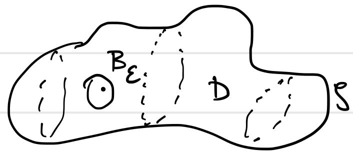
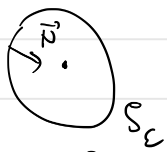
Solution: $D_\varepsilon = D \setminus B_\varepsilon$



Check that $\operatorname{div} \vec{F} = 0$ on D_ε

$$\oiint_S \vec{F} \cdot \vec{N}\, dS = \iiint_{D_\varepsilon} \operatorname{div} \vec{F}\, dV - \oiint_{S_\varepsilon} \vec{F} \cdot \vec{N}\, dS$$



$$\oiint_{S_\varepsilon} \frac{m\vec{r}}{|\vec{r}|^3} \cdot \left(-\frac{\vec{r}}{|r|}\right) dS = \oiint_{S_\varepsilon} \frac{m}{|\vec{r}|^2}\, dS = -\frac{m}{\varepsilon^2} \oiint_{S_\varepsilon} dS =$$

$$= -\frac{m}{\varepsilon^2} \cdot 4\pi\varepsilon^2 = -4\pi m$$

$$\Longrightarrow \oiint_S \vec{F} \cdot \vec{N}\, dS = 4\pi m.$$

Stokes's Theorem

Recall that Green's Theorem is

$$\oint_\gamma \vec{F} \cdot d\vec{r} = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\,dy$$

where $\vec{F} = (F_1, F_2)$

Curl $\vec{F}$ is defined only for $\vec{F} = (F_1, F_2, F_3)$

We can define Curl $\vec{F}$ as a <u>function</u> when $\vec{F} = (F_1(x,y), F_2(x,y))$ in the following way. Given $\vec{F} = (F_1, F_2)$ define $\vec{G} = (F_1, F_2, 0)$.

Then

$$\text{Curl } \vec{G} = \begin{vmatrix} \vec{e_1} & \vec{e_2} & \vec{e_3} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & 0 \end{vmatrix} =$$

$$= \begin{vmatrix} \partial/\partial y & \partial/\partial z \\ F_2 & 0 \end{vmatrix} \vec{e_1} - \begin{vmatrix} \partial/\partial x & \partial/\partial z \\ F_1 & 0 \end{vmatrix} \vec{e_2} + \begin{vmatrix} \partial/\partial x & \partial/\partial y \\ F_1 & F_2 \end{vmatrix} \vec{e_3}$$

$$= \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \vec{e_3} \underset{\text{definition}}{=} (\text{Curl } \vec{F}) \vec{e_3}$$

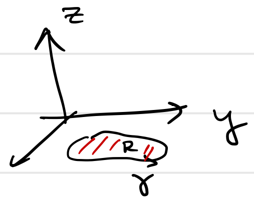
<u>Definition</u> If $\vec{F} = (F_1, F_2)$ then

$$\text{Curl } \vec{F} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

Green's Theorem can be written

$$\oint_\gamma \vec{F} \cdot d\vec{r} = \iint_R \text{Curl } \vec{F} \, dx\,dy$$

We can also formulate Green's Theorem in the following way:



R in xy-plane

γ in xy-plane

If $\vec{F} = (F_1(x,y), F_2(x,y), 0)$ then

$$\oint_\gamma \vec{F} \cdot d\vec{r} = \iint_R \text{Curl } \vec{F} \cdot \vec{e_3} \, dA$$

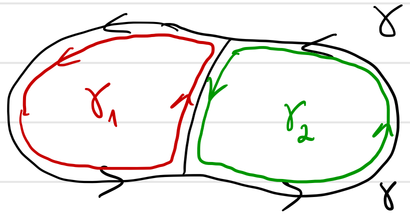
Stokes's Theorem generalizes this to non-planar surfaces.

Stokes's Theorem

Let S be a piecewise smooth, oriented surface in $\mathbb{R}^3$, having unit normal field $\vec{N}$ and boundary γ consisting of one or more piecewise smooth, closed curves with orientation inherited from S. If $\vec{F}$ is a smooth vector field defined on an open set containing S, then

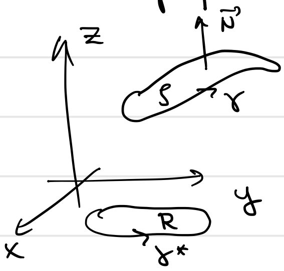
$$\oint_\gamma \vec{F}\cdot d\vec{r} = \iint_S \operatorname{Curl}\vec{F}\cdot\vec{N}\,dS$$

Proof: We cut the surface into pieces that project 1-1 onto a coordinate plane.



$$\oint_\gamma \vec{F}\cdot d\vec{r} =$$
$$= \oint_{\gamma_1} \vec{F}\cdot d\vec{r} + \oint_{\gamma_2} \vec{F}\cdot d\vec{r}$$

We assume that S projects 1-1 onto the xy-plane. We have



$z = g(x,y)$

$G(x,y,z) = z - g(x,y)$

$$\vec{N} = \frac{\nabla G}{|\nabla G|} \qquad dS = |\nabla G|\, dA$$

$$\iint_S \text{Curl}\,\vec{F}\cdot\vec{N}\,dS = \iint_R \text{Curl}\,\vec{F}\cdot\nabla G\,dA =$$

$$= \iint_R \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\left(-\frac{\partial g}{\partial x}\right) + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\left(-\frac{\partial g}{\partial y}\right) +$$

$$+ \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\cdot 1\,dA$$

Also $\oint_\gamma \vec{F}\cdot d\vec{r} = \oint_\gamma F_1\,dx + F_2\,dy + F_3\,dz =$

$$\left(z = g(x,y) \implies dz = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy\right)$$

$$\oint_{\gamma^*} F_1\,dx + F_2\,dy + F_3\left(\frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy\right) =$$

$$\oint_{\gamma^*}\left(F_1 + F_3\frac{\partial g}{\partial x}\right)dx + \left(F_2 + F_3\frac{\partial g}{\partial y}\right)dy = (*)$$

We apply Green's Theorem

$$(*) = \iint_R \frac{\partial}{\partial x}\left(F_2(x,y,g(x,y)) + F_3(x,y,g(x,y))\frac{\partial g}{\partial y}\right) -$$

$$-\frac{\partial}{\partial y}\left(F_1(x,y,g(x,y)) + F_3(x,y,g(x,y))\frac{\partial g}{\partial x}\right)\,dA =$$

$$= \iint_R \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial z}\cdot\frac{\partial g}{\partial x} + \frac{\partial F_3}{\partial x}\cdot\frac{\partial g}{\partial y} + \cancel{\frac{\partial F_3}{\partial z}\cdot\frac{\partial g}{\partial x}\cdot\frac{\partial g}{\partial y}} + \cancel{F_3\frac{\partial^2 g}{\partial x\partial y}}$$

$$-\left(\frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z}\cdot\frac{\partial g}{\partial y} + \frac{\partial F_3}{\partial y}\cdot\frac{\partial g}{\partial x} + \cancel{\frac{\partial F_3}{\partial z}\cdot\frac{\partial g}{\partial y}\cdot\frac{\partial g}{\partial x}} + \cancel{F_3\frac{\partial^2 g}{\partial y\partial x}}\right)dA$$

$$= \iint_R \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\left(-\frac{\partial g}{\partial x}\right) + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\left(-\frac{\partial g}{\partial y}\right) + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)dA$$

$\mathcal{Q}$

Recap: We have proven

Green's Theorem (in $\mathbb{R}^2$)

$$\oint_\gamma \vec{F} \cdot d\vec{r} = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA$$

$$\oint_\gamma \vec{F} \cdot d\vec{r} = \iint_R \operatorname{Curl} \vec{F} \cdot \vec{e_3} \, dA$$

Stokes's Theorem

$$\oint_\gamma \vec{F} \cdot d\vec{r} = \iint_S \operatorname{Curl} \vec{F} \cdot \vec{N} \, dS$$

$\operatorname{Curl} \vec{F}$ is used to calculate circulation around a curve

Divergence Theorem in the plane

$$\oint_\gamma \vec{F} \cdot \vec{N} \, ds = \iint_R \operatorname{div} \vec{F} \, dA$$

Gauss's Theorem

$$\oiint_S \vec{F} \cdot \vec{N} \, dS = \iiint_D \operatorname{div} \vec{F} \, dV$$

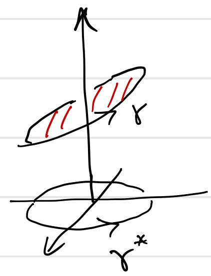
$\operatorname{div} \vec{F}$ is used to calculate flux / flow across a curve or surface

Really the same theorem (called Stokes's Theorem) for differential forms.

Ex Calculate $\oint_\gamma \vec{F}\cdot d\vec{r}$ where

$\vec{F}(x,y,z) = (-y^3, x^3, -z^3)$ and

γ is the curve of intersection of the cylinder $x^2+y^2=1$ and the plane $2x+2y+z=3$ oriented so it has a counterclockwise projection onto the xy-plane.



$$\oint_\gamma \vec{F}\cdot d\vec{r} = \iint_S \text{Curl}\,\vec{F}\cdot\vec{N}\,dS$$

$$\vec{N}\,dS = \frac{\nabla G}{|\nabla G|}|\nabla G|\,dx\,dy$$

$$= (2,2,1)\,dx\,dy$$

$$\text{Curl}\,\vec{F} = \begin{vmatrix} \vec{e_1} & \vec{e_2} & \vec{e_3} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y^3 & x^3 & -z^3 \end{vmatrix} = (0,0,3x^2+3y^2)$$

$$\oint_\gamma \vec{F}\cdot d\vec{r} = \iint_S \text{Curl}\,\vec{F}\cdot\vec{N}\,dS = \iint_R 3x^2+3y^2\,dx\,dy$$

$$= \int_0^{2\pi}\int_0^1 3r^2\cdot r\,dr\,d\theta = \left[\frac{3r^4}{4}\right]_0^1 2\pi = \frac{3\pi}{2}$$