$$\Delta = \frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_n^2}$$

A function satisfying $\Delta f = 0$ is called harmonic.

Proof g) (Do the rest by yourself)

$$\nabla \times \vec{F} = \left( \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

$$\nabla \cdot (\nabla \times \vec{F}) = \frac{\partial}{\partial x}\left( \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) + \frac{\partial}{\partial y}\left( \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) + \frac{\partial}{\partial z}\left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) = 0$$

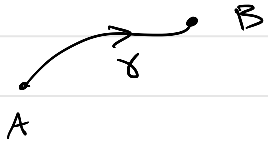
$\otimes$

Green's Theorem in the plane

This can be seen as a higher-dimensional version of the Fundamental Theorem of Calculus.

- Classical version $\int_a^b \frac{d}{dx} f(x)\, dx = f(b) - f(a)$.

- Version for line integrals in conservative fields



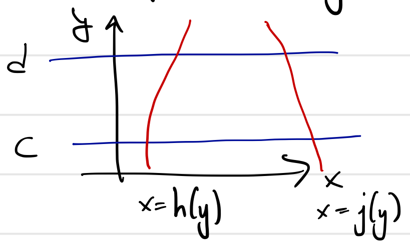

$$\int_\gamma \nabla\phi \cdot d\vec{r} = \phi(B) - \phi(A).$$

## Green's Theorem

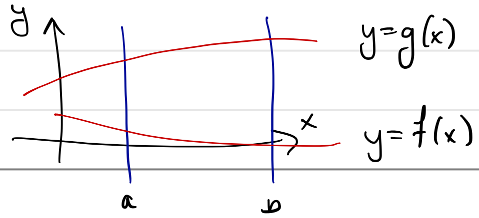
Let R be a regular, closed region in the plane whose boundary γ consists of one or more piecewise smooth curves. Also assume that γ is simple and positively oriented with respect to R. If $F(x,y) = (F_1, F_2)$ is a smooth vector field on R, then

$$\oint_\gamma F_1\,dx + F_2\,dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$$

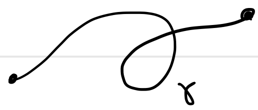
- Regular? : You can cut R into pieces that are x-simple and y-simple.
  - x-simple?




  - y-simple?

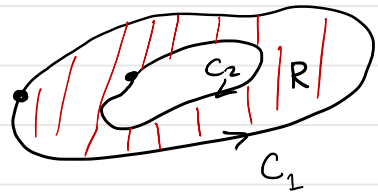


- $\gamma$ simple ?

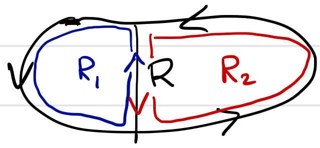



not simple

- Positively oriented ?




Proof:




If the theorem holds for $R_1$ & $R_2$ it holds for $R$.

Since $R$ is regular we get the theorem if we can show it for regions being both x-simple and y-simple.

We assume that

$$R = \{ (x,y) \in \mathbb{R}^2 ;\ a \leq x \leq b\ ,\ f(x) \leq y \leq g(x) \}$$
$$= \{ (x,y) \in \mathbb{R}^2 ;\ c \leq y \leq d\ ,\ h(y) \leq x \leq j(y) \}$$

$$\iint_R -\frac{\partial F_1}{\partial y}\, dx dy = -\int_a^b \left( \int_{f(x)}^{g(x)} \frac{\partial F_1}{\partial y}\, dy \right) dx =$$
$$= \int_a^b -F_1(x, g(x)) + F_1(x, f(x))\, dx$$

Now, $\oint_C F_1(x,y)\,dx = \int_a^b F_1(x,f(x)) - F_1(x,g(x))\,dx$

So $\oint_C F_1(x,y)\,dx = \iint_R -\frac{\partial F_1}{\partial y}\,dxdy$

Also $\oint_C F_2(x,y)\,dy = \iint_R \frac{\partial F_2}{\partial x}\,dxdy$

Why different signs?


$$\Longrightarrow \oint_\gamma F_1\,dx + F_2\,dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA. \quad \otimes$$

Ex Area bounded by a simple closed curve $\gamma$.
Try to find $(F_1, F_2)$ such that

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1.$$

$$\text{Area} = \iint_R 1\,dA = \oint_\gamma x\,dy = \oint_\gamma -y\,dx$$

$$= \frac{1}{2}\oint_\gamma x\,dy - y\,dx$$

Area of a disk with radius R.

$$\gamma(t) = (R\cos t, R\sin t)$$

$$\text{Area} = \oint_\gamma x\,dy = \int_0^{2\pi} R\cos t \cdot R\cos t\,dt =$$

$$= R^2\int_0^{2\pi}\cos^2 t\,dt = R^2\int_0^{2\pi}\frac{1+\cos 2t}{2}\,dt = \pi R^2$$

<u>Ex</u> Evaluate $I = \oint_\gamma (x-y^3)\,dx + (y^3+x^3)\,dy$

where $\gamma$ is the positively oriented boundary of the quarter disk $Q$ : $0 \le x^2+y^2 \le a^2$, $x\ge 0$, $y \ge 0$.


$$\vec{F} = (x-y^3,\ y^3+x^3)$$

$$I = \iint_Q \frac{\partial}{\partial x}(y^3+x^3) - \frac{\partial}{\partial y}(x-y^3)\,dA = \iint_Q 3x^2+3y^2\,dA$$

$$= \int_0^{\pi/2}\int_0^a 3r^2\cdot r\,dr\,d\theta = \frac{3\pi}{2}\int_0^a r^3\,dr = \frac{3\pi a^4}{8}.$$

Ex: Let $C$ be a positively oriented simple bounded curve in the plane bounding a regular region $R$ and not passing through the origin. Show that

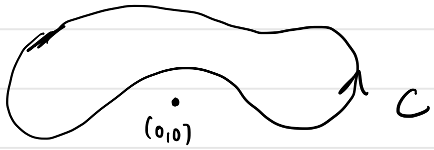
$$\oint_C \frac{-y\,dx + x\,dy}{x^2+y^2} = \begin{cases} 0 & \text{if } 0 \notin R \\ 2\pi & \text{if } 0 \in R \end{cases}$$

Solution: If $(x,y) \neq (0,0)$

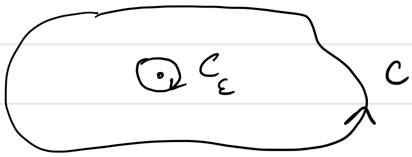
$$\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) =$$

$$= \frac{1}{x^2+y^2} - \frac{2x^2}{x^2+y^2} - \frac{2y^2}{x^2+y^2} + \frac{1}{x^2+y^2} = 0$$

Green's Theorem $\Longrightarrow \oint_C \frac{-y\,dx + x\,dy}{x^2+y^2} = 0$ if $0 \notin R$




Now assume that the origin is inside $R$

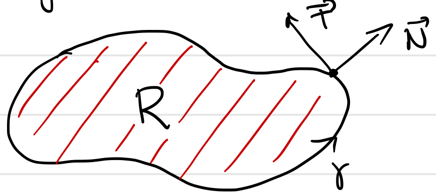



Put a small circle $C_\varepsilon$ around the origin

$$\oint_C \frac{-y\,dx + x\,dy}{x^2+y^2} = -\oint_{C_\varepsilon} \frac{-y\,dx + x\,dy}{x^2+y^2} = -(-2\pi) = 2\pi$$

Exercise

## Divergence Theorem in the plane




$\vec{T}$ = tangential unit vector field

$\vec{N}$ = unit normal outward (from R) vector field

Note that $\vec{T} = (T_1, T_2) \Rightarrow \vec{N} = (T_2, -T_1)$

Given $\vec{F} = (F_1, F_2)$ define $\vec{G} = (-F_2, F_1)$

We have $\vec{G} \cdot \vec{T} = -F_2 \cdot T_1 + F_1 \cdot T_2 = \vec{F} \cdot \vec{N}$

Now,
$$\iint_R \operatorname{div} \vec{F}\, dA = \iint_R \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}\, dA =$$
$$= \iint_R \frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y}\, dA \underset{\text{Green's Thm}}{=} \oint_\gamma \vec{G} \cdot d\vec{r} =$$
$$= \oint_\gamma \vec{G} \cdot \vec{T}\, ds = \underbrace{\oint_\gamma \vec{F} \cdot \vec{N}\, ds}_{\text{Flow out of } R.}$$

## Gauss's Theorem
### (Divergence Theorem in 3-space)

Let $D$ be a regular three-dimensional domain whose boundary $S$ is an oriented, closed surface with unit normal field $\vec{N}$ pointing out of $D$. If $\vec{F}$ is a smooth vector field defined on $D$, then

$$\iiint_D \operatorname{div} \vec{F}\, dV = \oiint_S \vec{F} \cdot \vec{N}\, dS$$