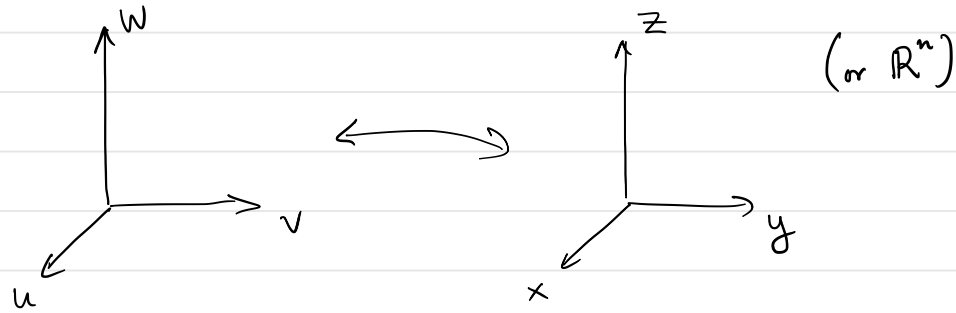
(Orthogonal) Curvilinear Coordinates



$$\begin{cases} x = x(u,v,w) \\ y = y(u,v,w) \\ z = z(u,v,w) \end{cases}$$

We consider maps that are locally 1-1 (and not necessarily injective (globally 1-1))

We aim to describe div, grad & Curl in (u,v,w). However first we need to be able to change coordinates in vector fields and for this we need the concept of local basis.

Coordinate Surfaces and Coordinate Curves

The image of $u=u_0$ ($v=v_0$, or $w=w_0$) in xyz-space is called a coordinate surface. Such surfaces has a parametrization via

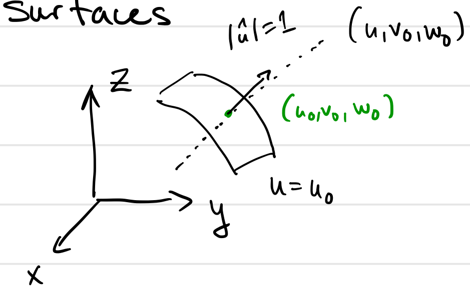
$$(v,w) \mapsto (x(u_0,v,w), y(u_0,v,w), z(u_0,v,w))$$

The intersection of coordinates surfaces are coordinate curves

$$w \longmapsto \left(x(u_0,v_0,w), y(u_0,v_0,w), z(u_0,v_0,w)\right)$$

for example

We call a curvilinear coordinate system orthogonal if the coordinate surfaces (curves) intersect at right angles at all points of intersection. In a orthogonal curvilinear coordinate system we can use coordinate curves to find normal vectors to coordinate surfaces



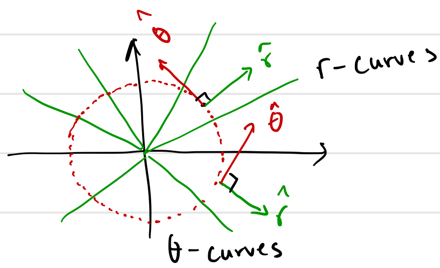
In a similar way we get $\hat{v}$ & $\hat{w}$

We get a local basis at (u_0, v_0, w_0)
$[\hat{u}, \hat{v}, \hat{w}]$

Ex Polar coordinates

$$(r,\theta) \longmapsto (r\cos\theta,\ r\sin\theta) \qquad 0<r<\infty$$

$$x = r\cos\theta \quad y = r\sin\theta \qquad \theta \in \mathbb{R}$$



$$\hat r = (\cos\theta, \sin\theta)$$

$$\hat\theta = \frac{1}{r}(-r\sin\theta, r\cos\theta)$$

$$= (-\sin\theta, \cos\theta)$$

$$\hat r \cdot \hat\theta = 0$$

Ex Express $F(x,y) = (-y,x)$ in polar coordinates using the local bases $[\hat r, \hat\theta]$

In general $F = (F_1, F_2) = F_1 \vec{e_1} + F_2 \vec{e_2}$

Note that $F_1 = F \cdot \vec{e_1}$ and $F_2 = F \cdot \vec{e_2}$

In the same way (since $\hat r \cdot \hat\theta = 0$) we have

$$F = \underbrace{(F\cdot\hat r)}_{F_r}\hat r + \underbrace{(F\cdot\hat\theta)}_{F_\theta}\hat\theta$$

$$F_r = F \cdot \hat{r} = (-r\sin\theta, r\cos\theta) \cdot (\cos\theta, \sin\theta) = 0$$

$$F_\theta = F \cdot \hat{\theta} = (-r\sin\theta, r\cos\theta) \cdot (-\sin\theta, \cos\theta) = r$$

$$\Longrightarrow \vec{F} = 0\,\hat{r} + r\,\hat{\theta} = r\,\hat{\theta}$$

<u>Ex</u> Cylindrical coordinates

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

Coordinate surface

$r = r_0$ cylinder with radius r_0

$\theta = \theta_0$ vertical half-planes radiating from x-axis

$z = z_0$ horizontal plane

$$\hat{r} = (\cos\theta, \sin\theta, 0) \qquad \hat{r}\cdot\hat{\theta} = \hat{r}\cdot\hat{z} = \hat{\theta}\cdot\hat{z} = 0$$

$$\hat{\theta} = (-\sin\theta, \cos\theta, 0)$$

$$\hat{z} = (0, 0, 1)$$

Ex Spherical coordinates

$$x = r\cos\theta\sin\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\phi$$

Coordinate surfaces

$r = r_0$ sphere with radius r_0

$\theta = \theta_0$ vertical half-planes radiating from the z-axis

$\phi = \phi_0$ cone with vertex at the origin.

It is easy to check that this is an orthogonal curvilinear coordinate system.

Scale factors and Differential Elements

The position vector in xyz-space

$$\vec{r}(u,v,w) = \big(x(u,v,w), y(u,v,w), z(u,v,w)\big)$$

We have $\frac{\partial \vec{r}}{\partial u} = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right)$ and $\frac{\partial \vec{r}}{\partial v}, \frac{\partial \vec{r}}{\partial w}$

The lengths of these vectors are called the scale factors of the coordinate system

$$h_u = \left|\frac{\partial \vec{r}}{\partial u}\right| \,,\quad h_v = \left|\frac{\partial \vec{r}}{\partial v}\right| \quad \text{and} \quad h_w = \left|\frac{\partial \vec{r}}{\partial w}\right|$$

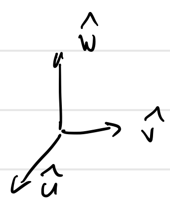
We assume that h_u, h_v and h_w all are non-zero and defines a right-handed coordinate system $\left[\frac{\partial \vec{r}}{\partial u}, \frac{\partial \vec{r}}{\partial v}, \frac{\partial \vec{r}}{\partial w}\right]$

For orthogonal systems one can use the scale factors to quickly calculate area elements and volume elements for the change of variables.

Notice that

$$\frac{\partial \vec{r}}{\partial u} = h_u \hat{u} \,,\quad \frac{\partial \vec{r}}{\partial v} = h_v \hat{v} \quad \text{and} \quad \frac{\partial \vec{r}}{\partial w} = h_w \hat{w}$$

Since $\hat{u}\cdot\hat{v} = \hat{u}\cdot\hat{w} = \hat{v}\cdot\hat{w} = 0$



$$\Longrightarrow dV = h_u h_v h_w \, du\,dv\,dw$$

You can also get surface area elements for coordinate surfaces.

<u>Ex</u> $u = u_0 \qquad dS = h_v h_w \, dv\,dw$

and so on.

Ex Cylindrical coordinates

$$\begin{cases} x = R\cos\theta \\ y = R\sin\theta \\ z = z \end{cases} \qquad (R,\theta,z) \longmapsto (x,y,z)$$

$$\frac{\partial \vec{r}}{\partial R} = (\cos\theta, \sin\theta, 0) \qquad h_R = 1$$

$$\frac{\partial \vec{r}}{\partial \theta} = (-R\sin\theta, R\cos\theta, 0) \qquad h_\theta = R$$

$$\frac{\partial \vec{r}}{\partial z} = (0,0,1) \qquad h_z = 1$$

$$dV = R \, dR \, d\theta \, dz$$

The gradient, divergence and Curl in orthogonal curvilinear coordinates

We begin with the gradient. We want to find $\nabla f = f_u \hat{u} + f_v \hat{v} + f_w \hat{w}$. Take a curve γ with parametrization $\gamma(s)$ in terms of arc length

$$\left(\left| \frac{d\gamma}{ds} \right| = 1 \right)$$

$$\frac{df}{ds} = \frac{\partial f}{\partial u} \cdot \frac{du}{ds} + \frac{\partial f}{\partial v} \cdot \frac{dv}{ds} + \frac{\partial f}{\partial w} \cdot \frac{dw}{ds}$$ because of chainrule