Aalto University
School of Electrical
Engineering

ELEC-E7450
Performance Analysis

Regenerative processes

Samuli Aalto
Department of Communications and Networking

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- Renewal sequences
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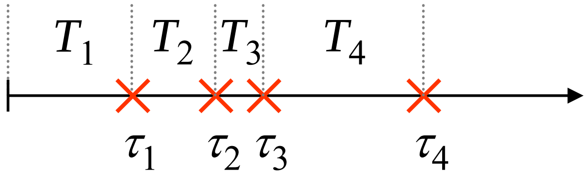
- Regenerative processes

Renewal sequence

- **Definition:**
  Let (T_n) be a sequence of independent and identically distributed (IID) positive random variables. Sequence (τ_n) of random variables defined by

$$\tau_0 := 0, \quad \tau_n := T_1 + \ldots + T_n$$

  is a renewal sequence. Variables T_n are called renewal intervals.



- **Example:**
  Poisson (point) process (τ_n) is a renewal sequence with exponentially distributed intervals T_n.

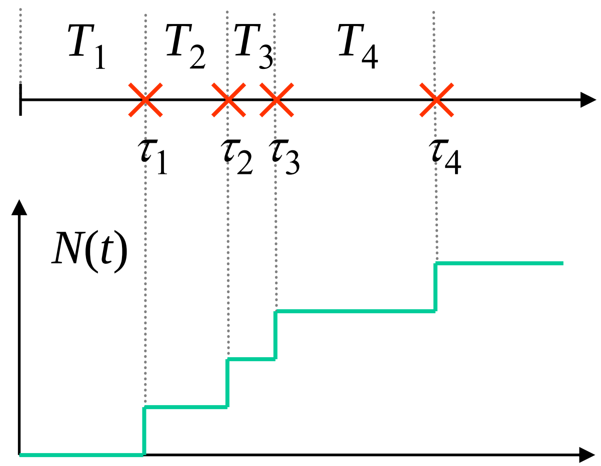
Renewal process

- **Definition:**
  Let (τ_n) be a renewal sequence.
  Counter process $N(t)$ defined by

$$N(0) := 0, \quad N(t) := \sum_{n=1}^{\infty} 1\{\tau_n \leq t\}$$

  is called the corresponding renewal process.

- **Example:**
  Poisson (counter) process $N(t)$ is a renewal process with exponentially distributed intervals T_n.



- Note that

$$\{N(t) \geq n\} = \{\tau_n \leq t\}$$

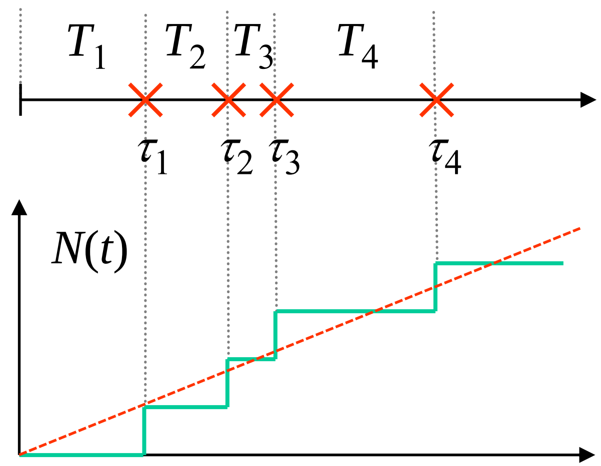
Elementary renewal theorem

- **Proposition:**
  Let (T_n) be an IID sequence with

$$E[T] < \infty$$

  and $N(t)$ the corresponding renewal process. Then

$$\lim_{t\to\infty} \frac{1}{t} N(t) \overset{\text{a.s.}}{=} \lim_{t\to\infty} \frac{1}{t} E[N(t)] = \frac{1}{E[T]}$$



- The latter part is called the elementary renewal theorem.

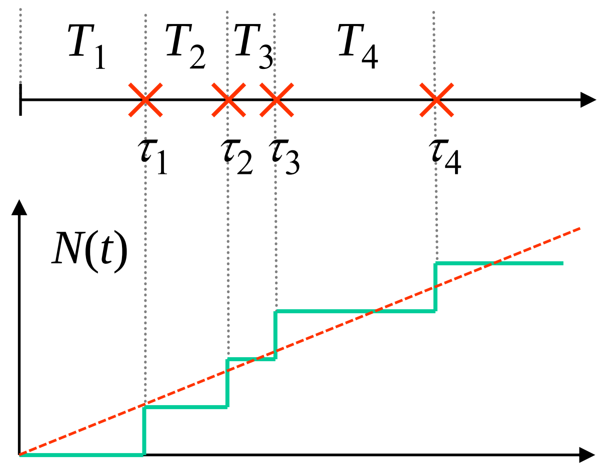
Blackwell's theorem

- **Proposition:**
  Let (T_n) be an IID sequence with a continuous distribution for which

$$E[T] < \infty$$

and $N(t)$ the corresponding renewal process. Then, for any $\Delta > 0$,

$$\lim_{t \to \infty} E[N(t+\Delta) - N(t)] = \frac{\Delta}{E[T]}$$



- This is called Blackwell's theorem.

Stopping times and Wald's equation

- **Definition:**
  Let (T_n) be an IID sequence.
  Random variable N is a stopping time with respect to sequence (T_n) if event $\{N = n\}$ depends on variables $T_1, \ldots, T_n$ but not on $T_{n+1}, T_{n+2}, \ldots$

- Note that $N(t) + 1$ is a stopping time of the IID sequence (T_n), while $N(t)$ is not, since

$$\{N(t) = n\} = \{\tau_n \le t, \tau_{n+1} > t\}$$

$$\{N(t) + 1 = n\} = \{\tau_{n-1} \le t, \tau_n > t\}$$

- **Proposition:**
  Let (T_n) be an IID sequence with

$$E[T] < \infty$$

  and N a stopping time of it. Then

$$E[\sum_{n=1}^{N} T_n] = E[N]E[T]$$

- The result is called Wald's equation.

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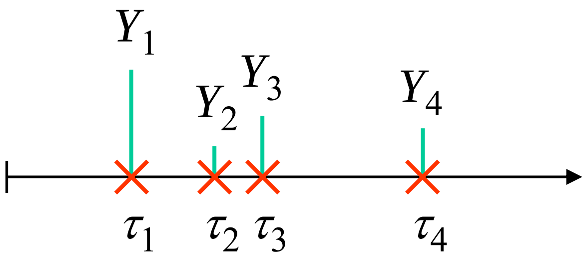
Renewal reward sequence

- **Definition:**
  Let (T_n, Y_n) be an IID sequence of pairs of positive random variables. Sequence (τ_n, Y_n) of pairs of random variables, where

$$\tau_0 := 0, \quad \tau_n := T_1 + \ldots + T_n$$

  is a renewal reward sequence.



- Random variables T_n and Y_n may depend on each other
- Sequence (τ_n) alone is clearly a renewal sequence

Renewal reward process

- **Definition:**
  Let (τ_n, Y_n) be a renewal reward sequence. The cumulative reward process $C(t)$ defined by

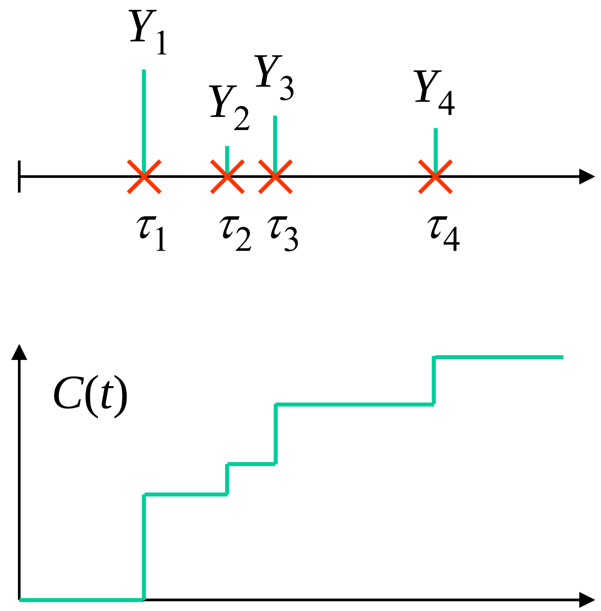
$$C(0) := 0, \quad C(t) := \sum_{n=1}^{\infty} Y_n 1\{\tau_n \le t\}$$

  is called the corresponding renewal reward process.

- Note that

$$C(t) = \sum_{n=1}^{N(t)} Y_n$$

  where $N(t)$ is the corresponding renewal process.



Elementary renewal reward theorem

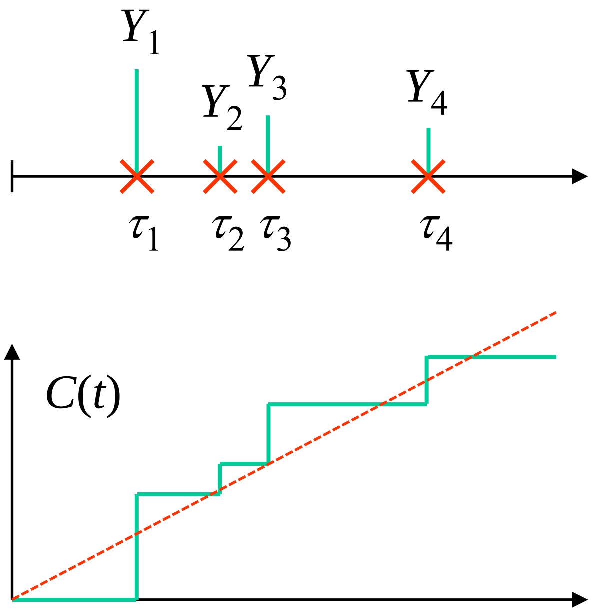
- **Proposition:**
  Let (τ_n, Y_n) be a renewal reward sequence with intervals T_n for which

$$E[T] < \infty$$

  and $C(t)$ the corresponding renewal reward process. Then

$$\lim_{t\to\infty} \frac{1}{t} C(t) \overset{\text{a.s.}}{=} \lim_{t\to\infty} \frac{1}{t} E[C(t)] = \frac{E[Y]}{E[T]}$$

- The latter part is called the elementary renewal reward theorem.



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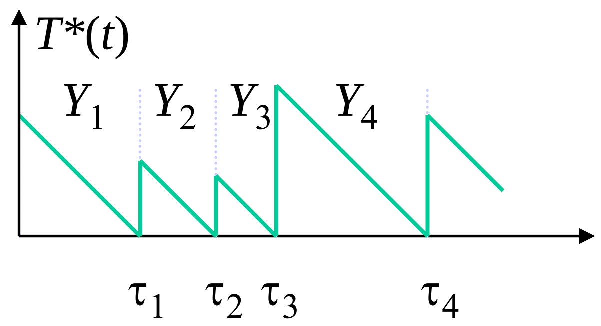
Regenerative process

- Consider a stochastic process $X(t)$, where $X(t) \geq 0$. Let (τ_n) be a renewal sequence with intervals T_n.
- Process $Y_n(t)$ defined by

$$Y_n(t) := X(\tau_{n-1} + t) 1_{\{\tau_{n-1}+t<\tau_n\}}$$

is called the nth cycle of process $X(t)$ and intervals T_n the corresponding cycle lengths.

- **Definition:**
  Process $X(t)$ is regenerative with respect to renewal sequence (τ_n) if cycles $Y_n(t)$ are IID.



- **Example:**
  Let (τ_n) be a renewal sequence. Residual lifetime process $T^*(t)$ with cycles Y_n defined by

$$Y_n(t) := (T_n - t) 1_{\{\tau_{n-1}+t<\tau_n\}}$$

is a regenerative process.

Elementary theorem for regenerative processes

- **Proposition:**
  Consider a regenerative process $X(t)$ with cycle lengths T_n for which

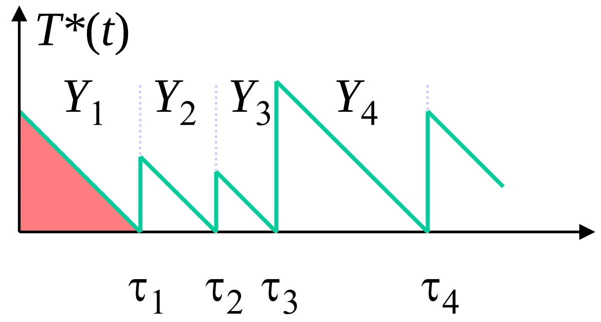
$$E[T] < \infty$$

Let $g(x)$ be a non-negative function and define integral

$$G(t) := \int_0^t g(X(s))ds$$

Then

$$\lim_{t\to\infty} \frac{1}{t} G(t) \overset{\text{a.s.}}{=} \lim_{t\to\infty} \frac{1}{t} E[G(t)] = \frac{E[G(T)]}{E[T]}$$



- In addition, if cycle lengths T_n have a continuous distribution, then

$$\lim_{t\to\infty} E[g(X(t))] = \frac{E[G(T)]}{E[T]}$$

Steady-state mean value

- **Corollary:**
  Consider a regenerative process $X(t)$ with continuously distributed cycle lengths T_n for which

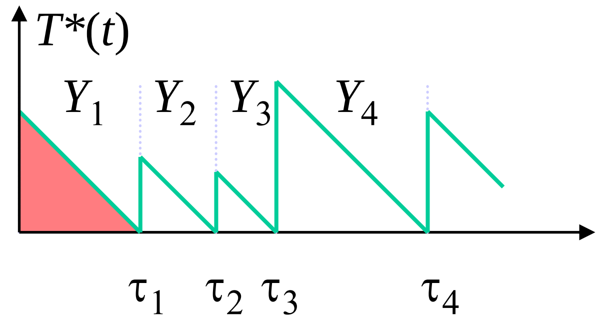
$$E[T] < \infty$$

Then

$$\lim_{t\to\infty} E[X(t)] = \frac{E[\int_0^T X(t)dt]}{E[T]}$$

- Define the steady-state mean value of regenerative process $X(t)$ by

$$E[X] := \frac{E[\int_0^T X(t)dt]}{E[T]}$$



- **Example:**
  The steady-state mean value of the residual lifetime process $T^*(t)$ is

$$E[T^*] = \frac{E[T^2]}{2E[T]}$$

Steady-state distribution

- **Corollary:**
  Consider a regenerative process $X(t)$ with continuously distributed cycle lengths T_n for which

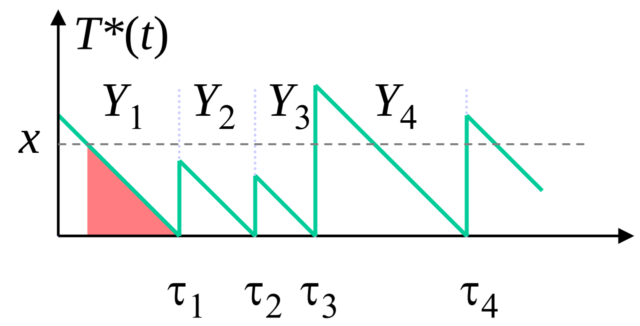
$$E[T] < \infty$$

Then

$$\lim_{t\to\infty} P\{X(t) \le x\} = \frac{E[\int_0^T 1\{X(t)\le x\}dt]}{E[T]}$$

- Define the steady-state distribution of regenerative process $X(t)$ by

$$P\{X \le x\} := \frac{E[\int_0^T 1\{X(t)\le x\}dt]}{E[T]}$$



- **Example:**
  The steady-state distribution of the residual lifetime process $T^*(t)$ is

$$P\{T^* \le x\} = \frac{E[\min\{T,x\}]}{E[T]} = \frac{\int_0^x P\{T>t\}dt}{E[T]}$$

Summary

- Renewal sequences
  - renewal sequence, renewal process, elementary renewal theorem, stopping times, Wald’s equation
- Renewal reward sequences
  - renewal reward sequence, renewal reward process, elementary renewal reward theorem
- Regenerative processes
  - regenerative process, elementary theorem, steady-state mean value, steady-state distribution