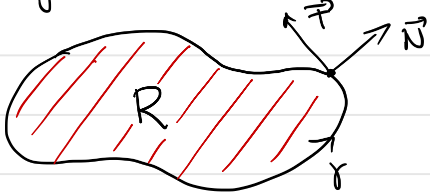
Divergence Theorem in the plane



$\vec{T}$ = tangential unit vector field

$\vec{N}$ = unit normal outward (from R) vector field

Note that $\vec{T} = (T_1, T_2) \Rightarrow \vec{N} = (T_2, -T_1)$

Given $\vec{F} = (F_1, F_2)$ define $\vec{G} = (-F_2, F_1)$

We have $\vec{G} \cdot \vec{T} = -F_2 \cdot T_1 + F_1 \cdot T_2 = \vec{F} \cdot \vec{N}$

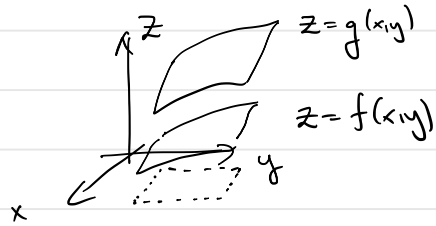
Now,
$$\iint_R \operatorname{div} \vec{F}\, dA = \iint_R \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}\, dA =$$
$$= \iint_R \frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y}\, dA \underset{\text{Green's Thm}}{\uparrow}{=} \oint_\gamma \vec{G} \cdot d\vec{r} =$$
$$= \oint_\gamma \vec{G} \cdot \vec{T}\, ds = \underbrace{\oint_\gamma \vec{F} \cdot \vec{N}\, ds}_{\text{Flow out of } R.}$$

Gauss's Theorem
(Divergence Theorem in 3-space)

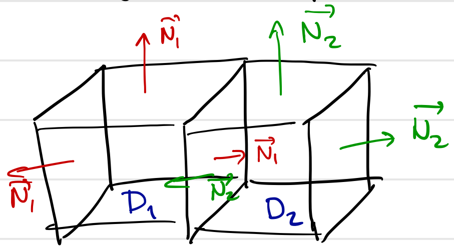
Let D be a regular three-dimensional domain whose boundary S is an oriented, closed surface with unit normal field $\vec{N}$ pointing out of D. If $\vec{F}$ is a smooth vector field defined on D, then
$$\iiint_D \operatorname{div} \vec{F}\, dV = \oiint_S \vec{F} \cdot \vec{N}\, dS$$

"Regular" = "x-simple, y-simple and z-simple

z − simple



"Proof": D regular if it can be cut into pieces that are x-simple, y-simple and z-simple.



$$\oint\!\!\oint_{S_1 \cup S^*} \vec{F}\cdot\vec{N}_1\, dS + \oint\!\!\oint_{S_2 \cup S^*} \vec{F}\cdot\vec{N}_2\, dS =$$

$$= \oint\!\!\oint_{S} \vec{F}\cdot\vec{N}\, dS \quad \text{since}$$

$$\iint_{S^*} \vec{F}\cdot\vec{N}_1\, dS = -\iint_{S^*} \vec{F}\cdot\vec{N}_2\, dS$$

Also $\displaystyle \iiint_{D_1} \operatorname{div} \vec{F}\, dV + \iiint_{D_2} \operatorname{div} \vec{F}\, dV = \iiint_{D} \operatorname{div} \vec{F}\, dV$

Assume D is x-simple, y-simple and z-simple

$$z\text{-simple} \Rightarrow D = \{(x,y,z) \in \mathbb{R}^3 ;\ (x,y) \in R,\ f(x,y) \le z \le g(x,y)\}$$

We look at a "third" of the vector field and the divergence.

$$\iiint_D \frac{\partial F_3}{\partial z}\, dV = \iint_R \left(\int_{f(x,y)}^{g(x,y)} \frac{\partial F_3}{\partial z}\, dz \right) dx\,dy =$$

$$= \iint_R F_3(x,y,g(x,y)) - F_3(x,y,f(x,y))\, dx\,dy$$

$$\oiint_S F_3(x,y,z)\,\vec{e_3} \cdot \vec{N}\, dS = \iint_{top} + \iint_{bottom} + \iint_{sides}$$

$$\vec{e_3} \cdot \vec{N} = 0 \quad \text{on sides}$$

$$\text{On top} \quad \vec{N}\, dS = \left(-\frac{\partial g}{\partial x}\vec{e_1} - \frac{\partial g}{\partial y}\vec{e_2} + \vec{e_3} \right) dx\,dy$$

$$\iint_{top} F_3\ \vec{e_3} \cdot \vec{N}\, dS = \iint_R F_3(x,y,g(x,y))\, dx\,dy$$

$$\iint_{bottom} F_3\,\vec{e_3} \cdot \vec{N}\, dS = -\iint_R F_3(x,y,f(x,y))\, dx\,dy$$

$\vec{N}$ points downwards

$$\Longrightarrow \iiint_D \frac{\partial F_3}{\partial z}\, dV = \oiint_S F_3(x,y,z)\,\vec{e_3} \cdot \vec{N}\, dS$$

Repeat and get

$$\iiint_D \operatorname{div} \vec{F}\, dV = \oiint_S \vec{F} \cdot \vec{N}\, dS$$

$\otimes$

<u>Ex</u> $F(x,y,z) = \left(bxy^2, bx^2y, (x^2+y^2)z^2\right)$

and let S be the closed bounding $x^2+y^2 \le a^2$ and $0 \le z \le b$. Find

$$\oiint_S \vec{F} \circ \vec{N}\, dS$$

<u>Solution:</u> $D = \{ x^2+y^2 \le a^2;\ 0 \le z \le b \}$

$$\oiint_S \vec{F} \circ \vec{N}\, dS = \iiint_D \operatorname{div} \vec{F}\, dV =$$

$$= \iiint_D by^2 + bx^2 + 2z(x^2+y^2)\, dV =$$

$$= \iiint_D (2z+b)(x^2+y^2)\, dV =$$

$$\underset{\substack{\text{Cylindrical} \\ \text{coordinates}}}{=} \int_0^b \int_0^{2\pi} \int_0^a (2z+b)\, r^2\, r\, dr\, d\theta\, dz =$$

$$= \frac{2\pi a^4}{4} \int_0^b 2z + b\, dz =$$

$$= \frac{\pi a^4}{2} \left[z^2 + bz \right]_0^b = \pi a^4 b^2$$

Ex Calculate $\oiint_S x^2+y^2 \, dS$ where S is the sphere $x^2+y^2+z^2=a^2$.

Solution: We use Gauss's Theorem

$\vec{N} = \frac{1}{a}(x,y,z)$. Now find $\vec{F}$ so that

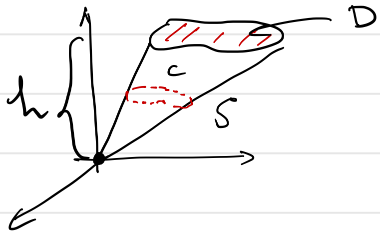
$\vec{F}\cdot\vec{N} = x^2+y^2$!

Choose $\vec{F} = (ax, ay, 0)$.

$$\oiint_S x^2+y^2 \, dS = \oiint_S \vec{F}\cdot\vec{N} \, dS = \iiint_{x^2+y^2+z^2 \le a^2} \operatorname{div} F \, dV =$$

$$= 2a \iiint_{x^2+y^2+z^2 \le a^2} 1 \, dV = 2a \frac{4\pi a^3}{3} = \frac{8\pi a^4}{3}$$

Ex Calculate the volume of a cone C with base area and height h.



area $D = A$

Solution: Use $F(x,y,z) = (x,y,z)$

$$\operatorname{div} F = 1+1+1 = 3$$

$$F\cdot\vec{N} = 0 \text{ on } S \qquad F\cdot\vec{N} = \underset{\underset{h}{\shortparallel}}{z} \text{ on } D$$

$$3V = \iiint_C \operatorname{div} F \, dV = \iint_D F\cdot\vec{N}\, dS$$

$$= h\iint_D 1\, dA = hA$$

$$\Longrightarrow \quad V = \frac{hA}{3}$$

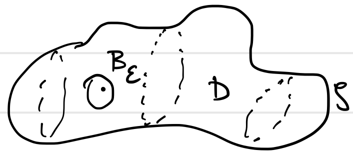
Ex Let S be the boundary surface of an arbitrary regular domain D in $\mathbb{R}^3$ that contains the origin in it's interior. Find

$$\oint\!\!\!\oint_S \vec{F}\cdot\vec{N}\, dS \quad \text{where } \vec{r} = (x,y,z),$$

$$\vec{F}(\vec{r}) = \frac{m\vec{r}}{|\vec{r}|^3} \quad \text{and } \vec{N} \text{ is the}$$

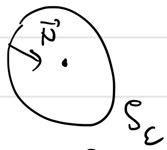
unit outward normal field on S.

Solution: $D_\varepsilon = D \setminus B_\varepsilon$



Check that $\operatorname{div} \vec{F} = 0$ on D_ε

$$\oiint_S \vec{F} \cdot \vec{N}\, dS = \iiint_{D_\varepsilon} \operatorname{div} \vec{F}\, dV - \oiint_{S_\varepsilon} \vec{F} \cdot \vec{N}\, dS$$



$$\oiint_{S_\varepsilon} \frac{m\vec{r}}{|\vec{r}|^3} \cdot \left(-\frac{\vec{r}}{|r|}\right) dS = \oiint_{S_\varepsilon} \frac{m}{|\vec{r}|^2}\, dS = -\frac{m}{\varepsilon^2} \oiint_{S_\varepsilon} dS =$$

$$= -\frac{m}{\varepsilon^2} \cdot 4\pi\varepsilon^2 = -4\pi m$$

$$\Longrightarrow \oiint_S \vec{F} \cdot \vec{N}\, dS = 4\pi m.$$

Stokes's Theorem

Recall that Green's Theorem is

$$\oint_\gamma \vec{F} \cdot d\vec{r} = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\,dy$$

where $\vec{F} = (F_1, F_2)$