Regenerative processes

Samuli Aalto
Department of Communications and Networking
Aalto University

April 21, 2022

1 Renewal sequences

Let (T_n) be a sequence of independent and identically distributed (*IID*) positive random variables. A sequence (τ_n) of random variables defined by

$$\tau_0 := 0, \quad \tau_n := T_1 + \ldots + T_n, \tag{1}$$

is a *renewal sequence*. The counter process $N(t)$ defined by

$$N(0) := 0, \quad N(t) := \sum_{n=1}^{\infty} 1_{\{\tau_n \leq t\}} \tag{2}$$

is called the corresponding *renewal process*.

Note that

$$\{N(t) \geq n\} = \{\tau_n \leq t\}. \tag{3}$$

Proposition 1

Let (T_n) be an IID sequence with $E[T] < \infty$ and $N(t)$ the corresponding renewal process. Then

$$\lim_{t\to\infty} \frac{1}{t} N(t) \stackrel{a.s.}{=} \lim_{t\to\infty} \frac{1}{t} E[N(t)] = \frac{1}{E[T]}. \tag{4}$$

Proof We prove the first part by the Strong Law of Large Numbers (SLLN). Since

$$\tau_{N(t)} \leq t < \tau_{N(t)+1},$$

we have

$$\frac{\tau_{N(t)}}{N(t)} \leq \frac{t}{N(t)} < \frac{\tau_{N(t)+1}}{N(t)} = \frac{\tau_{N(t)+1}}{N(t)+1}\frac{N(t)+1}{N(t)}.$$

Letting $t \to \infty$, we get

$$E[T] \overset{a.s.}{\leq} \lim_{t\to\infty} \frac{t}{N(t)} \overset{a.s.}{\leq} E[T],$$

since, $\tau_n/n \overset{a.s.}{\to} E[T]$ (as $n \to \infty$) and $N(t) \overset{a.s.}{\to} \infty$ (as $t \to \infty$) by SLLN.

The proof of the latter part can be found, e.g., in [1, Sect. 3.5]. □

The latter part is known as the *Elementary Renewal Theorem*, see, e.g., [1, Sect. 3.5].

Proposition 2

Let (T_n) be an IID sequence with a continuous distribution for which $E[T] < \infty$ and $N(t)$ the corresponding renewal process. Then, for any $\Delta > 0$,

$$\lim_{t\to\infty}(E[N(t+\Delta)] - E[N(t)]) = \frac{\Delta}{E[T]}. \tag{5}$$

The result is known as *Blackwell's Theorem*, see, e.g., [1, Sect. 3.5].

Let (T_n) be an IID sequence. Random variable N is a *stopping time* with respect to sequence (T_n) if event $\{N = n\}$ depends on $T_1, \ldots, T_n$ but not on $T_{n+1}, T_{n+2}, \ldots$, for any n.

Note that $N(t) + 1$ is a stopping time of an IID sequence (T_n) while $N(t)$ is not, since

$$\{N(t) = n\} = \{\tau_n \leq t, \tau_{n+1} > t\}$$
$$\{N(t) + 1 = n\} = \{\tau_{n-1} \leq t, \tau_n > t\}.$$

Proposition 3

Let (T_n) be an IID sequence with $E[T] < \infty$ and N a stopping time with respect to (T_n). Then

$$E[\sum_{n=1}^{N} T_n] = E[N]E[T]. \tag{6}$$

Proof By the Monotone Convergence Theorem, we have

$$E[\sum_{n=1}^{N} T_n] = E[\sum_{n=1}^{\infty} T_n 1_{\{N\geq n\}}] = \sum_{n=1}^{\infty} E[T_n 1_{\{N\geq n\}}].$$

Since

$$1_{\{N\geq n\}} = \prod_{i=1}^{n-1}(1 - 1_{\{N=i\}}),$$

and N is a stopping time with respect to sequence (T_n), variable $1_{\{N\geq n\}}$ depends on $T_1, \ldots, T_{n-1}$ but *not* on T_n. Thus,

$$E[\sum_{n=1}^{N} T_n] = \sum_{n=1}^{\infty} E[T_n]E[1_{\{N\geq n\}}] = E[T]\sum_{n=1}^{\infty} P\{N \geq n\} = E[T]E[N],$$

which completes the proof. □

The result above is known as *Wald's equation*, see, e.g., [1, Sect. 3.4]. We note that Wald's equation is also valid when random variable N is independent of sequence (T_n).

2 Renewal reward sequences

Let (T_n, Y_n) be an IID sequence of pairs of positive random variables. A sequence (τ_n, Y_n) of pairs of random variables, where

$$\tau_0 := 0, \quad \tau_n := T_1 + \ldots + T_n, \tag{7}$$

is a *renewal reward sequence*. The cumulative process $C(t)$, defined by

$$C(0) := 0, \quad C(t) := \sum_{n=1}^{\infty} Y_n 1_{\{\tau_n \leq t\}}, \tag{8}$$

is called the corresponding *renewal reward process.*

Random variables T_n and Y_n may be dependent on each other. Note also that sequence (τ_n) alone is a renewal sequence and

$$C(t) = \sum_{n=1}^{N(t)} Y_n, \tag{9}$$

where $N(t)$ is the renewal process corresponding to sequence (τ_n).

Proposition 4

Let (τ_n, Y_n) be a renewal reward sequence with intervals T_n for which $E[T] < \infty$ and $C(t)$ the corresponding renewal reward process. Then

$$\lim_{t\to\infty} \frac{1}{t} C(t) \overset{a.s.}{=} \lim_{t\to\infty} \frac{1}{t} E[C(t)] = \frac{E[Y]}{E[T]}. \tag{10}$$

Proof We prove the first part by the Strong Law of Large Numbers (SLLN) and Proposition 1. Now

$$\frac{1}{t} C(t) = \frac{1}{t} \sum_{n=1}^{N(t)} Y_n = \frac{N(t)}{t} \frac{1}{N(t)} \sum_{n=1}^{N(t)} Y_n$$

Letting $t \to \infty$, we get

$$\lim_{t\to\infty} \frac{1}{t} C(t) \overset{a.s.}{=} \frac{E[Y]}{E[T]},$$

since, $(1/n) \sum_{m=1}^{n} Y_m \overset{a.s.}{\to} E[Y]$ (as $n \to \infty$) and $N(t) \overset{a.s.}{\to} \infty$ (as $t \to \infty$) by SLLN, and $N(t)/t \overset{a.s.}{\to} 1/E[T]$ (as $t \to \infty$) by Proposition 1.

The proof of the latter part can be found, e.g., in [1, Sect. 3.9]. □

3 Regenerative processes

Consider a stochastic process $X(t)$, where $X(t) \geq 0$. Let (τ_n) be a renewal sequence with intervals T_n. The process $Y_n(t)$, defined by

$$Y_n(t) = X(\tau_{n-1} + t) 1_{\{\tau_{n-1}+t<\tau_n\}}, \tag{11}$$

is called the nth *cycle* of process $X(t)$ and intervals T_n the corresponding *cycle lengths*. Process $X(t)$ is *regenerative* with respect to the renewal sequence (τ_n) if cycles $Y_n(t)$ are IID.

Proposition 5

Consider a regenerative process $X(t)$ with cycle lengths T_n for which $E[T] < \infty$. Let $g(x)$ be a non-negative function defined on $[0, \infty)$. Then

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t g(X(s))\,ds \overset{a.s.}{=} \lim_{t\to\infty} \frac{1}{t} E[\int_0^t g(X(s))\,ds] = \frac{E[\int_0^T g(X(t))\,dt]}{E[T]}. \tag{12}$$

In addition, if cycle lengths T_n have a continuous distribution, then

$$\lim_{t\to\infty} E[g(X(t))] = \frac{E[\int_0^T g(X(t))\,dt]}{E[T]}. \tag{13}$$

It follows that, under all assumptions of the previous proposition, the corresponding limiting distribution is well defined,

$$P\{X \le x\} := \lim_{t\to\infty} P\{X(t) \le x\} = \frac{E[\int_0^T 1_{\{X(t)\le x\}}\,dt]}{E[T]}. \tag{14}$$

However, even for non-continuous distributions (e.g., the deterministic distribution) of T_n, for which the limit (13) does not necessarily exist, the steady-state distribution is well defined,

$$P\{X \le x\} := \frac{E[\int_0^T 1_{\{X(t)\le x\}}\,dt]}{E[T]}, \tag{15}$$

referring to the proportion of time that the process spends below level x in the long run. Variable X is called, in any case, the corresponding *steady-state* variable. Its mean value is clearly

$$E[X] = \frac{E[\int_0^T X(t)\,dt]}{E[T]}. \tag{16}$$

References

[1] S.M. Ross, *Applied Probability Models with Optimization Applications*, Holden-Day, 1970 [also available as paperback, Dover, 1992].

[2] S. Asmussen, *Applied Probability and Queues*, Wiley, 1987.