the superconducting quantum LC oscillator we have

$$\widehat{H} = \frac{\widehat{p}^2}{2L} + \frac{\widehat{q}^2}{2C} \,. \tag{46}$$

We aim to diagonalize $\widehat{H}$ into a form involving only one (general, not Hermitian) operator $\widehat{a}$. This can be done by a change of variables. Specifically, because the expression (46) is quadratic, we may use a linear transformation:

$$\widehat{p} = \sqrt{\frac{\hbar\omega L}{2}} \left(\widehat{a} + \widehat{a}^\dagger \right) \,,$$

$$\widehat{q} = \sqrt{\frac{\hbar\omega C}{2}} i \left(\widehat{a} - \widehat{a}^\dagger \right) \,.$$

Here ω is a free scalar parameter, which we will choose later. The square root factors have been inserted for convenience. Note that for a general $\widehat{a}$, the combinations $(\widehat{a}+\widehat{a}^\dagger)$ and $i(\widehat{a}-\widehat{a}^\dagger)$ are Hermitian and independent. The Hamiltonian (46) becomes

$$\begin{aligned}\widehat{H} &= \frac{\left[\sqrt{\frac{\hbar\omega L}{2}}\left(\widehat{a}+\widehat{a}^\dagger\right)\right]^2}{2L} + \frac{\left[\sqrt{\frac{\hbar\omega C}{2}}i\left(\widehat{a}-\widehat{a}^\dagger\right)\right]^2}{2C} \\ &= \frac{\hbar\omega}{4}\left(\widehat{a}\widehat{a}^\dagger + \widehat{a}^\dagger\widehat{a} + \widehat{a}\widehat{a}^\dagger + \widehat{a}^\dagger\widehat{a}\right) \\ &= \frac{\hbar\omega}{2}\left(\widehat{a}\widehat{a}^\dagger + \widehat{a}^\dagger\widehat{a}\right) \ .\end{aligned}$$

In order to explicitly find $\widehat{a}$ in terms of $\widehat{p}$ and $\widehat{q}$, we invert the transformation. We first scale each equation:

$$\begin{aligned}\frac{1}{\sqrt{2\hbar\omega L}}\widehat{p} &= \frac{1}{\sqrt{2\hbar\omega L}}\sqrt{\frac{\hbar\omega L}{2}}\left(\widehat{a}+\widehat{a}^\dagger\right) = \frac{1}{2}\left(\widehat{a}+\widehat{a}^\dagger\right) \ , \\ i\frac{1}{\sqrt{2\hbar\omega C}}\widehat{q} &= i\frac{1}{\sqrt{2\hbar\omega C}}\sqrt{\frac{\hbar\omega C}{2}}i\left(\widehat{a}-\widehat{a}^\dagger\right) = \frac{1}{2}\left(\widehat{a}^\dagger-\widehat{a}\right) \ .\end{aligned}$$

Then summing the results, we obtain

$$i\frac{1}{\sqrt{2\hbar\omega C}}\widehat{q} + \frac{1}{\sqrt{2\hbar\omega L}}\widehat{p} = \widehat{a}^\dagger \ ,$$

and by Hermitian conjugation,

$$-i\frac{1}{\sqrt{2\hbar\omega C}}\widehat{q} + \frac{1}{\sqrt{2\hbar\omega L}}\widehat{p} = \widehat{a} \ . \tag{47}$$

Let us next calculate the commutator $[\widehat{a},\widehat{a}^\dagger]$. We have

$$\begin{aligned}\widehat{a}\widehat{a}^\dagger &= \left(-i\frac{1}{\sqrt{2\hbar\omega C}}\widehat{q} + \frac{1}{\sqrt{2\hbar\omega L}}\widehat{p}\right)\left(i\frac{1}{\sqrt{2\hbar\omega C}}\widehat{q} + \frac{1}{\sqrt{2\hbar\omega L}}\widehat{p}\right) \\ &= \frac{1}{2\hbar\omega C}\widehat{q}^2 + i\frac{1}{\sqrt{2\hbar\omega C}\sqrt{2\hbar\omega L}}\left(\widehat{p}\widehat{q}-\widehat{q}\widehat{p}\right) + \frac{1}{2\hbar\omega L}\widehat{p}^2 \\ &= \frac{1}{2\hbar\omega C}\widehat{q}^2 + i\frac{1}{\sqrt{2\hbar\omega C}\sqrt{2\hbar\omega L}}\left(-i\hbar\right) + \frac{1}{2\hbar\omega L}\widehat{p}^2 \\ &= \frac{1}{2\hbar\omega C}\widehat{q}^2 + \frac{1}{\sqrt{2\omega C}\sqrt{2\omega L}} + \frac{1}{2\hbar\omega L}\widehat{p}^2 \\ &= \frac{1}{2\hbar\omega C}\widehat{q}^2 + \frac{1}{2\omega\sqrt{LC}} + \frac{1}{2\hbar\omega L}\widehat{p}^2 \ ,\end{aligned} \tag{48}$$

where we have used (42). Similarly,

$$\begin{aligned}\widehat{a}^\dagger\widehat{a} &= \left(i\frac{1}{\sqrt{2\hbar\omega C}}\widehat{q} + \frac{1}{\sqrt{2\hbar\omega L}}\widehat{p}\right)\left(-i\frac{1}{\sqrt{2\hbar\omega C}}\widehat{q} + \frac{1}{\sqrt{2\hbar\omega L}}\widehat{p}\right) \\ &= \frac{1}{2\hbar\omega C}\widehat{q}^2 + i\frac{1}{\sqrt{2\hbar\omega C}\sqrt{2\hbar\omega L}}\left(\widehat{q}\widehat{p}-\widehat{p}\widehat{q}\right) + \frac{1}{2\hbar\omega L}\widehat{p}^2 \\ &= \frac{1}{2\hbar\omega C}\widehat{q}^2 - i\frac{1}{\sqrt{2\hbar\omega C}\sqrt{2\hbar\omega L}}\left(\widehat{p}\widehat{q}-\widehat{q}\widehat{p}\right) + \frac{1}{2\hbar\omega L}\widehat{p}^2 \\ &= \frac{1}{2\hbar\omega C}\widehat{q}^2 - i\frac{1}{\sqrt{2\hbar\omega C}\sqrt{2\hbar\omega L}}\left(-i\hbar\right) + \frac{1}{2\hbar\omega L}\widehat{p}^2 \\ &= \frac{1}{2\hbar\omega C}\widehat{q}^2 - \frac{1}{\sqrt{2\omega C}\sqrt{2\omega L}} + \frac{1}{2\hbar\omega L}\widehat{p}^2 \\ &= \frac{1}{2\hbar\omega C}\widehat{q}^2 - \frac{1}{2\omega\sqrt{LC}} + \frac{1}{2\hbar\omega L}\widehat{p}^2 \ .\end{aligned} \tag{49}$$

Hence we find that

$$[\widehat{a}, \widehat{a}^\dagger] \equiv \widehat{a}\widehat{a}^\dagger - \widehat{a}^\dagger\widehat{a} = \frac{1}{\omega\sqrt{LC}} \, .$$

By choosing $\omega = 1/\sqrt{LC}$ (in analogy to the classical counterpart), we have

$$[\widehat{a}, \widehat{a}^\dagger] = 1 \, . \tag{50}$$

Using (48), we may write

$$\hbar\omega \, \widehat{a}\widehat{a}^\dagger = \frac{\widehat{q}^2}{2C} + \frac{\widehat{p}^2}{2L} + \frac{\hbar}{2\sqrt{LC}} = \frac{\widehat{q}^2}{2C} + \frac{\widehat{p}^2}{2L} + \frac{\hbar\omega}{2} \, ,$$

which lets us to represent the Hamiltonian as

$$\widehat{H} = \frac{\widehat{q}^2}{2C} + \frac{\widehat{p}^2}{2L} = \hbar\omega(\widehat{a}\widehat{a}^\dagger - \frac{1}{2}) \, . \tag{51}$$

Note that by (50), which gives $\widehat{a}\widehat{a}^\dagger = \widehat{a}^\dagger\widehat{a} + 1$, this can be also written as

$$\widehat{H} = \hbar\omega(\widehat{a}^\dagger\widehat{a} + \frac{1}{2}) \, .$$

In the exercises, we will see that this form lends itself to a physical interpretation [see p. 5 in the solution sheet].

Remark. Another way to compute $[\widehat{a}, \widehat{a}^\dagger]$ is to use the algebraic rules for commutators. For any operators $\widehat{A}$, $\widehat{B}$ and $\widehat{C}$, it holds that

$$\begin{aligned}
[\widehat{A}, \widehat{A}] &= 0 \,, \\
[\widehat{A}, \widehat{B}] &= -[\widehat{B}, \widehat{A}] \,, \\
[\widehat{A},\, \widehat{B} + \widehat{C}] &= [\widehat{A}, \widehat{B}] + [\widehat{A}, \widehat{C}] \,.
\end{aligned}$$

We have

$$\begin{aligned}
[\widehat{a}, \widehat{a}^\dagger] &= \left[-i\frac{1}{\sqrt{2\hbar\omega C}}\widehat{q} + \frac{1}{\sqrt{2\hbar\omega L}}\widehat{p},\; i\frac{1}{\sqrt{2\hbar\omega C}}\widehat{q} + \frac{1}{\sqrt{2\hbar\omega L}}\widehat{p} \right] \\
&= \left(-i\frac{1}{\sqrt{2\hbar\omega C}}\right)\left(i\frac{1}{\sqrt{2\hbar\omega C}}\right)[\widehat{q}, \widehat{q}] + \left(-i\frac{1}{\sqrt{2\hbar\omega C}}\right)\left(\frac{1}{\sqrt{2\hbar\omega L}}\right)[\widehat{q}, \widehat{p}] \\
&+ \left(\frac{1}{\sqrt{2\hbar\omega L}}\right)\left(i\frac{1}{\sqrt{2\hbar\omega C}}\right)[\widehat{p}, \widehat{q}] + \left(\frac{1}{\sqrt{2\hbar\omega L}}\right)\left(\frac{1}{\sqrt{2\hbar\omega L}}\right)[\widehat{p}, \widehat{p}] \\
&= \left(-i\frac{1}{\sqrt{2\hbar\omega C}}\right)\left(\frac{1}{\sqrt{2\hbar\omega L}}\right)[\widehat{q}, \widehat{p}] + \left(\frac{1}{\sqrt{2\hbar\omega L}}\right)\left(i\frac{1}{\sqrt{2\hbar\omega C}}\right)[\widehat{p}, \widehat{q}] \\
&= \left(i\frac{1}{\sqrt{2\hbar\omega C}}\right)\left(\frac{1}{\sqrt{2\hbar\omega L}}\right)[\widehat{p}, \widehat{q}] + \left(\frac{1}{\sqrt{2\hbar\omega L}}\right)\left(i\frac{1}{\sqrt{2\hbar\omega C}}\right)[\widehat{p}, \widehat{q}] \\
&= 2i\frac{1}{\sqrt{2\hbar\omega C}}\frac{1}{\sqrt{2\hbar\omega L}}[\widehat{p}, \widehat{q}] \\
&= i\frac{1}{\hbar\omega\sqrt{LC}}[\widehat{p}, \widehat{q}] \\
&= i\frac{1}{\hbar\omega\sqrt{LC}}(-i\hbar) \\
&= \frac{1}{\omega\sqrt{LC}} \\
&= 1 \,,
\end{aligned}$$

again by choosing $\omega = 1/\sqrt{LC}$.

3.2 Energy considerations

Recall the time-dependent Schrödinger equation (45),

$$i\hbar\,\partial_t\,|\psi\rangle = \widehat{H}\,|\psi\rangle \ .$$

Let us consider a state $|\psi\rangle$ of the form

$$|\psi\rangle = e^{-i(E/\hbar)t}|\psi_E\rangle \ ,$$

where $|\psi_E\rangle$ does not depend on time (i.e. only the phase depends on time). For such a state,

$$i\hbar\,\partial_t|\psi\rangle = E|\psi\rangle \ .$$

By inserting this into the time-dependent Schrödinger equation and discarding a common scalar factor of $e^{-i(E/\hbar)t}$,

$$\widehat{H}\,|\psi_E\rangle = E\,|\psi_E\rangle \ , \tag{52}$$

which is known as the steady-state Schrödinger equation. The state $|\psi_E\rangle$ is the eigenstate corresponding to the eigenenergy E.

This allows us to think of a general state $|\psi\rangle$ in terms of a spectral decomposition:

$$|\psi\rangle = \sum_k c_k(t)|\psi_k\rangle \ ,$$

where we require the normalization (at each fixed t)

$$\sum_k |c_k(t)|^2 = 1 \ .$$

We observe that the coefficients $c_k(t)$ are the projections of the general state $|\psi\rangle$ onto each eigenstate $|\psi_m\rangle$:

$$\langle\psi_m|\psi\rangle = \sum_k c_k(t)\langle\psi_m|\psi_k\rangle = \sum_k c_k(t)\,\delta_{mk} = c_m(t) \ .$$

On the other hand, because the coefficients $c_k(t)$ are the only time-dependent part of the spectral decomposition, and for each eigenstate $|\psi_k\rangle$ only the phase depends on time, we may write

$$c_k(t) = c_k(0)\,e^{-i(E_k/\hbar)t} \ ,$$

where $c_k(0)$ is a complex number. Then the normalization condition becomes

$$\begin{aligned}
1 &= \sum_k |c_k(t)|^2 \\
&= \sum_k \left|c_k(0)\,e^{-i(E_k/\hbar)t}\right|^2 \\
&= \sum_k \left[c_k(0)\,e^{-i(E_k/\hbar)t}\right]^* \cdot \left[c_k(0)\,e^{-i(E_k/\hbar)t}\right] \\
&= \sum_k c_k(0)^* c_k(0)\,e^{+i(E_k/\hbar)t}\,e^{-i(E_k/\hbar)t} \\
&= \sum_k c_k(0)^* c_k(0) \\
&= \sum_k |c_k(0)|^2 \ ,
\end{aligned}$$

so we need to require normalization only at $t = 0$. Observe also that

$$i\hbar\,\partial_t|\psi(t)\rangle = \widehat{H}|\psi(t)\rangle = \widehat{H}\left[\sum_k c_k(t)|\psi_k\rangle\right] = \sum_k c_k(t)\,\widehat{H}|\psi_k\rangle = \sum_k c_k(t)E_k|\psi_k\rangle \ .$$

In other words, knowing the eigenstates is sufficient in order to know all possible time evolutions of the system.

What is the physical interpretation of the operator $\widehat{a}$? For our system, if the state was $\widehat{a}\,|\psi_E\rangle$ instead of just $|\psi_E\rangle$, in the steady-state Schrödinger equation (52), we would have

$$\begin{aligned}
\widehat{H}\widehat{a}\,|\psi_E\rangle &= \hbar\omega(\widehat{a}\widehat{a}^\dagger - \frac{1}{2})\widehat{a}\,|\psi_E\rangle \\
&= \hbar\omega\widehat{a}\left(\widehat{a}^\dagger\widehat{a} - \frac{1}{2}\right)|\psi_E\rangle \\
&= \hbar\omega\widehat{a}\left[(\widehat{a}\widehat{a}^\dagger - 1) - \frac{1}{2}\right]|\psi_E\rangle \\
&= \hbar\omega\widehat{a}\left[(\widehat{a}\widehat{a}^\dagger - \frac{1}{2}) - 1\right]|\psi_E\rangle \\
&= \widehat{a}\left(\widehat{H} - \hbar\omega\right)|\psi_E\rangle \\
&= \widehat{a}\,(E - \hbar\omega)\,|\psi_E\rangle \\
&= (E - \hbar\omega)\,\widehat{a}\,|\psi_E\rangle \ . \qquad (53)
\end{aligned}$$

Here we have used the commutation relation (50) and the fact that E and $\hbar\omega$ are scalar constants. We observe that $\widehat{a}\,|\psi_E\rangle$ is also an eigenstate of $\widehat{H}$, corresponding to the eigenvalue $(E - \hbar\omega)$; the operator $\widehat{a}$ thus decreases the energy by $\hbar\omega$. It is called the *annihilation operator*. It annihilates one quantum of energy, which in our case of electromagnetic excitations, is a photon in the microwave range.

Because the energy is bounded from below, it follows that there exists a ground state $|0\rangle$, for which $\widehat{a}\,|0\rangle = 0$. In this state, the number of photons is zero, and it is called the *vacuum*. The energy level corresponding to the ground state, called the *zero-point energy*, is obtained from (50), (51) and (52):

$$\begin{aligned}
\widehat{H}\,|0\rangle &= \hbar\omega(\widehat{a}\widehat{a}^\dagger - \frac{1}{2})\,|0\rangle \\
&= \hbar\omega\left[(1 + \widehat{a}^\dagger\widehat{a}) - \frac{1}{2}\right]|0\rangle \\
&= \hbar\omega\left(\frac{1}{2} + \widehat{a}^\dagger\widehat{a}\right)|0\rangle \\
&= \frac{\hbar\omega}{2}\,|0\rangle + \hbar\omega\,\widehat{a}^\dagger\,\widehat{a}\,|0\rangle \\
&= \frac{\hbar\omega}{2}\,|0\rangle \ , \qquad (54)
\end{aligned}$$

giving the eigenvalue $\hbar\omega/2$.

Typically, the zero-point energy can be neglected. For example, in a qubit register, it results only in an overall phase factor, which is not measurable. However, there are some special physical setups in which the zero-point energy can be observable (keyword: *Casimir effect*).

Similarly to (53), it can be shown that the operator $\widehat{a}^\dagger$ increases the energy by $\hbar\omega$, creating a photon. We have

$$\widehat{H}\widehat{a}^\dagger\,|\psi_E\rangle = (E + \hbar\omega)\,\widehat{a}^\dagger\,|\psi_E\rangle \ . \qquad (55)$$

The operator $\widehat{a}^\dagger$ is called the *creation operator*. For example, the first excited state is $|1\rangle = \widehat{a}^\dagger\,|0\rangle$. The steady-state Schrödinger equation for the state $|1\rangle$ reads $\widehat{H}|1\rangle = (3/2)\hbar\omega|1\rangle$. Provided that we use the normalization $\langle 0|0\rangle = 1$, we see that the first excited state obtained this way is also normalized:

$$\langle 1|1\rangle = \langle 0|\widehat{a}\widehat{a}^\dagger|0\rangle = \langle 0|(\widehat{a}^\dagger\widehat{a} + 1)|0\rangle = \langle 0|\widehat{a}^\dagger\widehat{a}|0\rangle + \langle 0|1|0\rangle = 0 + 1 = 1 \ .$$

Note here that $\widehat{a}\,|0\rangle = 0$. Furthermore, the state $|1\rangle$ is seen to be orthogonal to state $|0\rangle$:

$$\langle 1|0\rangle \equiv (|1\rangle)^\dagger\,|0\rangle = \left(\widehat{a}^\dagger|0\rangle\right)^\dagger\,|0\rangle = \langle 0|\widehat{a}|0\rangle = 0 \ .$$

The quantized superconducting LC circuit just discussed is a quantum-mechanical harmonic oscillator. In such a system, using a resonant drive at energy $\hbar\omega$ it is difficult to excite just the transition $|0\rangle \to |1\rangle$, because the spacing of the energy levels is $\hbar\omega$ for any transition $|n\rangle \to |n+1\rangle$. This is why in practical qubit circuits, a nonlinear component is needed: it changes the spacing of the energy levels, allowing one to prevent the system from entering energy levels other than those corresponding to $|0\rangle =: |g\rangle$ (ground state) and $|1\rangle =: |e\rangle$ (first excited state). Such nonlinearity can be introduced with the help of a Josephson junction, which we discuss next.

Exercises

1. Show that if the state $|n\rangle$ is normalized, i.e. $\langle n|n\rangle = 1$, then in order for the state $|n+1\rangle$ to also be normalized, the operator $\widehat{a}^\dagger$ must have the property

$$\widehat{a}^\dagger |n\rangle = \sqrt{n+1}\ |n+1\rangle\ .$$

   Hint:

$$\widehat{a}\widehat{a}^\dagger - \widehat{a}^\dagger\widehat{a} = 1\ , \quad |1\rangle = \widehat{a}^\dagger |0\rangle\ , \quad \widehat{a}|0\rangle = 0\ .$$

2. Show that the operator $\widehat{a}^\dagger$ can be represented as

$$\widehat{a}^\dagger = \sum_{m=0}^{\infty} \sqrt{m+1}\ |m+1\rangle\langle m|\ .$$

3. Let

$$|n\rangle = \widehat{A}\,|0\rangle\ .$$

   Determine $\widehat{A}$ in terms of $\widehat{a}^\dagger$.

   Hint: represent as $\widehat{A} = \alpha\left(\widehat{a}^\dagger\right)^n$, and find the value of the scalar α.

4 Quantum electronic components for qubits

Before we move on to discuss qubits, let us first look at the Josephson junction, which plays an important part in the construction of qubits. We also make some brief observations on capacitors and coils in a quantum setting.

4.1 The Josephson junction

The Josephson junction is shown schematically in Figure 3. There are two Josephson effects, one that occurs for alternating current (AC) and one that occurs for direct current (DC). For the AC Josephson effect with instantaneous voltage U, it holds that

$$\dot{\varphi} = \frac{2eU}{\hbar}\ , \tag{56}$$

where e is the elementary charge ($e = 1.602176565(35) \times 10^{-19}$ C). The factor of $2e$ comes from the fact that the charge of a Cooper pair is $-2e$, and Cooper pairs transport charge in a superconductor. The quantity φ is the phase of the superconducting ordering parameter describing the Cooper pairs.

For the DC Josephson effect,

$$I_\mathrm{s} = I_0 \sin\varphi\ , \tag{57}$$

where I_s is the supercurrent flowing through the Josephson junction and I_0 is the so-called critical current that is a characteristic property of a junction and can be controlled in its fabrication.

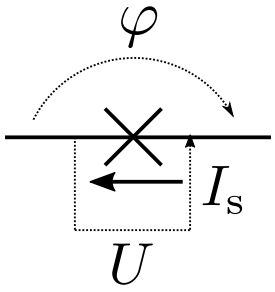

Figure 3: Circuit diagram for a Josephson junction.

Using (56) and (57), the energy related to supercurrent flowing through a Josephson junction can be written as

$$\begin{aligned} H_{\mathrm{J}} &= \int P \,\mathrm{d}t \\ &= \int U I_{\mathrm{s}} \,\mathrm{d}t \\ &= \int \left(\frac{\hbar}{2e} \dot{\varphi} \right) \left(I_0 \sin \varphi \right) \,\mathrm{d}t \\ &= \frac{\hbar I_0}{2e} \int \dot{\varphi} \sin \varphi \,\mathrm{d}t \\ &= -\frac{\hbar I_0}{2e} \int \left(\frac{\partial}{\partial t} \cos \varphi \right) \,\mathrm{d}t \\ &= -\frac{\hbar I_0}{2e} \cos \varphi \; + \; \text{const.} \\ &=: -E_{\mathrm{J}} \cos \varphi \; + \; \text{const.} \end{aligned} \tag{58}$$

The coefficient E_{J} is called the Josephson energy. The constant of integration can be taken as zero, as it represents an arbitrary reference level for the energy.

Remark. Recall that the phase factor in the steady-state Schrödinger equation is

$$e^{-i(E/\hbar)t} =: e^{i\varphi} \,.$$

Considering the energy difference $E = -2eU$ across the junction (the minus sign comes from the negative charge of the Cooper pair $-2e$), we obtain the corresponding phase difference φ as

$$\varphi = \frac{2eUt}{\hbar} \,.$$

Equation (56) follows by differentiating both sides with respect to time.

The derivation of equation (57) is much more involved and is omitted here.

From the AC and DC Josephson relations (56) and (57) we see that supercurrent can flow without voltage, although the AC current will then be zero.

4.2 Capacitors and coils in a quantum setting

Let us next discuss some topics that will be needed for circuit quantum electrodynamics.

The electrostatic energy stored in a capacitor (the charging energy) can be written as [recall equation (20)]

$$H_{\mathrm{C}} = \frac{Q^2}{2C} = \frac{2e^2}{C} n^2 \,, \tag{59}$$

where we have expressed the charge Q as

$$Q = -2ne \,, \tag{60}$$

i.e. as an integer multiple of the Cooper pair charge $-2e$. This quantization of charge occurs on islands, i.e. in wires connecting e.g. two Josephson junctions or a junction and a capacitor.

For a coil, as was previously discussed in the classical setting,

$$LI = \Phi \quad \Rightarrow \quad U = L\dot{I} = \dot{\Phi} \,.$$

Thus by (56),

$$\dot{\varphi} = \frac{2eU}{\hbar} = \frac{2e\dot{\Phi}}{\hbar} \,, \tag{61}$$

whence the phase φ is obtained as

$$\varphi = \frac{2e\Phi}{\hbar} =: 2\pi \frac{\Phi}{\Phi_0} \tag{62}$$

(plus an additive constant). In (62), we have defined the quantum of flux as

$$\Phi_0 := \frac{h}{2e} \,. \tag{63}$$

Its numerical value is $\Phi_0 = 2.067833903 \cdot 10^{-15}$ Wb. Note the use of h instead of $\hbar$.

The kinetic energy related to the current flowing in the coil (inductive energy) is [recall equation (24)]

$$H_\mathrm{L} = \frac{\Phi^2}{2L} = \frac{\Phi_0^2 \varphi^2}{8\pi^2 L} \,. \tag{64}$$

where we have used (62) to represent Φ.

5 The qubit

We first treat a single qubit classically. We then apply quantization, and finally discuss a simple basic qubit type, the charge qubit.

5.1 Classical treatment

A schematic of a simple superconducting qubit is shown in Figure 4. We choose the free coordinate as

$$q = \varphi \,.$$

From the viewpoint of the phase variable φ, potential energy stored in the Josephson junction is

$$V = -E_\mathrm{J} \cos\varphi \,, \tag{65}$$

and the kinetic energy (i.e. energy related to $\dot{\varphi}$) of the system is

$$T = \frac{Q_\mathrm{g}^2}{2C_\mathrm{g}} + \frac{Q_\mathrm{J}^2}{2C_\mathrm{J}} + U_\mathrm{g} Q_\mathrm{g} \,. \tag{66}$$

In (66), the first two terms account for the capacitors, and the last term accounts for the kinetic energy contribution of the power supply. Note the sign of the last term: if Q_g increases, the voltage source chrages up in agreement with Figure 4. In an actual circuit, the capacitance C_J is part of the physical realization of the Josephson junction; this capacitance and the ideal junction connected in parallel thus model one physical component.

From the circuit diagram, we read off the relations

$$Q_\mathrm{g} = C_\mathrm{g}(U - U_\mathrm{g}) \,, \tag{67}$$

$$Q_\mathrm{J} = C_\mathrm{J} U \,, \tag{68}$$

$$Q_\mathrm{g} + Q_\mathrm{J} = -2ne \,, \tag{69}$$

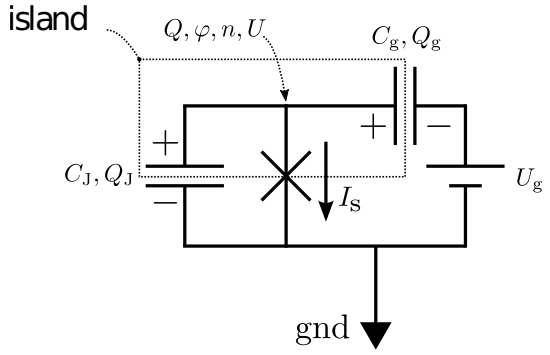

Figure 4: Circuit diagram of a simple superconducting qubit.

where n is the number of Cooper pairs on the island (refer to Figure 4). In this classical picture, n is a non-negative integer, but in the quantum picture to follow later, superposition states such as $(1/\sqrt{2})(|0\rangle + |1\rangle)$ will also appear.

Let us determine how n relates to our free coordinate φ. Combining equations (67)–(69),

$$\begin{aligned} -2ne &= (C_\mathrm{g} + C_\mathrm{J})U - C_\mathrm{g}U_\mathrm{g} \\ &=: C_\Sigma U - C_\mathrm{g}U_\mathrm{g} \,, \end{aligned} \tag{70}$$

where we denote the sum of the capacitances by

$$C_\Sigma := C_\mathrm{g} + C_\mathrm{J} \,. \tag{71}$$

Equation (70) yields

$$n = -\frac{C_\Sigma U}{2e} + \frac{C_\mathrm{g}U_\mathrm{g}}{2e} \,, \tag{72}$$

Using (56), the voltage U can be expressed as

$$U = \frac{\hbar}{2e}\dot{\varphi} \,. \tag{73}$$

Inserting (73) into (72) gives

$$\begin{aligned} n &= -\frac{\hbar C_\Sigma}{4e^2}\dot{\varphi} + \frac{C_\mathrm{g}U_\mathrm{g}}{2e} \\ &=: -\frac{\hbar C_\Sigma}{4e^2}\dot{\varphi} + n_\mathrm{g} \,, \end{aligned} \tag{74}$$

where we have defined the gate charge

$$n_\mathrm{g} := \frac{C_\mathrm{g}U_\mathrm{g}}{2e} \,. \tag{75}$$

The inverse relation, representing $\dot{\varphi}$ in terms of the number of Cooper pairs n, is

$$\dot{\varphi} = \frac{4e^2}{\hbar C_\Sigma}\left(n_\mathrm{g} - n\right) \,. \tag{76}$$

Using (67), (68), (71), and (73), the kinetic energy can be written as

$$\begin{aligned} T &= \frac{Q_\mathrm{g}^2}{2C_\mathrm{g}} + \frac{Q_\mathrm{J}^2}{2C_\mathrm{J}} + U_\mathrm{g}Q_\mathrm{g} \\ &= \frac{1}{2}C_\mathrm{g}(U - U_\mathrm{g})^2 + \frac{1}{2}C_J U^2 + U_\mathrm{g}C_\mathrm{g}(U - U_\mathrm{g}) \\ &= \frac{1}{2}C_\mathrm{g}U^2 - C_\mathrm{g}UU_\mathrm{g} + \frac{1}{2}C_\mathrm{g}U_\mathrm{g}^2 + \frac{1}{2}C_\mathrm{J}U^2 + U_\mathrm{g}C_\mathrm{g}U - U_\mathrm{g}C_\mathrm{g}U_\mathrm{g} \\ &= \frac{1}{2}(C_\mathrm{g} + C_\mathrm{J})U^2 - \frac{1}{2}C_\mathrm{g}U_\mathrm{g}^2 \\ &= \frac{1}{2}C_\Sigma U^2 - \frac{1}{2}C_\mathrm{g}U_\mathrm{g}^2 \\ &= \frac{1}{2}C_\Sigma\left(\dot{\varphi}\frac{\hbar}{2e}\right)^2 - \frac{1}{2}C_\mathrm{g}U_\mathrm{g}^2 \\ &= \frac{\hbar^2}{8e^2}C_\Sigma\dot{\varphi}^2 - \frac{1}{2}C_\mathrm{g}U_\mathrm{g}^2 \,. \end{aligned} \tag{77}$$

The last form expresses the kinetic energy in terms of (strictly speaking, the time derivative of) our free coordinate φ.

Using (65) and (66), the Lagrangian is obtained as

$$L = T - V = \frac{\hbar^2}{8e^2}C_\Sigma\dot{\varphi}^2 - \frac{1}{2}C_\mathrm{g}U_\mathrm{g}^2 + E_\mathrm{J}\cos\varphi \,. \tag{78}$$

With $q = \varphi$, the generalized momentum coordinate becomes

$$p = \frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial \dot{\varphi}} = \frac{\hbar^2}{4e^2} C_\Sigma \dot{\varphi} \, . \tag{79}$$

Note that alternatively, by (56), in terms of the voltage U we may write

$$p = \frac{\hbar^2}{4e^2} C_\Sigma \left(\frac{2eU}{\hbar} \right) = \frac{\hbar C_\Sigma}{2e} U \, . \tag{80}$$

Inserting (76) into (79) gives another representation, now in terms of n:

$$p = \frac{\hbar^2}{4e^2} C_\Sigma \left[\frac{4e^2}{\hbar C_\Sigma} \left(n_\mathrm{g} - n \right) \right] = \hbar \left(n_\mathrm{g} - n \right) \, . \tag{81}$$

The Hamiltonian is, as usual,

$$H = \dot{q} p - L = \dot{q} p - T + V \, .$$

Observe that as far as the free coordinate φ is concerned,

$$\begin{aligned} T &= \frac{\hbar^2}{8e^2} C_\Sigma \dot{\varphi}^2 - \frac{1}{2} C_\mathrm{g} U_\mathrm{g}^2 \\ &= \frac{\hbar^2}{8e^2} C_\Sigma \dot{\varphi}^2 + \text{const.} \end{aligned}$$

Also, we have

$$\dot{q} p = \dot{\varphi} \, \frac{\hbar^2}{4e^2} C_\Sigma \dot{\varphi} = \frac{\hbar^2}{4e^2} C_\Sigma \dot{\varphi}^2 \, ,$$

so it follows that

$$\begin{aligned} \dot{q} p - T &= \frac{\hbar^2}{4e^2} C_\Sigma \dot{\varphi}^2 - \left(\frac{\hbar^2}{8e^2} C_\Sigma \dot{\varphi}^2 - \frac{1}{2} C_\mathrm{g} U_\mathrm{g}^2 \right) \\ &= + \frac{\hbar^2}{8e^2} C_\Sigma \dot{\varphi}^2 + \frac{1}{2} C_\mathrm{g} U_\mathrm{g}^2 \\ &= \frac{\hbar^2}{8e^2} C_\Sigma \dot{\varphi}^2 + \text{const.} \\ &= T + \text{const.} \end{aligned}$$

We may thus write the Hamiltonian as

$$H = \dot{q} p - T + V = T + V + \text{const.} \, ,$$

and finally discard the constant, since it corresponds to an arbitrary offset of the energy and cannot be physically distinguished.

Thus, particularly for our system, we have (up to an additive constant)

$$\begin{aligned} H &= T + V \\ &= \frac{\hbar^2}{8e^2} C_\Sigma \dot{\varphi}^2 - E_\mathrm{J} \cos \varphi \, . \end{aligned} \tag{82}$$

Using (76) to eliminate $\dot{\varphi}$ in favor of n, we obtain

$$\begin{aligned} H &= \frac{\hbar^2}{8e^2} C_\Sigma \left[\frac{4e^2}{\hbar C_\Sigma} \left(n_\mathrm{g} - n \right) \right]^2 - E_\mathrm{J} \cos \varphi \\ &= \frac{\hbar^2}{8e^2} C_\Sigma \frac{16 e^4}{\hbar^2 C_\Sigma^2} \left(n_\mathrm{g} - n \right)^2 - E_\mathrm{J} \cos \varphi \\ &= \frac{2e^2}{C_\Sigma} \left(n_\mathrm{g} - n \right)^2 - E_\mathrm{J} \cos \varphi \\ &=: E_\mathrm{C} \left(n_\mathrm{g} - n \right)^2 - E_\mathrm{J} \cos \varphi \, , \end{aligned} \tag{83}$$

where we have defined the charging energy as

$$E_{\mathrm{C}} := \frac{2e^2}{C_\Sigma} \,. \tag{84}$$

5.2 Quantization

As discussed above, in the quantum-mechanical treatment, we replace the classical generalized coordinates by their operator counterparts as

$$q \to \widehat{q} = \widehat{\varphi} \,, \tag{85}$$
$$p \to \widehat{p} = \hbar(n_{\rm g} - \widehat{n}) \,. \tag{86}$$

For the generalized momentum variable p, we have used the representation in terms of the number of Cooper pairs n, as given by equation (81). The gate charge $n_{\rm g}$ is defined by (75).

For (85) and (86), the standard commutation relation (42) yields

$$-i\hbar = [\widehat{p}, \widehat{q}] = [\,\hbar(n_{\rm g} - \widehat{n}),\, \widehat{\varphi}\,] = -\hbar[\widehat{n}, \widehat{\varphi}] \,,$$

i.e.,

$$[\widehat{n}, \widehat{\varphi}] = i \,. \tag{87}$$

Analogously to (83), in terms of the operators $\widehat{n}$ and $\widehat{\varphi}$, the Hamiltonian of the quantized system is

$$\widehat{H} = E_{\rm C}(n_{\rm g} - \widehat{n})^2 - E_{\rm J}\cos\widehat{\varphi} \,, \tag{88}$$

where $n_{\rm g}$ is given by (75) and $E_{\rm C}$ by (84).

Remark. In the literature, often one encounters a different sign in the commutation relation (87). However, one can check that the sign in (87) is correct by calculating that $\langle \hat{I}_{\rm s}\rangle = 2e\partial_t\langle\hat{n}\rangle = -2ei\langle[\hat{n}, \hat{H}]\rangle/\hbar = 2e\langle E_{\rm J}\sin(\hat{\varphi})\rangle/\hbar = I_0\langle\sin(\hat{\varphi})\rangle$, which equivalent to the DC Josephson relation (57). Thus our definitions are consistent.

Exercises

1. Verify that in the charge basis $\{|m\rangle\}_\infty$, the operator $\widehat{n}$ becomes

$$\widehat{n} = \sum_{m=-\infty}^{\infty} m|m\rangle\langle m| \,.$$

   (The eigenstates $\{|m\rangle\}$ of the number operator are orthonormal.)

2. Let

$$\widehat{n}|m\rangle = m|m\rangle \,.$$

   Compute the phase-shifted state $e^{\pm i\widehat{\varphi}}|m\rangle$.

   Hint: Compute $\widehat{n}e^{\pm i\widehat{\varphi}}|m\rangle$, and use the commutation relation (87).

3. Write $\cos(\widehat{\varphi})$ in terms of the eigenstates of the charge operator.

5.3 The charge qubit

A simple basic qubit type is the *charge qubit*, for which

$$E_{\rm C} \gg E_{\rm J} \,, \quad n_{\rm g} \in (0, 1) \,. \tag{89}$$

The charge qubit approximation allows us to limit the consideration to two charge states, $|0\rangle$ and $|1\rangle$. For the charge qubit, the Hamiltonian (88) becomes

$$\widehat{H} = E_{\rm C}\left(|1\rangle\langle 1| - n_{\rm g}\widehat{I}\right)^2 - \frac{1}{2}E_{\rm J}\Big(|1\rangle\langle 0| + |0\rangle\langle 1|\Big) \,, \tag{90}$$

where $\widehat{I}$ is the identity operator,

$$\widehat{I} := \Big(|0\rangle\langle 0| + |1\rangle\langle 1|\Big) \,. \tag{91}$$

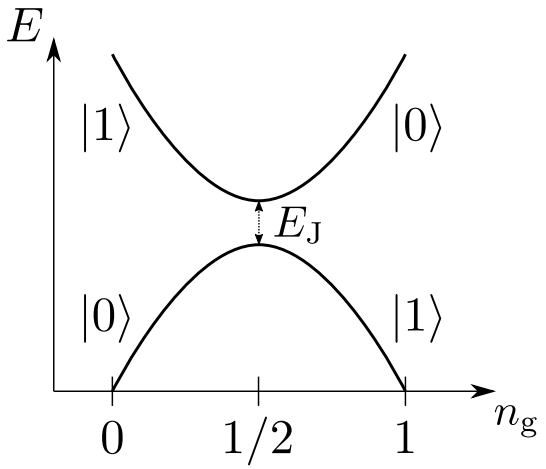

Figure 5: Sketch of the energies E of the charge qubit as a function of n_g. The symbols $|0\rangle$ and $|1\rangle$ represent the eigenstates of charge that are accurately reached at integer values of the gate charge. Note the energy gap E_J.

We have used the results of exercises 5.2.1 and 5.2.3 to represent $\widehat{n}$ and $\cos\widehat{\varphi}$ in terms of the charge basis, and retained only terms relating to the states $|0\rangle$ and $|1\rangle$. In the first parenthetical expression, the term $|0\rangle\langle 0|$ does not appear, because its coefficient is zero. Also observe that

$$\Big(|1\rangle\langle 1|\Big)^2 = |1\rangle\langle 1|\,|1\rangle\langle 1| = |1\rangle\langle 1|1\rangle\langle 1| = |1\rangle\langle 1|\,,$$

because the states are normalized ($\langle 1|1\rangle = 1$). Equation (90) can be written as

$$\begin{aligned}\widehat{H} &= E_\mathrm{C}(1-2n_\mathrm{g})|1\rangle\langle 1| + E_\mathrm{C}n_\mathrm{g}^2\widehat{I} - E_\mathrm{J}\widehat{\sigma}_x \\ &= E_\mathrm{C}(1-2n_\mathrm{g})\widehat{\sigma}_z - E_\mathrm{J}\widehat{\sigma}_x + \text{const.}\times\widehat{I}\,, \end{aligned} \tag{92}$$

where we have defined

$$\widehat{\sigma}_x := \frac{1}{2}\Big(|1\rangle\langle 0| + |0\rangle\langle 1|\Big)\,, \tag{93}$$

$$\widehat{\sigma}_z := \frac{1}{2}\Big(|1\rangle\langle 1| - |0\rangle\langle 0|\Big)\,. \tag{94}$$

In the charge qubit, gate operations are performed by either a piecewise constant $n_\mathrm{g}(t)$, or by $n_\mathrm{g}(t) = A(t)\cos\left[w_\mathrm{CS}t + \varphi(t)\right]$, where $w_\mathrm{CS} := (E_\mathrm{C} - E_\mathrm{S})/\hbar$. The latter type of control induces *Rabi oscillations*. The energies of the charge qubit are sketched in Figure 5.

Also other types of qubits have been found and in practice they work better than the charge qubit. One example is the *transmon qubit*, for which $E_J \gtrapprox E_C$.

Exercise

1. Solve the eigenvalues (energy levels) and eigenstates of the charge qubit.

   Hint: write as a 2×2 *matrix in the charge basis* $\{|0\rangle, |1\rangle\}$.

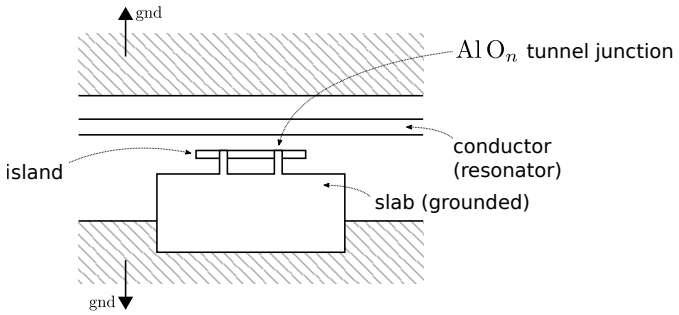

Figure 6: Qubit coupled to a superconducting resonator. Physical system.

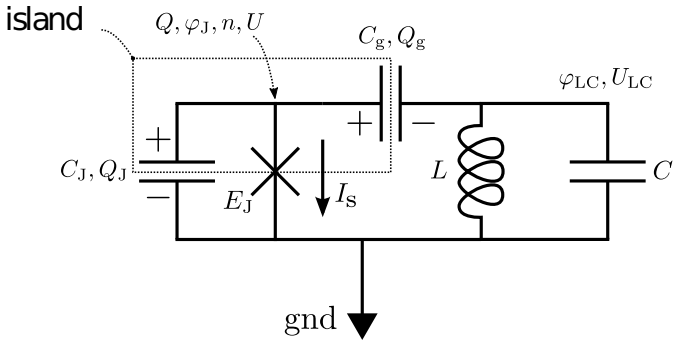

Figure 7: Qubit coupled to a superconducting LC oscillator.

6 Circuit quantum electrodynamics (cQED)

In this section, we consider one of the simplest practical one-qubit circuits: a qubit coupled to a tunable superconducting LC oscillator. Tuning the qubit into resonance with the oscillator (by adjusting the gate charge of Josephson energy) allows one to exchange quantum information between the two. Tuning the qubit back out of resonance decouples the qubit from the oscillator, allowing the implementation of single-qubit gates.

The physical system is sketched in Figure 6, and an idealized circuit diagram is shown in Figure 7. Two tunnel junctions with flux going through them can be modelled as a single junction that has a controllable Josephson energy E_{J}. The resonator is modelled as a superconducting LC oscillator.

Let us choose the free coordinates as $q_{\mathrm{J}} = \varphi_{\mathrm{J}}$ and $q_{\mathrm{LC}} = \varphi_{\mathrm{LC}}$. The potential energy of the system is given by

$$V = -E_{\mathrm{J}} \cos \varphi_{\mathrm{J}} + \frac{\Phi_0^2}{8\pi^2 L} \varphi_{\mathrm{LC}}^2 \,, \tag{95}$$

and the kinetic energy is [recall the voltage in the AC Josephson effect in terms of φ, equation (73)]

$$\begin{aligned} T &= \frac{1}{2} C_{\mathrm{J}} U_{\mathrm{J}}^2 + \frac{1}{2} C U_{\mathrm{LC}}^2 + \frac{1}{2} C_{\mathrm{g}} (U_{\mathrm{J}} - U_{\mathrm{LC}})^2 \\ &= \frac{\hbar^2 C_{\mathrm{J}}}{8e^2} \dot{\varphi}_{\mathrm{J}}^2 + \frac{\hbar^2 C}{8e^2} \dot{\varphi}_{\mathrm{LC}}^2 + \frac{\hbar^2 C_{\mathrm{g}}}{8e^2} \left(\dot{\varphi}_{\mathrm{J}} - \dot{\varphi}_{\mathrm{LC}} \right)^2 \,. \end{aligned} \tag{96}$$

The generalized momentum variables are

$$p_{\mathrm{J}} \equiv \frac{\partial L}{\partial \dot{\varphi}_{\mathrm{J}}} = \frac{\hbar^2 C_{\mathrm{J}}}{4e^2} \dot{\varphi}_{\mathrm{J}} + \frac{\hbar^2 C_{\mathrm{g}}}{4e^2} \left(\dot{\varphi}_{\mathrm{J}} - \dot{\varphi}_{\mathrm{LC}} \right) \,, \tag{97}$$

$$p_{\mathrm{LC}} \equiv \frac{\partial L}{\partial \dot{\varphi}_{\mathrm{LC}}} = \frac{\hbar^2 C}{4e^2} \dot{\varphi}_{\mathrm{LC}} + \frac{\hbar^2 C_{\mathrm{g}}}{4e^2} \left(\dot{\varphi}_{\mathrm{LC}} - \dot{\varphi}_{\mathrm{J}} \right) \,. \tag{98}$$

The classical Hamiltonian is

$$\begin{aligned} H &= \left(\sum_k \dot{q}_k p_k\right) - L \\ &= \dot{\varphi}_{\mathrm{J}} p_{\mathrm{J}} + \dot{\varphi}_{\mathrm{LC}} p_{\mathrm{LC}} - L \,. \end{aligned} \tag{99}$$

In the sum, $k = \{\mathrm{J}, \mathrm{LC}\}$. Let us define conjugate charges and fluxes as

$$\widehat{Q}_{\mathrm{J}} = \widehat{p}_{\mathrm{J}} \frac{2e}{\hbar} \,, \quad \widehat{Q}_{\mathrm{LC}} = \widehat{p}_{\mathrm{LC}} \frac{2e}{\hbar} \,, \tag{100}$$

$$\widehat{\Phi}_{\mathrm{J}} = \frac{\hbar}{2e} \widehat{\varphi}_{\mathrm{J}} \,, \quad \widehat{\Phi}_{\mathrm{LC}} = \frac{\hbar}{2e} \widehat{\varphi}_{\mathrm{LC}} \,. \tag{101}$$

Using (100) and (101), we have

$$\widehat{H} = \frac{\widehat{Q}_{\mathrm{J}}^2}{2\widetilde{C}_{\mathrm{J}}} + \frac{\widehat{Q}_{\mathrm{LC}}^2}{2\widetilde{C}_{\mathrm{LC}}} + \frac{\widehat{Q}_{\mathrm{J}} \widehat{Q}_{\mathrm{LC}}}{C_{\mathrm{int}}} + \frac{\widehat{\Phi}_{\mathrm{LC}}^2}{2L} - E_{\mathrm{J}} \cos\left(\frac{2e}{\hbar} \widehat{\Phi}_{\mathrm{J}}\right) \,, \tag{102}$$

where

$$\widetilde{C}_{\mathrm{J}} = \frac{CC_{\mathrm{g}} + C_{\mathrm{g}}C_{\mathrm{J}} + CC_{\mathrm{J}}}{C + C_{\mathrm{g}}} \,, \tag{103}$$

$$\widetilde{C}_{\mathrm{LC}} = \frac{CC_{\mathrm{g}} + C_{\mathrm{g}}C_{\mathrm{J}} + CC_{\mathrm{J}}}{C_{\mathrm{J}} + C_{\mathrm{g}}} \,, \tag{104}$$

$$C_{\mathrm{int}} = \frac{CC_{\mathrm{g}} + C_{\mathrm{g}}C_{\mathrm{J}} + CC_{\mathrm{J}}}{C_{\mathrm{g}}} \,. \tag{105}$$

For simplicity, consider the system in the transmon limit $E_{\mathrm{J}} \gtrapprox E_{\mathrm{C}}$. By Taylor approximation, we have

$$\begin{aligned} -E_{\mathrm{J}} \cos(\frac{2e}{\hbar} \widehat{\Phi}_{\mathrm{J}}) &\approx -E_{\mathrm{J}} + \frac{1}{2} E_{\mathrm{J}} \left(\frac{2e}{\hbar} \widehat{\Phi}_{\mathrm{J}}\right)^2 \\ &=: -E_{\mathrm{J}} + \frac{\widehat{\Phi}_{\mathrm{J}}^2}{2L_{\mathrm{J}}} \,, \end{aligned} \tag{106}$$

where we have defined the Josephson inductance as

$$L_{\mathrm{J}} := \frac{\hbar^2}{4e^2 E_{\mathrm{J}}} \,. \tag{107}$$

This allows us to represent the whole system as a harmonic oscillator. We must however keep in mind that the qubit has only two states, $|g\rangle$ (ground) and $|e\rangle$ (excited), differing by the energy $\hbar\omega_q = \sqrt{2E_C E_J}$.

We thus have the *Rabi Hamiltonian*

$$\begin{aligned} \widehat{H} = \frac{\hbar\omega_q}{2} \Big(|e\rangle\langle e| - |g\rangle\langle g|\Big) + \hbar\omega_{\mathrm{LC}} \left(\widehat{a}_{\mathrm{LC}}^\dagger \widehat{a}_{\mathrm{LC}} + \frac{1}{2}\right) \\ + \hbar g \Big(|e\rangle\langle g| + |g\rangle\langle e|\Big) \otimes \left(\widehat{a}_{\mathrm{LC}} + \widehat{a}_{\mathrm{LC}}^\dagger\right) \,, \end{aligned} \tag{108}$$

where g is the coupling coefficient. When $\hbar g \ll \hbar\omega_q \approx \hbar\omega_{\mathrm{LC}}$, the interaction term can be approximated such that

$$\begin{aligned} |e\rangle\langle g|\, \widehat{a}^\dagger &\approx 0 \,, \\ |g\rangle\langle e|\, \widehat{a} &\approx 0 \,, \end{aligned}$$

which leads to the *Jaynes–Cummings Hamiltonian*. This can be justified with the help of a rotating coordinate system. Physically, the Jaynes–Cummings Hamiltonian follows from the Rabi Hamiltonian by neglecting high-energy terms.

Exercise

1. Solve the three lowest eigenenergies of the Jaynes–Cummings Hamiltonian.

   Hint: write the Hamiltonian as a matrix in the subspace $\{|g,0\rangle, |e,0\rangle, |g,1\rangle\}$*, and diagonalize.*

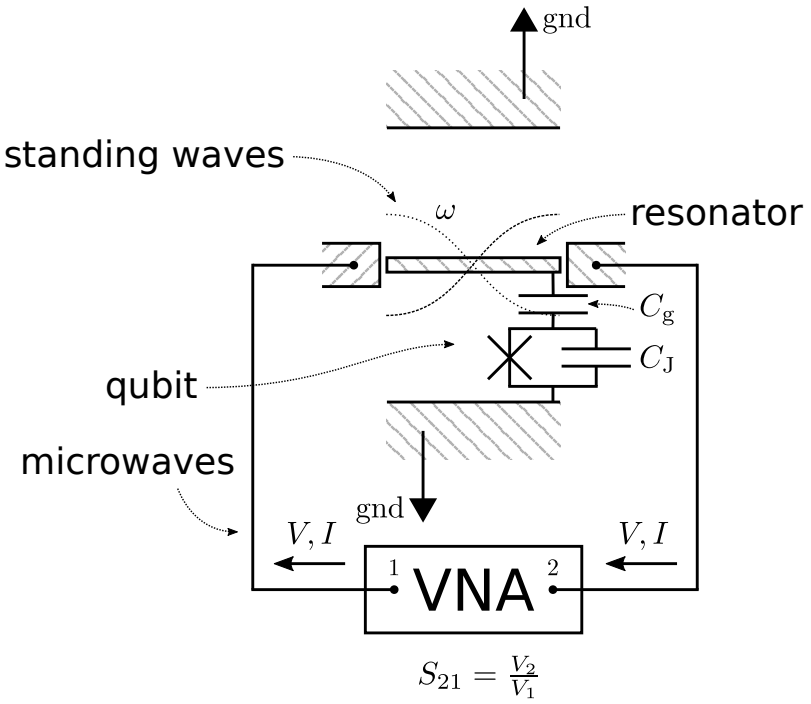

Figure 8: Measurement setup using a vector network analyzer for a qubit-resonator circuit.

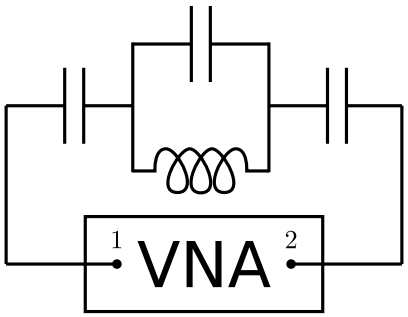

Figure 9: Schematic of the idealized measurement setup.

7 Measurement and control of the qubit; closing words

In a qubit-resonator circuit, such as that considered in the previous section, measurements are performed on the qubit by measuring the resonant frequency of the resonator.

Consider Figure 8, which shows a measurement setup for a qubit-resonator circuit. A vector network analyzer (labeled VNA in the Figure) sends microwaves from port 1. Some of the waves are reflected back, and some pass through the resonator, which in the figure is drawn as a distributed element. Standing waves, with angular frequency ω, form in the resonator. The voltage V and current I (indicated in the figure near the ports) are related to the microwaves. The VNA measures the response at port 2, and determines the voltage transmission coefficient $S_{21} = V_2/V_1$. As the frequency of the microwaves is varied, peaks in the response indicate a resonance.

To simplify the analysis, the measurement setup can be idealized as shown in Figure 9. The resonator is represented by a superconducting LC circuit having just one resonant angular frequency ω, and the qubit is omitted.

When the qubit and resonator frequencies are far from resonance, i.e.

$$\Delta := \omega_{\mathrm{LC}} - \omega_q \gg g \,, \tag{109}$$

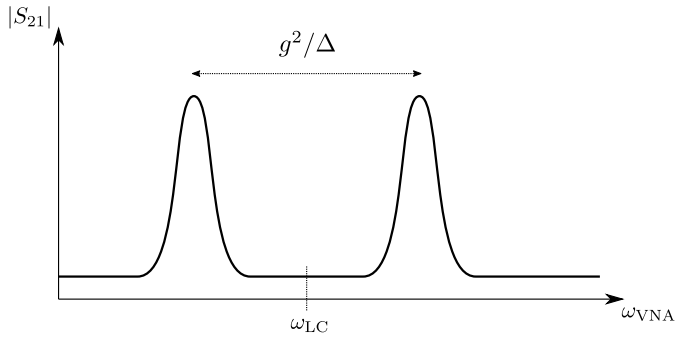

Figure 10: Sketch of the change in the resonator frequency as the qubit changes its state.

the Hamiltonian can be approximated as

$$\widehat{H} = \frac{\hbar}{2}\left(\omega_q - \frac{g^2}{\Delta}\right)\Big(|e\rangle\langle e| - |g\rangle\langle g|\Big) + \hbar\omega_{\mathrm{LC}}\left(\widehat{a}^\dagger_{\mathrm{LC}}\widehat{a}_{\mathrm{LC}} + \frac{1}{2}\right) + \hbar\frac{g^2}{2\Delta}\Big(|e\rangle\langle e| - |g\rangle\langle g|\Big)\widehat{a}^\dagger_{\mathrm{LC}}\widehat{a}_{\mathrm{LC}}\,. \tag{110}$$

Looking at the terms having the operator $\widehat{a}^\dagger_{\mathrm{LC}}\widehat{a}_{\mathrm{LC}}$ indicates that the resonator frequency changes by g^2/Δ when the qubit changes its state, as represented in Figure 10.

Initialization of the qubit is usually performed by waiting for it to return to its ground state, but also active initialization methods exist. (This is the topic of the talk on Friday in the quantum workshop.)

Qubits are controlled by driving them with a voltage input at a resonant frequency, inducing Rabi oscillations.

Practical quantum computing requires many-qubit systems. They have been investigated by e.g.:

- Fowler, A. G., Mariantoni, M., Martinis, J. M. & Cleland, A. N. *Surface codes: towards practical large-scale quantum computation.* Phys. Rev. A **86**, 032324 (2012) doi:10.1103/PhysRevA.86.032324
- Kelly, J. et al. *State preservation by repetitive error detection in a superconducting quantum circuit*. Nature **519**, 66–69. (2015) doi:10.1038/nature14270