Lecture 8

Lecturer: G. S. Paraoanu

Department of Applied Physics, School of Science,

Aalto University, P.O. Box 15100, FI-00076 AALTO, Finland

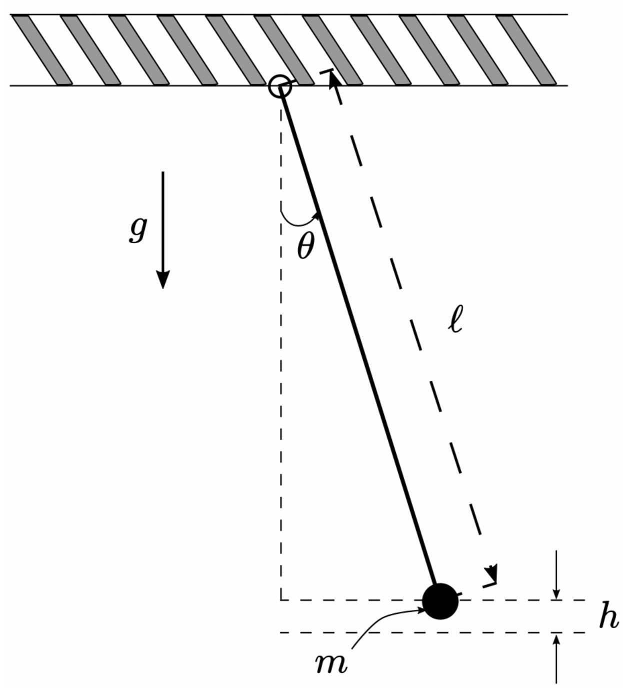
I. QUANTIZATION OF ELECTRICAL NETWORKS

DiVincenzo's Criteria:

1. A scalable physical system with well characterized qubit
2. The ability to initialize the state of the qubits to a simple fiducial state
3. Long relevant decoherence times
4. A "universal" set of quantum gates
5. A qubit-specific measurement capability

Harmonic Oscillator:

The harmonic oscillator is an important primer for studying quantum circuits. In particular, we will see that the canonical position and momentum in a classical harmonic oscillator are analogues of charge and flux in an LC circuit.



We consider a pendulum of length ℓ and mass m that subtends an angle θ with respect to the center.

The kinetic energy is $T = \frac{1}{2}mv^2 = \frac{1}{2}m\ell^2\dot{\theta}^2$, and the potential energy is $V = mgh$, where $h = (1 - \cos\theta)\ell$. Considering only small θ, we have $h \approx (1 - 1 + \frac{\theta}{2})\ell = \ell\frac{\theta}{2}$. Therefore, the potential energy is $V = \frac{1}{2}mg\ell\theta^2$. Since the Lagrangian is $L \equiv T - V$, the Lagrangian of our system is

$$L = \frac{1}{2}m\ell^2\dot{\theta}^2 - \frac{1}{2}mg\ell\theta^2 \tag{1}$$

Upon introducing generalized coordinates q and p:

$$q \equiv \ell\theta \tag{2}$$

$$p \equiv \frac{\partial L}{\partial \dot{q}} \approx \frac{\partial}{\partial \dot{q}}\Big(\frac{1}{2}m\dot{q}^2 - \frac{mg}{2\ell}q^2\Big) = m\dot{q} = m\ell\dot{\theta} \tag{3}$$

The Euler-Lagrange equation is

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}}\Big) - \frac{L}{q} = 0\ , \tag{4}$$

and since $T(q) = 0$, i.e., there is no dependence of the kinetic energy on q, only $\dot{q}$, we have $\frac{\partial L}{\partial q} = -\frac{\partial V}{\partial q}$. Further, since $p = \frac{\partial L}{\partial \dot{q}}$, we have

$$\frac{dp}{dt} = -\frac{\partial V}{\partial q}\ . \tag{5}$$

Now, inserting Eq. 2 and Eq. 3 into Eq. 4, we have

$$\dot{p} = -mg\theta\ . \tag{6}$$

However, directly differentiating 3 with respect to time, yields

$$\dot{p} = m\ell\ddot{\theta}\ . \tag{7}$$

Upon equating Eq. 6 and Eq. 7, we have

$$m\ell\ddot{\theta} + mg\theta = 0\ , \tag{8}$$

i.e., the equation of motion:

$$\ddot{\theta} + \frac{g}{\ell}\theta = 0\ . \tag{9}$$

Taking as a usual trial function $\theta = C\exp(i\omega t)$, where $C \neq 0$ is a constant, and ω is the angular frequency, Eq. 9 becomes:

$$\Big(-\omega^2 + \frac{g}{\ell}\Big)C\exp(i\omega t) = 0\ . \tag{10}$$

This valid at all t only iff

$$-\omega^2 + \frac{g}{\ell} = 0\ . \tag{11}$$

This is the so-called characteristic polynomial of the linearized equation of motion, and solving for ω we find the natural angular frequency of small vibrations of the pendulum:

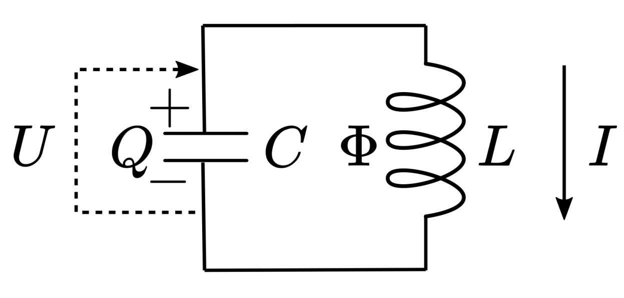
$$\omega = \sqrt{g/\ell}\ . \tag{12}$$

Starting from energy considerations, we have derived the eigenfrequency of the system!

Superconducting LC oscillator:

Once you understand the harmonic oscillator, you can easily apply the concept

Let us consider a classical superconducting LC oscillator. The electrical energy will oscillate between the potential energy stored in the capacitor U, and the kinetic energy associated with the magnetic flux in the coil $\Phi = LI$, where L is the inductance of the coil, and $I = -\dot{Q}$ is the current (direction consistent with the figure below).



From the above circuit figure, we also see that the voltage drop across the inductor is U; hence, from Lenz's law we have $\dot{\Phi} = U$. Therefore,

$$U = L\dot{I} \tag{13}$$

And since the instantaneous power fed into an electric circuit is simply the product of the voltage across circuit times the current flowing into the positive voltage node, we have

$$P = U\dot{Q} \ . \tag{14}$$

The electrical kinetic energy in the coil is then

$$T = \int_{t_0}^{t_1} P dt = \int_{t_0}^{t_1} U I dt = \int_{t_0}^{t_1} (L\dot{I}) I dt = \int_0^I L I^{'} dI^{'} = \frac{1}{2} L I^2 = \frac{\Phi^2}{2L} \ , \tag{15}$$

and the electrical potential energy stored in the capacitor is

$$V = \int_{t_0}^{t_1} P dt = \int_{t_0}^{t_1} U\dot{Q} dt = \int_{t_0}^{t_1} U\Big(\frac{dQ}{dt} dt\Big) = \int_0^Q \frac{Q^{'}}{C} dQ^{'} = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2} Q U = \frac{1}{2} C U^2 \ . \tag{16}$$

From Eq. 15 and Eq. 16 we have $T = \frac{1}{2}LI^2 = \frac{1}{2}L\dot{Q}^2$ and $V = \frac{1}{2}CU^2 = \frac{Q^2}{2C}$.

The Lagrangian is therefore

$$L = T - V = \frac{1}{2} L\dot{Q}^2 - \frac{Q^2}{2C} \ . \tag{17}$$

To find the equation of motion, we choose the generalized coordinates

$$q = Q \ , \tag{18}$$

$$p \equiv \frac{\partial L}{\partial \dot{q}} = L\dot{Q} = -LI = -\Phi \ . \tag{19}$$

Indeed the equation of motion is

$$\ddot{Q} + \frac{1}{LC} Q = 0 \ , \tag{20}$$

where the natural angular frequency of oscillations is

$$\omega = \frac{1}{\sqrt{LC}} \ . \tag{21}$$

Compare this with $\omega = \sqrt{g/\ell}$ for the pendulum.

To summarize, the pendulum and LC oscillator analogues are

$$\text{Momentum} \quad \hat{p} \quad \longleftrightarrow \quad \text{Charge} \quad \hat{q}$$

$$\text{Position} \quad \hat{x} \quad \longleftrightarrow \quad \text{Flux} \quad \hat{\Phi}$$

$$\text{Mass} \quad m \quad \longleftrightarrow \quad \text{Capacitance} \quad C$$

$$\text{Resonance frequency} \quad \omega_r \longleftrightarrow \quad \omega_r = \sqrt{\frac{1}{LC}}$$

Legendre transformation to Hamiltonian:

The general definition for a Hamiltonian is

$$H \equiv \dot{q}p - L\ . \tag{22}$$

We take the total time derivative to analyze the system dynamically

$$\frac{dH}{dt} = \ddot{q}p + \dot{q}\dot{p} - \frac{\partial L}{\partial q}\dot{q} - \frac{\partial L}{\partial \dot{q}}\ddot{q} - \dot{L}\ . \tag{23}$$

Further, we take $p \equiv \partial L/\partial q$, and $dp/dt = \dot{p}$, as $p = p(t)$ only.

Therefore, the total time derivative of the Hamiltonian is

$$\frac{dH}{dt} = \ddot{q}p + \dot{q}\dot{p} - \frac{\partial L}{\partial q}\dot{q} - p\ddot{q} - \dot{L} \tag{24}$$

Simplifying this formula further yields

$$\frac{dH}{dt} = \dot{q}\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q}\right] - \dot{L}\ . \tag{25}$$

Since the Lagrangian is time-independent, $\dot{L} = 0$, due to the Euler-Lagrange equation, we have

$$\frac{dH}{dt} = 0\ . \tag{26}$$

That is to say, the Hamiltonian is a constant of motion, i.e., energy is conserved in the system.

Furthermore, in terms of our generalized coordinates, Eq. 22 is

$$H = \dot{Q}(L\dot{Q}) - \left(\frac{1}{2}L\dot{Q}^2 - \frac{Q^2}{2C}\right) = \frac{1}{2}L\dot{Q}^2 + \frac{Q^2}{2C}\ , \tag{27}$$

Further, from our standard definitions, we have

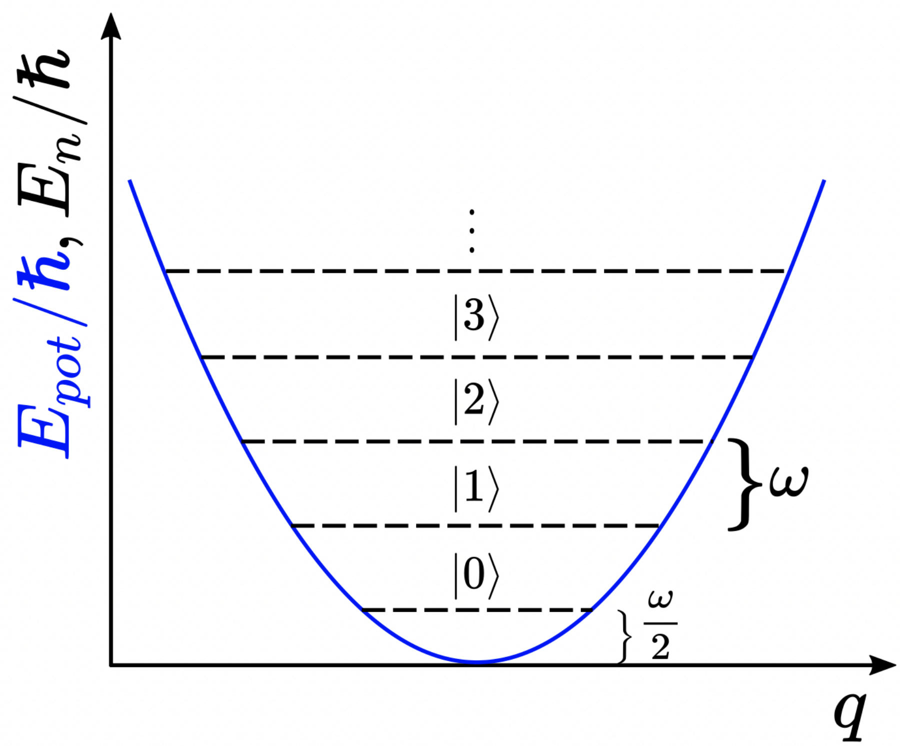
$$H = \frac{\Phi^2}{2L} + \frac{Q^2}{2C} \ . \tag{28}$$

Therefore, the Hamiltonian represents the total energy of the system

$$H = T + V \ . \tag{29}$$

We have derived the total energy of the system starting from the Lagrangian. This is necessary to derive energy quantization!

Quantization of Oscillators



Quantization means we see the effects of single particles of excitations or excitations, e.g. the photoelectric effect, where the electromagnetic field is quantized and hence the energy $E = \hbar\omega$ is quantized.

- In a harmonic oscillator, the energy is quantized equidistantly.
- Energy quantization can be seen as counting the number of photons stored in the oscillator.

In quantum mechanics, variables are replaced by operators, i.e.

$$
\begin{aligned}
q &\to \hat{q} : \mathcal{H} \to \mathcal{H} \ , \\
p &\to \hat{p} : \mathcal{H} \to \mathcal{H} \ ,
\end{aligned}
$$

some examples being the charge $\hat{q}$ and flux $\hat{\Phi}$ operators.

For practical reasons, we often use matrix representations, e.g.,

$$
q_{k\ell} = \langle e_k | \hat{q} | e_\ell \rangle \ .
$$

Generically, an operator acting on a state is a matrix times a vector and may be represented as

$$
\begin{pmatrix} (O\psi)_1 \\ (O\psi)_2 \\ \vdots \\ (O\psi)_i \\ \vdots \end{pmatrix} = \begin{pmatrix} O_{11} & O12 & \cdots & O_{1j} & \cdots \\ O_{21} & O_{22} & \cdots & O_{2j} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ O_{i1} & O_{i2} & \cdots & O_{ij} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_j \\ \vdots \end{pmatrix} \tag{30}
$$

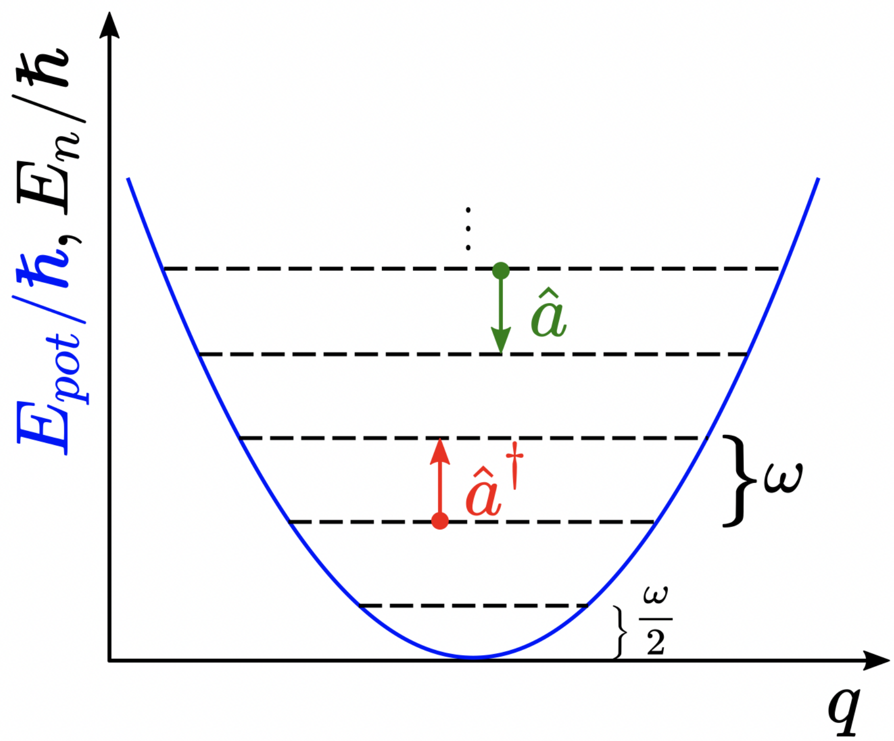
Two conjugate variables obey the commutation relation:

$$
[\hat{p}, \hat{q}] \equiv \hat{p}\hat{q} - \hat{q}\hat{p} = -i\hbar \ . \tag{31}
$$

For pedagogical purposes, it is convenient to transform systems into the basis of number states (give matrix representation of a)

$$
a^\dagger = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & \cdots \\ \sqrt{1} & 0 & 0 & \cdots & 0 & \cdots \\ 0 & \sqrt{2} & 0 & \cdots & 0 & \cdots \\ 0 & 0 & \sqrt{3} & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \cdots \\ \vdots & \vdots & \vdots & \cdots & \sqrt{n} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{32}
$$

$$a = \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \cdots & 0 & \cdots \\ 0 & 0 & \sqrt{2} & 0 & \cdots & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{3} & \cdots & 0 & \cdots \\ 0 & 0 & 0 & 0 & \ddots & \vdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \sqrt{n} & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{33}$$



These can be interpreted as ladder operators raising and lowering the excitation number

$$\hat{a} \equiv \frac{\omega_r C\hat{\Phi} + i\hat{q}}{\sqrt{2\omega_r C\hbar}} \qquad \text{is the } \underline{\text{annihilation operator.}} \tag{34}$$

$$\hat{a}^\dagger \equiv \frac{\omega_r C\hat{\Phi} - i\hat{q}}{\sqrt{2\omega_r C\hbar}} \qquad \text{is the } \underline{\text{creation operator.}} \tag{35}$$

Their product gives the excitation number of a system

$$\hat{n} \equiv \hat{a}^\dagger \hat{a} \ . \tag{36}$$

Quantization of the LC oscillator

For the superconducting resonator, we have

$$\hat{H} = \frac{\hat{p}^2}{2L} + \frac{\hat{q}^2}{2C} \ . \tag{37}$$

We aim to diagonalize H into a form involving only one operator. This can be achieved via a change of variables:

$$\hat{p} = \sqrt{\frac{\hbar\omega L}{2}}\Big(\hat{a} + \hat{a}^\dagger\Big), \quad \hat{q} = \sqrt{\frac{\hbar\omega C}{2}} i\Big(\hat{a} - \hat{a}^\dagger\Big) \ , \tag{38}$$

where ω is a free scalar parameter, which we will choose later. The square root factors have been inserted for convenience.

Note that $(\hat{a} + \hat{a}^\dagger)$ and $i(\hat{a} - \hat{a}^\dagger)$ are Hermitian and independent.

Eq. is now

$$\hat{H} = \frac{\left[\sqrt{\frac{\hbar\omega L}{2}}(\hat{a} + \hat{a}^\dagger)\right]^2}{2L} + \frac{\left[\sqrt{\frac{\hbar\omega C}{2}} i(\hat{a} - \hat{a}^\dagger)\right]^2}{2C} \tag{39}$$

$$= \frac{\hbar\omega}{4}\Big(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}\Big) \tag{40}$$

$$= \frac{\hbar\omega}{2}\Big(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}\Big) \ . \tag{41}$$

Using $[\hat{a}, \hat{a}^\dagger] = 1$, it follows that $\hat{a}\hat{a}^\dagger = \hat{a}^\dagger\hat{a} + 1$, we obtain

$$\hat{H} = \hbar\omega\Big(\hat{a}^\dagger\hat{a} + \frac{1}{2}\Big) \ . \tag{42}$$

This tells us that the total energy of the system is given by vacuum fluctuations $(+1/2)$ and the number of photons stored at frequency ω!

References