Solution: Problem set 5, Lecture 9, Question 3.

(a) and (b) are almost covered in tutorial notes 9-10 on Mycourse webpage

(c). You can find the steps up to the differential equation part in the tutorial notes.

$\Rightarrow$ We will solve for a general case of Δ, that is, not assume any particular values for Δ. Later, we can plug in the values for $\Delta = 0$, 0.5 GHz and 1 GHz.

We begin with the differential equations, we have obtained using Schrödinger equation from interaction picture.

where, we take the Ansatz

$$|\psi^{in}\rangle = c_i|0, n+1\rangle + c_f|1, n\rangle$$

$$\dot{c}_i = -ig\sqrt{n+1}\, e^{i\Delta t}\, c_f \quad \text{—— (1)}$$

$$\dot{c}_f = -ig\sqrt{n+1}\, e^{-i\Delta t}\, c_i \quad \text{—— (2)}$$

We assume the form for

$$c_i = e^{i\Delta t} x(t) \; ; \text{ where } x(t) \text{ is some time-dependent function.}$$

Plugging this form in eq^n (2) yields,

$$\Rightarrow \dot{c}_f = -ig\sqrt{n+1}\; x(t) \quad \text{—— (3)}$$

and in eq^n (1) yields,

$$\frac{dc_i}{dt} = \frac{d}{dt}\left[e^{i\Delta t} x(t)\right] \Rightarrow \dot{c}_i(t) = i\Delta e^{i\Delta t} x(t) + e^{i\Delta t}\dot{x}(t)$$

$$= \left(i\Delta x(t) + \dot{x}(t)\right) e^{i\Delta t} \quad \text{—— (4)}$$

Plugging (4) in (1), we get,

$$\Rightarrow \left(i\Delta x(t) + \dot{x}(t)\right)\cancel{e^{i\Delta t}} = -ig\sqrt{n+1}\,\cancel{e^{i\Delta t}}\,c_f$$

$$\Rightarrow i\Delta\dot{x}(t) + \ddot{x}(t) = -ig\sqrt{n+1}\,\dot{c}_f$$ | Taking $\frac{d}{dt}$ on both sides.

$$\Rightarrow i\Delta\dot{x}(t) + \ddot{x}(t) = -ig\sqrt{n+1}\cdot\left(-ig\sqrt{n+1}\,x(t)\right)$$ | Using (3)

$$\Rightarrow i\Delta\dot{x}(t) + \ddot{x}(t) = -g^2(n+1)\,x(t)$$

$$\Rightarrow \ddot{x}(t) + i\Delta\dot{x}(t) + g^2(n+1)\,x(t) = 0 \quad \longrightarrow (5)$$

Characteristic function to solve the Second-order ODE

$$r^2 + i\Delta r + g^2(n+1) = 0$$

$$\Rightarrow r = \frac{-i\Delta \pm \sqrt{(i\Delta)^2 - 4g^2(n+1)}}{2}$$

$$= \frac{-i\Delta \pm \sqrt{-\Delta^2 - 4g^2(n+1)}}{2}$$

$$\therefore r = \frac{-i\Delta \pm i\sqrt{\Delta^2 + 4g^2(n+1)}}{2};$$

Define these to make things compact!

$$\Omega_\Delta^2 = \Delta^2 + 4g^2(n+1)$$

$$4g^2(n+1) = \Delta^2 - \Omega_\Delta^2$$

$$2g\sqrt{n+1} = \sqrt{\Delta^2 - \Omega_\Delta^2}$$

This leads to a general solution to (5),

$$x(t) = A e^{-i(\Delta + \Omega_\Delta)t/2} + B e^{-i(\Delta - \Omega_\Delta)t/2} \quad ; A, B \in \mathbb{R}.$$

Plugging the general solution to the form for C_i.

$$\therefore \quad c_i(t) = e^{i\Delta t}\left(A e^{-i(\Delta+\Omega_\Delta)t/2} + B e^{-i(\Delta-\Omega_\Delta)t/2}\right) \quad \text{---} \ (6)$$

Taking the derivative,

$$\dot{c}_i(t) = i\Delta e^{i\Delta t}\left[A e^{-i(\Delta+\Omega_\Delta)t/2} + B e^{-i(\Delta-\Omega_\Delta)t/2}\right]$$
$$+ e^{i\Delta t}\left[-\frac{i(\Delta+\Omega_\Delta)}{2} A e^{-i(\Delta+\Omega_\Delta)t/2} - \frac{i(\Delta-\Omega_\Delta)}{2} B e^{-i(\Delta-\Omega_\Delta)t/2}\right]$$

$$= e^{i\Delta t}\left[\left(i\Delta - \frac{i(\Delta+\Omega_\Delta)}{2}\right) A e^{-i(\Delta+\Omega_\Delta)t/2} + \left(i\Delta - \frac{i(\Delta-\Omega_\Delta)}{2}\right) B e^{-i(\Delta-\Omega_\Delta)t/2}\right]$$

$$= e^{i\Delta t}\left[\frac{2i\Delta - i\Delta - i\Omega_\Delta}{2} A e^{-i(\Delta+\Omega_\Delta)t/2} + \left(\frac{2i\Delta - i\Delta + i\Omega_\Delta}{2}\right) B e^{-i(\Delta-\Omega_\Delta)t/2}\right]$$
$$= e^{i\Delta t}\left[\frac{i\Delta - i\Omega_\Delta}{2} A e^{-i(\Delta+\Omega_\Delta)t/2} + \frac{i\Delta + i\Omega_\Delta}{2} B e^{-i(\Delta-\Omega_\Delta)t/2}\right]$$

Using the derivative in eq^n (1),

$$\Rightarrow \quad \dot{c}_i(t) = -ig\sqrt{n+1}\, e^{i\Delta t} c_f(t)$$

$$\Rightarrow \quad c_f(t) = \frac{1}{-ig\sqrt{n+1}\, e^{i\Delta t}} \dot{c}_i(t)$$

$$= \frac{1}{-ig\sqrt{n+1}\, \cancel{e^{i\Delta t}}} \cancel{e^{i\Delta t}}\left[\frac{i\Delta - i\Omega_\Delta}{2} A e^{-i(\Delta+\Omega_\Delta)t/2} + \frac{i\Delta + i\Omega_\Delta}{2} B e^{-i(\Delta-\Omega_\Delta)t/2}\right]$$

$$= \frac{i(\Delta-\Omega_\Delta)}{-ig\sqrt{n+1}\cdot 2} A e^{-i(\Delta+\Omega_\Delta)t/2} + \frac{i(\Delta+\Omega_\Delta)}{-ig\sqrt{n+1}\cdot 2} B e^{-i(\Delta-\Omega_\Delta)t/2}$$

$$= -\frac{(\Delta-\Omega_\Delta)}{2g\sqrt{n+1}} A e^{-i(\Delta+\Omega_\Delta)t/2} - \frac{\Delta+\Omega_\Delta}{2g\sqrt{n+1}} B e^{-i(\Delta-\Omega_\Delta)t/2}$$

$$\boxed{c_f(t) = \frac{-(\Delta-\Omega_\Delta)}{\sqrt{\Delta^2-\Omega_\Delta^2}} A e^{-i(\Delta+\Omega_\Delta)t/2} - \frac{(\Delta+\Omega_\Delta)}{\sqrt{\Delta^2-\Omega_\Delta^2}} B e^{-i(\Delta-\Omega_\Delta)t/2}} \quad \text{---} \ (7)$$

initial conditions : $|c_i|^2 = 1$ and $|c_f|^2 = 0$

$$\therefore |C_i(t)|^2 = |e^{i\Delta t}|^2 \left| A e^{-i(\Delta+\Omega_\Delta)t/2} + B e^{-i(\Delta-\Omega_\Delta)t/2} \right|^2$$

$$= |e^{-i(\Delta+\Omega_\Delta)t/2}|^2 \left| A + B e^{-i(\Delta-\Omega_\Delta)t/2} e^{i(\Delta+\Omega_\Delta)t/2} \right|^2$$

$$= \left| A + B e^{-i(\cancel{\Delta}-\Omega_\Delta-\cancel{\Delta}-\Omega_\Delta)t/2} \right|^2$$

$$= \left| A + B e^{i\Omega_\Delta t} \right|^2$$

$$|c_i(0)|^2 = 1$$

$$\Rightarrow |A+B|^2 = 1 \quad \text{—— (8)}$$

Another condition : $|C_f(0)|^2 = 0$

$$\Rightarrow |C_f(t)|^2 = \left| \frac{-(\Delta-\Omega_\Delta)}{\sqrt{\Delta^2-\Omega_\Delta^2}} A e^{-i(\Delta+\Omega_\Delta)t/2} - \frac{(\Delta+\Omega_\Delta)}{\sqrt{\Delta^2-\Omega_\Delta^2}} B e^{-i(\Delta-\Omega_\Delta)t/2} \right|^2$$

$$= |e^{-i(\Delta+\Omega_\Delta)t/2}|^2 \left| \frac{-(\Delta-\Omega_\Delta)}{\sqrt{\Delta^2-\Omega_\Delta^2}} A - \frac{(\Delta+\Omega_\Delta)}{\sqrt{\Delta^2-\Omega_\Delta^2}} B e^{i\Omega_\Delta t} \right|^2$$

$$|C_f(0)|^2 = \left| \frac{-(\Delta-\Omega_\Delta)}{\sqrt{\Delta^2-\Omega_\Delta^2}} A - \frac{(\Delta+\Omega_\Delta)}{\sqrt{\Delta^2-\Omega_\Delta^2}} B \right|^2$$

$$\Rightarrow \left| \frac{-(\Delta-\Omega_\Delta)}{\sqrt{\Delta^2-\Omega_\Delta^2}} A - \frac{(\Delta+\Omega_\Delta)}{\sqrt{\Delta^2-\Omega_\Delta^2}} B \right|^2 = 0$$

$$\Rightarrow -(\Delta-\Omega_\Delta) A = (\Delta+\Omega_\Delta) B$$

$$\Rightarrow A = -\frac{(\Delta+\Omega_\Delta)}{(\Delta-\Omega_\Delta)} B \quad \text{—— (9)}$$

Plugging equation (9) in eq^n (8),

$$|A+B|^2 = 1$$

$$\Rightarrow \left| -\frac{(\Delta+\Omega_\Delta)}{(\Delta-\Omega_\Delta)} B + B \right|^2 = 1$$

$$\Rightarrow |B|^2 \left| \frac{-\cancel{\Delta}-\Omega_\Delta + \cancel{\Delta}-\Omega_\Delta}{\Delta-\Omega_\Delta} \right|^2 = 1$$

$$\Rightarrow |B|^2 \left| \frac{-2\Omega_\Delta}{\Delta-\Omega_\Delta} \right|^2 = 1$$

$$\Rightarrow B = \frac{\Delta-\Omega_\Delta}{2\Omega_\Delta} \quad \Rightarrow A = -\frac{(\Delta+\Omega_\Delta)}{\cancel{(\Delta-\Omega_\Delta)}} \cdot \frac{\cancel{(\Delta-\Omega_\Delta)}}{2\Omega_\Delta} = -\frac{(\Delta+\Omega_\Delta)}{2\Omega_\Delta}$$

$$\therefore C_f(t) = \frac{-(\Delta-\Omega_\Delta)}{\sqrt{\Delta^2-\Omega_\Delta^2}} A e^{-i(\Delta+\Omega_\Delta)t/2} - \frac{(\Delta+\Omega_\Delta)}{\sqrt{\Delta^2-\Omega_\Delta^2}} B e^{-i(\Delta-\Omega_\Delta)t/2}$$

$$= \frac{-(\Delta-\Omega_\Delta)}{\sqrt{\Delta^2-\Omega_\Delta^2}} \cdot \frac{-(\Delta+\Omega_\Delta)}{2\Omega_\Delta} e^{-i(\Delta+\Omega_\Delta)t/2} - \frac{(\Delta+\Omega_\Delta)}{\sqrt{\Delta^2-\Omega_\Delta^2}} \cdot \frac{\Delta-\Omega_\Delta}{2\Omega_\Delta} e^{-i(\Delta-\Omega_\Delta)t/2}$$

$$= \frac{(\Delta^2-\Omega_\Delta^2)}{\sqrt{\Delta^2-\Omega_\Delta^2}} \cdot \frac{1}{2\Omega_\Delta} e^{-i(\Delta+\Omega_\Delta)t/2} - \frac{(\Delta^2-\Omega_\Delta^2)}{\sqrt{\Delta^2-\Omega_\Delta^2}} \cdot \frac{1}{2\Omega_\Delta} e^{-i(\Delta-\Omega_\Delta)t/2}$$

$$= \frac{(\Delta^2-\Omega_\Delta^2)}{\sqrt{\Delta^2-\Omega_\Delta^2}} \cdot \frac{1}{2\Omega_\Delta} \left(e^{-i(\Delta+\Omega_\Delta)t/2} - e^{-i(\Delta-\Omega_\Delta)t/2} \right)$$

$$= \frac{\sqrt{\Delta^2-\Omega_\Delta^2}}{2\Omega_\Delta} \left(e^{-i(\Delta+\Omega_\Delta)t/2} - e^{-i(\Delta-\Omega_\Delta)t/2} \right)$$

$$= \frac{2g\sqrt{n+1}}{2\sqrt{\Delta^2+4g^2(n+1)}} \left(e^{-i(\Delta+\Omega_\Delta)t/2} - e^{-i(\Delta-\Omega_\Delta)t/2} \right)$$

$$\therefore \quad \boxed{C_f(t) = \frac{g\sqrt{n+1}}{\sqrt{\Delta^2 + 4g^2(n+1)}}\left(e^{-i(\Delta+\Omega_\Delta)t/2} - e^{-i(\Delta-\Omega_\Delta)t/2}\right)}$$

final form for excited state probability coefficient !

We find the probability for excited state as a function of time.

$$|C_f(t)|^2 = \left|\frac{g\sqrt{n+1}}{\sqrt{\Delta^2 + 4g^2(n+1)}}\left(e^{-i(\Delta+\Omega_\Delta)t/2} - e^{-i(\Delta-\Omega_\Delta)t/2}\right)\right|^2$$

$$= \frac{g^2(n+1)}{(\Delta^2 + 4g^2(n+1))}\left|\cancel{e^{-i(\Delta+\Omega_\Delta)t/2}}^{\,1}\right|^2 \left|1 - e^{i\Omega_\Delta t}\right|^2$$

$$= \frac{g^2(n+1)}{(\Delta^2 + 4g^2(n+1))}\left|(1-\cos\Omega_\Delta t) + i\sin\Omega_\Delta t\right|^2$$

$$= \frac{g^2(n+1)}{(\Delta^2 + 4g^2(n+1))}\left[(1-\cos\Omega_\Delta t)^2 + \sin^2\Omega_\Delta t\right]$$

$$= \frac{g^2(n+1)}{(\Delta^2 + 4g^2(n+1))}\left[1 - 2\cos\Omega_\Delta t + \cos^2\Omega_\Delta t + \sin^2\Omega_\Delta t\right]$$

$$= \frac{2g^2(n+1)}{(\Delta^2 + 4g^2(n+1))}\left(1 - \cos\Omega_\Delta t\right) \qquad \left[\cos 2\cdot\left(\frac{\Omega_\Delta}{2}\right)t = 1 - 2\sin^2\left(\frac{\Omega_\Delta t}{2}\right)\right]$$

$$= \frac{2g^2(n+1)}{(\Delta^2 + 4g^2(n+1))}\, 2\sin^2\left(\frac{\Omega_\Delta}{2}t\right)$$

$$\boxed{\therefore \; |C_f(t)|^2 = \frac{4g^2(n+1)}{(\Delta^2 + 4g^2(n+1))}\sin^2\left(\frac{1}{2}\sqrt{\Delta^2 + 4g^2(n+1)}\, t\right)} \; := \text{probability amplitude.}$$

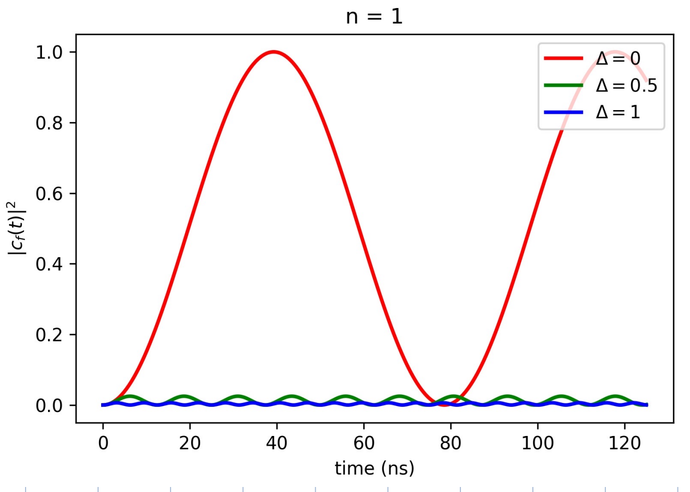
Verify that for detuning $\Delta = 0$; you obtain

$$|C_f(t)|^2 = \sin^2\left(\frac{\Omega_\Delta}{2}t\right) = \sin^2\left(g\sqrt{n+1}\, t\right)$$

(i) $n = 1$ photons in the resonator.

x-axis : time

Y-axis : probability of excited state $|c_f|^2$ as a function of time.

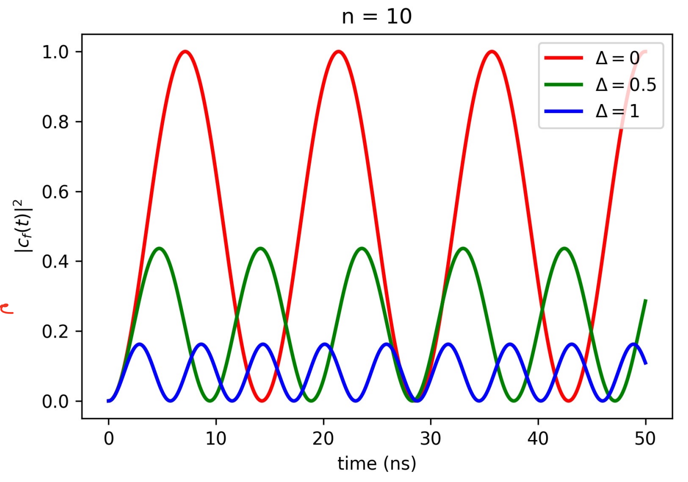


(ii) $n = 10$ photons in the resonator

<u>Observation # 1</u>

Observe what happens as you increase the photon number in the resonator.

(Check the frequency of oscillation)



(iii) $n = 100$ photons in the resonator

<u>Observation # 2</u>

Observe what happens as you increase the detuning

(notice the probability amplitude)