Ex Cylindrical coordinates

$$\begin{cases} x = R\cos\theta \\ y = R\sin\theta \\ z = z \end{cases} \qquad (R,\theta,z) \longmapsto (x,y,z)$$

$$\frac{\partial \vec{r}}{\partial R} = (\cos\theta, \sin\theta, 0) \qquad h_R = 1$$

$$\frac{\partial \vec{r}}{\partial \theta} = (-R\sin\theta, R\cos\theta, 0) \qquad h_\theta = R$$

$$\frac{\partial \vec{r}}{\partial z} = (0,0,1) \qquad h_z = 1$$

$$dV = R\, dR\, d\theta\, dz$$

The gradient, divergence and Curl in <u>orthogonal</u> curvilinear coordinates

We begin with the gradient. We want to find $\nabla f = f_u \hat{u} + f_v \hat{v} + f_w \hat{w}$. Take a curve γ with parametrization $\gamma(s)$ in terms of arc length

$$\left(\left| \frac{d\gamma}{ds} \right| = 1 \right)$$

$$\frac{df}{ds} = \frac{\partial f}{\partial u} \cdot \frac{du}{ds} + \frac{\partial f}{\partial v} \cdot \frac{dv}{ds} + \frac{\partial f}{\partial w} \cdot \frac{dw}{ds}$$ because of chainrule

We can also calculate

$$\frac{df}{ds} = \nabla f \cdot \hat{T}$$ where $\hat{T}$ is the unit tangent vector of γ

$$\hat{T} = \frac{d\gamma}{ds} = \frac{\partial \gamma}{\partial u} \cdot \frac{du}{ds} + \frac{\partial \gamma}{\partial v} \cdot \frac{dv}{ds} + \frac{\partial \gamma}{\partial w} \cdot \frac{dw}{ds}$$

$$= h_u \frac{du}{ds} \hat{u} + h_v \frac{dv}{ds} \hat{v} + h_w \frac{dw}{ds} \hat{w}$$

So $$\frac{df}{ds} = f_u h_u \frac{du}{ds} + f_v h_v \frac{dv}{ds} + f_w h_w \frac{dw}{ds}$$

(u,v,w) orthogonal

$$\Rightarrow f_u h_u = \frac{\partial f}{\partial u}, \quad f_v h_v = \frac{\partial f}{\partial v} \quad \& \quad f_w h_w = \frac{\partial f}{\partial w}$$

$$\Rightarrow \nabla f = \frac{1}{h_u} \frac{\partial f}{\partial u} \hat{u} + \frac{1}{h_v} \frac{\partial f}{\partial v} \hat{v} + \frac{1}{h_w} \frac{\partial f}{\partial w} \hat{w}$$

<u>Ex</u> In polar coordinates

$$\begin{cases} x = R\cos\theta \\ y = R\sin\theta \end{cases} \qquad \begin{aligned} h_R &= |(\cos\theta, \sin\theta)| = 1 \\ h_\theta &= |(-R\sin\theta, R\cos\theta)| = R \end{aligned}$$

$$\Rightarrow \nabla f = \frac{\partial f}{\partial R} \hat{R} + \frac{1}{R} \frac{\partial f}{\partial \theta} \hat{\theta}$$

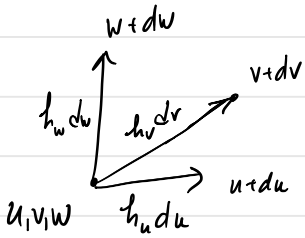
In cylindrical coordinates $(h_R = 1,\ h_\theta = R,\ h_z = 1)$ we get

$$\nabla f = \frac{\partial f}{\partial R}\hat{R} + \frac{1}{R}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{\partial f}{\partial z}\hat{z}$$

Divergence in orthogonal curvilinear coordinates

$$F(u,v,w) = F_u \hat{u} + F_v \hat{v} + F_w \hat{w}$$

Remember that $\operatorname{div} F$ is the outward flux per unit volume



On u and $u+du$ surface the flux is

$$F(u+du,v,w)\cdot\hat{u}\, dS_u - F(u,v,w)\cdot\hat{u}\, dS_u$$

$$= F_u(u+du,v,w)\, h_v(u+du,v,w)\, h_w(u+du,v,w)\, dv\,dw -$$

$$- F_u(u,v,w)\, h_v(u,v,w)\, h_w(u,v,w)\, dv\,dw =$$

$$= \frac{\partial}{\partial u}(F_u h_v h_w)\, du\,dv\,dw + \text{higher order terms}$$

Add the other 4 surfaces and divide by the volume ($h_u h_v h_w \, du dv dw$)

$$\Rightarrow \operatorname{div} F = \frac{1}{h_u h_v h_w}\left(\frac{\partial}{\partial u}(h_v h_w F_u) + \frac{\partial}{\partial v}(h_u h_w F_v) + \frac{\partial}{\partial w}(h_u h_v F_w)\right)$$

<u>Ex</u> Cylindrical coordinates

$$h_R = h_z = 1 \quad h_\theta = R$$

$$F(R,\theta,z) = F_R \hat{R} + F_\theta \hat{\theta} + F_z \hat{z}$$

$$\operatorname{div} F = \frac{1}{R}\left(\frac{\partial}{\partial R}(RF_R) + \frac{\partial}{\partial \theta} F_\theta + \frac{\partial}{\partial z}(RF_z)\right)$$

$$= \frac{1}{R}\left(F_R + R\frac{\partial F_R}{\partial R} + \frac{\partial F_\theta}{\partial \theta} + R\frac{\partial F_z}{\partial z}\right) =$$

$$= \frac{1}{R}F_R + \frac{\partial F}{\partial R} + \frac{1}{R}\frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z}$$

<u>Finally Curl F</u>

We begin by showing that

$$\operatorname{Curl}(f\nabla g) = \nabla f \times \nabla g$$

We have (c) $\text{Curl}(\phi F) = (\nabla\phi)\times F + \phi(\nabla\times F)$

and (h) $\text{Curl}(\nabla g) = \nabla\times(\nabla g) = \vec{0}$

$$\Rightarrow \text{Curl}(f\nabla g) = \nabla f\times\nabla g + f\,\nabla\times(\nabla g) = \nabla f\times\nabla g.$$

Now study $f(u,v,w) = u$. What is ∇f ?

$$\nabla f = \frac{1}{h_u}\frac{\partial f}{\partial u}\hat{u} = \frac{1}{h_u}\hat{u} \quad \text{or} \quad \nabla u = \frac{1}{h_u}\hat{u}$$

$$\Rightarrow \hat{u} = h_u\nabla u$$

Also $\hat{v} = h_v\nabla v$ and $\hat{w} = h_w\nabla w$

We get $F = F_u\hat{u} + F_v\hat{v} + F_w\hat{w} =$

$$= F_u h_u\nabla u + F_v h_v\nabla v + F_w h_w\nabla w$$

and $\text{Curl}\,F = \nabla\times(F_u h_u\nabla u) + \nabla\times(F_v h_v\nabla v) + \nabla\times(F_w h_w\nabla w)$

$$\nabla\times(F_u h_u\nabla u) = \nabla(F_u h_u)\times\nabla u =$$

$$= \left(\frac{1}{h_u}\frac{\partial}{\partial u}(F_u h_u)\hat{u} + \frac{1}{h_v}\frac{\partial}{\partial v}(F_u h_u)\hat{v} + \frac{1}{h_w}\frac{\partial}{\partial w}(F_u h_u)\hat{w}\right)\times\frac{1}{h_u}\hat{u} = \left[\begin{array}{l}\hat{u}\times\hat{u} = 0\\ \hat{v}\times\hat{u} = -\hat{w}\\ \hat{w}\times\hat{u} = \hat{v}\end{array}\right. =$$

$$= \frac{1}{h_u h_v h_w}\left[\frac{\partial}{\partial w}(F_u h_u)h_v\hat{v} - \frac{\partial}{\partial v}(F_u h_u)h_w\hat{w}\right]$$

Do the same for the other terms and you get

$$\text{Curl } F = \frac{1}{h_u h_v h_w} \begin{vmatrix} h_u \hat{u} & h_v \hat{v} & h_w \hat{w} \\ \partial/\partial u & \partial/\partial v & \partial/\partial w \\ F_u h_u & F_v h_v & F_w h_w \end{vmatrix}$$

<u>Ex</u> Cylindrical coordinates $(h_R = h_z = 1,\ h_\theta = R)$

$[\hat{R}, \hat{\theta}, \hat{z}]$ right-handed

$$F = F_R \hat{R} + F_\theta \hat{\theta} + F_z \hat{z}$$

$$\text{Curl } F = \frac{1}{R} \begin{vmatrix} \hat{R} & R\hat{\theta} & \hat{z} \\ \partial/\partial R & \partial/\partial\theta & \partial/\partial z \\ F_R & RF_\theta & F_z \end{vmatrix} =$$

$$= \frac{1}{R}\left(\left(\frac{\partial F_z}{\partial \theta} - \frac{\partial}{\partial z}(RF_\theta)\right)\hat{R} - \left(\frac{\partial F_z}{\partial R} - \frac{\partial F_R}{\partial z}\right)R\hat{\theta} + \left(\frac{\partial}{\partial R}(RF_\theta) - \frac{\partial F_R}{\partial \theta}\right)\hat{z}\right) =$$

$$= \left(\frac{1}{R}\frac{\partial F_z}{\partial \theta} - \frac{\partial F_\theta}{\partial z}\right)\hat{R} + \left(\frac{\partial F_R}{\partial z} - \frac{\partial F_z}{\partial R}\right)\hat{\theta} + $$

$$+ \left(\frac{1}{R}F_\theta + \frac{\partial F_\theta}{\partial R} - \frac{1}{R}\frac{\partial F_R}{\partial \theta}\right)\hat{z}$$

Ex $\vec{r} = (x, y, z)$ and $F(x,y,z) = m \dfrac{\vec{r}}{|\vec{r}|^3}$

With correct choice of m this vector field describe gravity (and with another choice electromagnetism).

It is annoying to verify that $\operatorname{div} F = 0$ (outside the origin where F is undefined) in xyz-coordinates. But in spherical coordinates it is easier. Let $[\hat{R}, \hat{\phi}, \hat{\theta}]$ be a local basis (notice the order of the basis vectors!)

In spherical coordinates the vector field is

$$F(R, \phi, \theta) = \frac{m}{R^2} \hat{R}$$

Recall

$$x = R \sin\phi \cos\theta$$
$$y = R \sin\phi \sin\theta$$
$$z = R \cos\phi$$

and

$$h_R = |(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)| = 1$$

$$h_\phi = |(R\cos\phi\cos\theta, R\cos\phi\sin\theta, -R\sin\phi)| = R$$

$$h_\theta = \left|\left(-R\sin\phi\sin\theta,\ R\sin\phi\cos\theta,\ 0\right)\right| = R\sin\phi$$

$$\operatorname{div} F = \frac{1}{h_R h_\phi h_\theta}\left(\frac{\partial}{\partial R}\left(F_R \cdot h_\phi \cdot h_\theta\right) + \underbrace{\cdots}_{=0 \text{ in this case}}\right)$$

$$= \frac{1}{R^2\sin\phi}\,\frac{\partial}{\partial R}\left(\frac{m}{R^2}R^2\sin\phi\right) = 0$$