Aalto University
School of Electrical Engineering

ELEC-E7450
Performance Analysis

Queueing networks

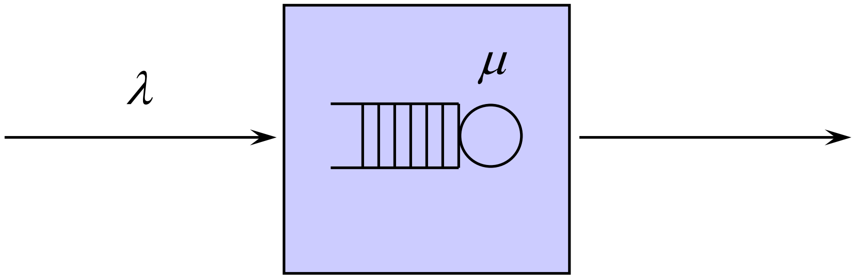
Samuli Aalto
Department of Communications and Networking

Contents

- Burke’s theorem
- Open queueing networks
- Closed queueing networks

M/M/1

- Customers arrive according to a Poisson process at rate λ
  - IID inter-arrival times
  - exponential inter-arrival time distribution with mean $1/\lambda$
- Customers are served by 1 server
  - IID service times
  - exponential service time distribution with mean $E[S] = 1/\mu$
- There are ∞ customer places in the system



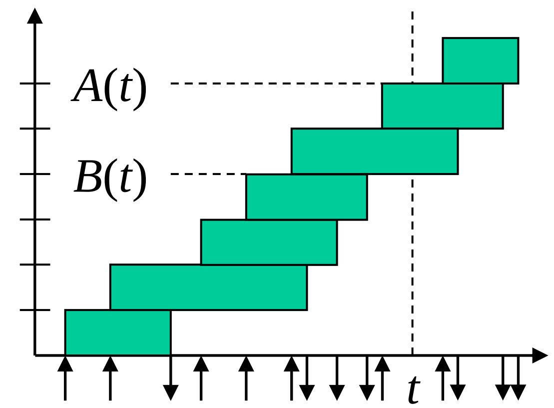
Departure process

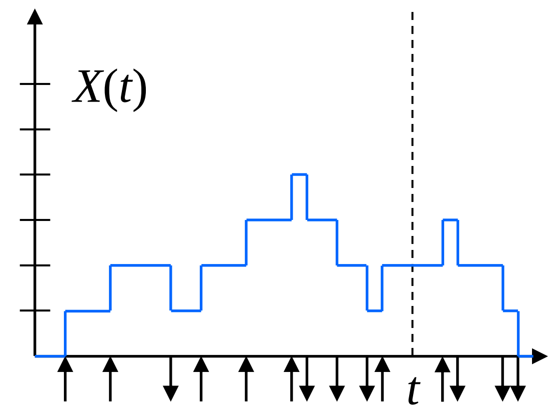
- $A(t)$ = number of arrivals until time t
  = arrival process
- $B(t)$ = number of departures until time t
  = departure process

$$A(t) := \max\{i : \alpha_i \le t\}$$
$$B(t) := |\{i : \beta_i \le t\}|$$

- $X(t)$ = number of customers at time t
  = queue length process

$$X(t) := A(t) - B(t)$$

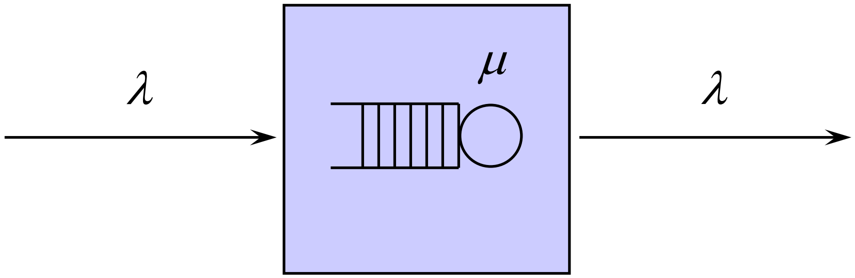




Burke's theorem

- **Theorem**:
  Consider a stationary M/M/1 queue ($\rho < 1$).
  - (i) The departure process $B(t)$ is a Poisson process at rate λ.
  - (ii) For any t, the queue length $X(t)$ at time t is independent of the departure process $(B(s); s < t)$ prior to time t.

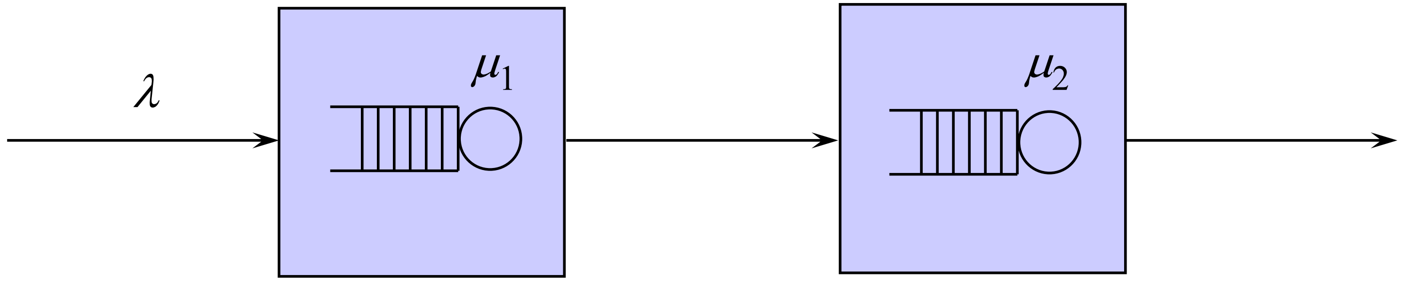
- The result can be proved by a reversibility argument, see Kelly (1979)



Tandem queue

- Customers arrive according to a Poisson process at rate λ
- There are M single-server queues in tandem
  - departure process $B_i(t)$ of queue i is the arrival process $A_{i+1}(t)$ of queue $i+1$
  - independent and exponentially distributed service times S_i in each queue i with mean $E[S_i] = 1/\mu_i$
- Let ρ_i denote the load in queue i defined by

$$\rho_i := \frac{\lambda}{\mu_i}$$



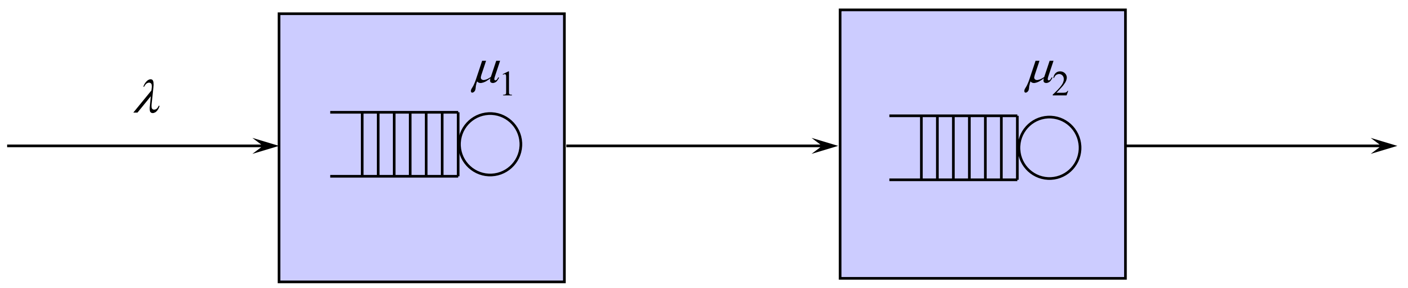
Steady-state queue length distribution

- Let $N_i(t)$ denote the number of customers in queue i at time t

- **Corollary**:
  Consider a tandem queue. If $\rho_i < 1$ for all i, then the system is stable and the steady-state queue length distribution is

$$P\{N_1 = n_1, \ldots, N_M = n_M\} = \prod_{i=1}^{M} (1-\rho_i)\rho_i^{n_i}$$

- **Note**: Both queues behave as independent M/M/1 queues



Contents

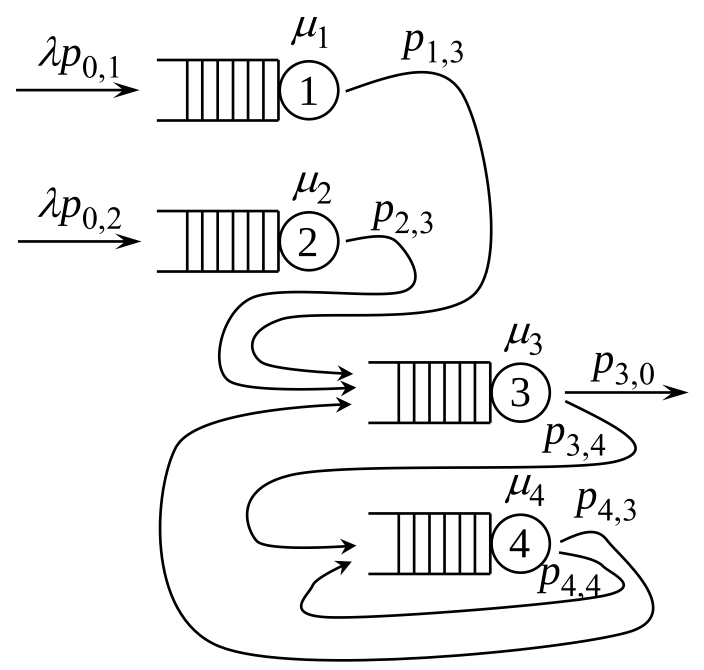
Jackson network

- Open network of M single-server queues
- Arrivals from outside to queue i:
  - independent Poisson process at rate $\lambda p_{0,i}$
- Service times in queue i:
  - IID exponentially distributed with mean $E[S_i] = 1/\mu_i$
- Moving from queue i to queue j:
  - after service completion with probability $p_{i,j}$
- Departures to outside from queue i:
  - after service completion with probability $p_{i,0}$

$$\sum_{i=1}^{M} p_{0,i} = 1$$

$$\sum_{j=0}^{M} p_{i,j} = 1$$

$$\sum_{i=1}^{M} p_{i,0} > 0$$



Assumption: Routing probabilities are such that each customer eventually leaves the network.

Queue length process

- Denote

$$N(t) = (N_1(t), \ldots, N_M(t))$$

where $N_i(t)$ refers to the total number of customers in queue i at time t

- Process $N(t)$ is an irreducible Markov process with state space

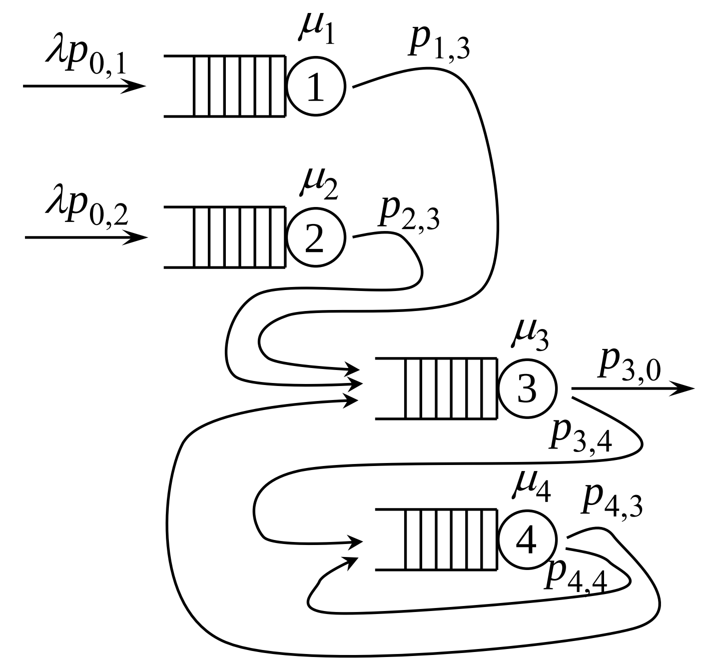
$$S = \{n = (n_1, \ldots, n_M) \mid n_i \in \{0,1,2,\ldots\}\}$$

and transition rates

$$q(n, n + e_i) = \lambda p_{0,i}$$

$$q(n + e_i, n + e_j) = \mu_i p_{i,j}$$

$$q(n + e_i, n) = \mu_i p_{i,0}$$



Flow conservation equations

- Let θ_i denote the average rate at which customers leave queue i, (i.e., throughput)
- Single-server queue i is stable if and only if

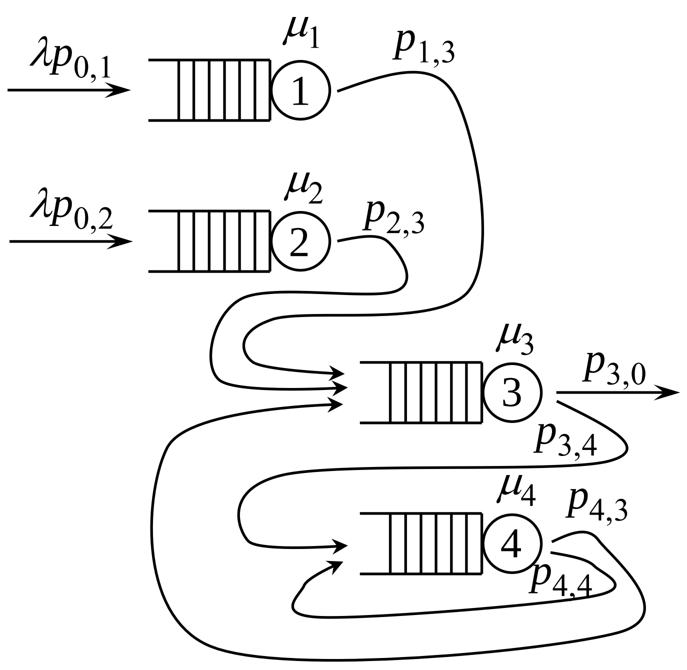
$$\theta_i < \mu_i$$

- For a stable system, the θ_j clearly satisfy the following flow conservation equations (FCE) for any j:

$$\theta_j = \lambda p_{0,j} + \sum_{i=1}^{M} \theta_i p_{i,j}$$

- This is a linear system of equations with a unique solution
- By summing up all (FCE) equations, we get

$$\lambda = \sum_{i=1}^{M} \theta_i p_{i,0}$$

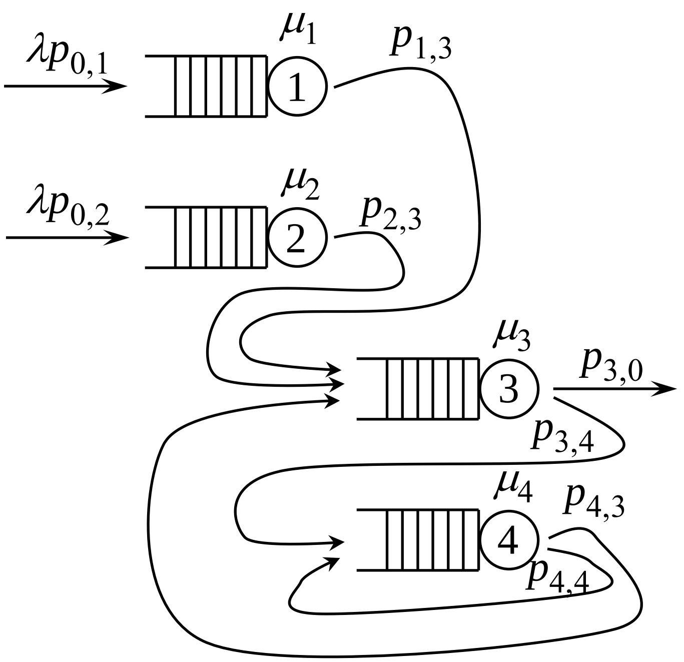


Note that (FCE) equations do not depend on the service rates μ_i.

Stability

- **Proposition:**
  Consider a Jackson network. The system is stable if and only if the θ_i uniquely determined from the flow conservation equations (FCE) satisfy

$$\rho_i := \frac{\theta_i}{\mu_i} < 1 \quad \text{for all } i$$



It follows that, in a stable system, the throughputs θ_i do not depend on the service rates μ_i.

Jackson's theorem (1)

- **Theorem:**
  Consider a stable Jackson network. The steady-state distribution of process $N(t)$ is

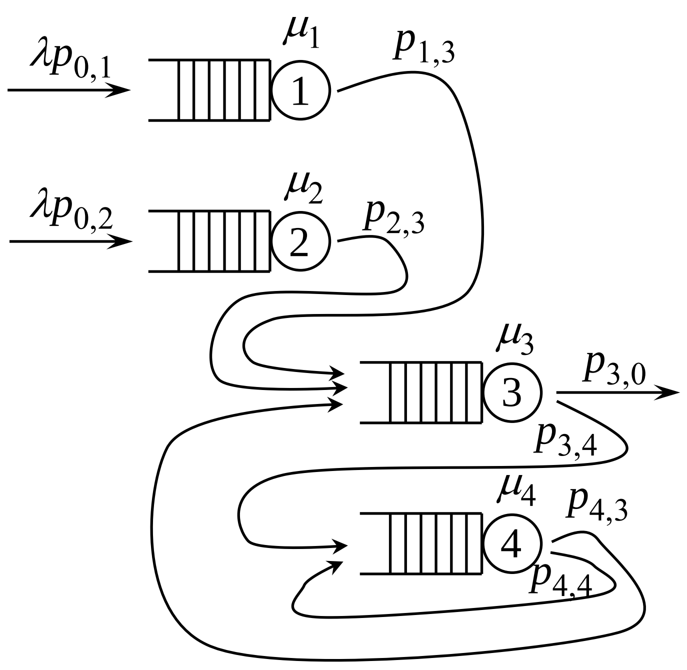
$$P\{N=n\}=\prod_{i=1}^{M}(1-\rho_i)\rho_i^{n_i}$$

- This result is known as Jackson's theorem

- **Note:**
  In steady state, all queues behave as independent M/M/1 queues

- Note also that for any i, we have the following recursive equation:

$$P\{N=n+e_i\}=P\{N=n\}\rho_i$$



Jackson's theorem (2)

- **Corollary:**
  Consider a stable Jackson network. The steady-state queue length N_i of queue i satisfies

$$P\{N_i = n_i\} = (1-\rho_i)\rho_i^{n_i}$$

$$E[N_i] = \frac{\rho_i}{1-\rho_i}$$

- Let $X(t)$ denote the total number of customers in the whole network at time t,

$$X(t) = N_1(t) + \ldots + N_M(t)$$

- Let X and T denote the steady-state variables for the total number of customers in the whole network and the time that a customer spends in the whole system, respectively.

- **Corollary:**
  For a stable Jackson network,

$$E[X] = \sum_{i=1}^{M} \frac{\rho_i}{1-\rho_i}$$

$$E[T] = \sum_{i=1}^{M} \frac{\theta_i}{\lambda} \frac{1}{\mu_i - \theta_i}$$

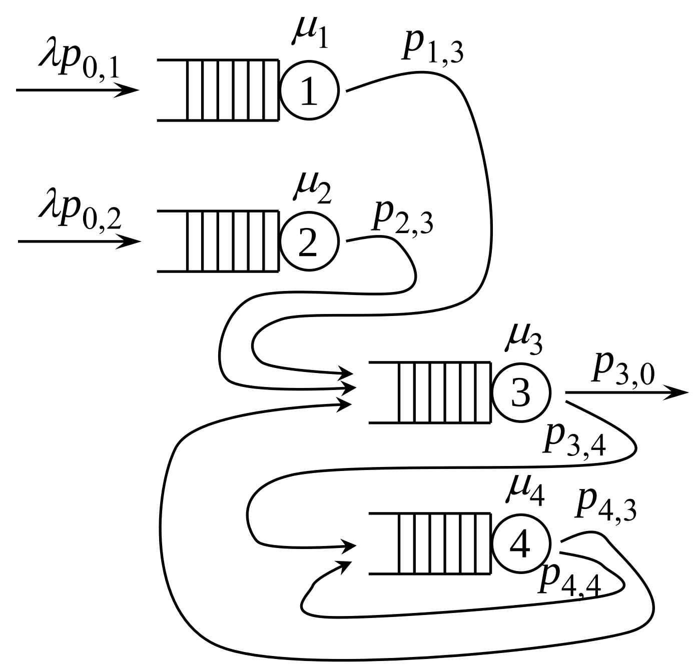
- Note that θ_i/λ can be interpreted as the mean number of visits to queue i

Arrival theorem

- Let N^* denote the steady-state variable describing the state of the system seen by a customer entering any queue (with the entering customer excluded).

- **Theorem:**
  Consider a stable Jackson network. The steady-state distribution seen by a customer entering any queue is the same as the steady-state distribution of the Markov process $N(t)$,

$$P\{N^* = n\} = P\{N = n\}$$



- This result is known as the arrival theorem for Jackson networks

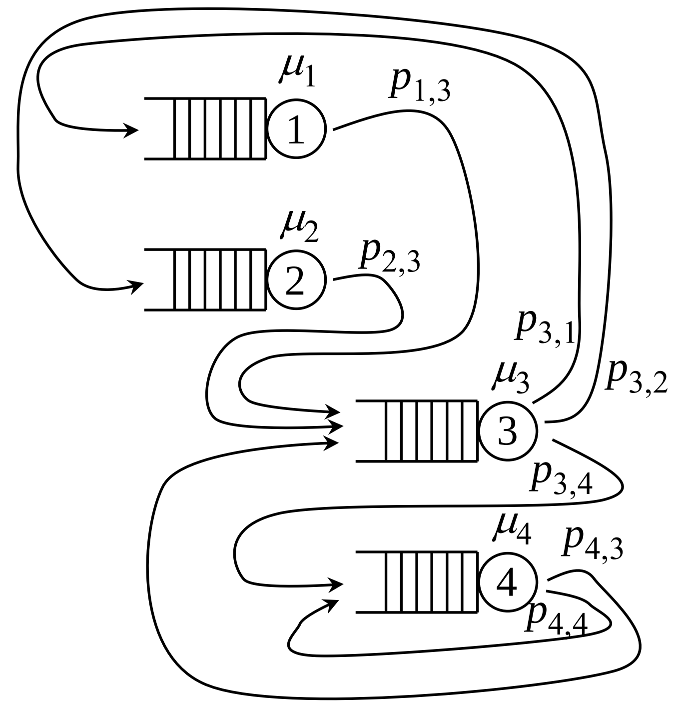
Contents

- Burke’s theorem
- Open queueing networks
- Closed queueing networks

Gordon-Newell network

- Closed network of M single-server queues
- No arrivals from outside but a fixed number of customers denoted by K
- Service times in queue i:
  - IID exponentially distributed with mean $E[S_i] = 1/\mu_i$
- Moving from queue i to queue j:
  - after service completion with probability $p_{i,j}$
- No departures to outside

$$\sum_{j=1}^{M} p_{i,j} = 1$$



Assumption: Routing probabilities are such that each customer finally visits each queue.

Queue length process

- Denote

$$N(t) = (N_1(t), \ldots, N_M(t))$$

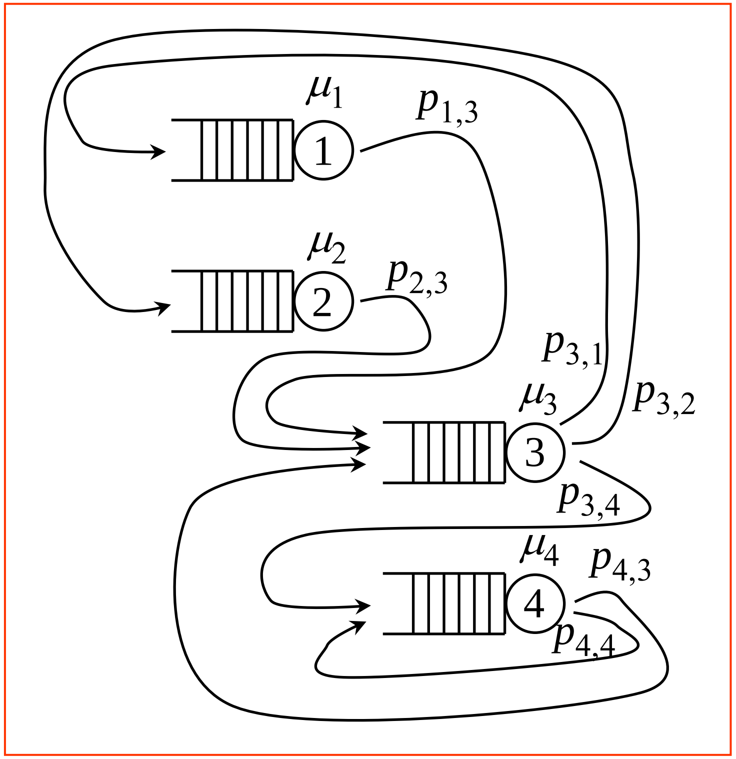
where $N_i(t)$ refers to the total number of customers in queue i at time t

- Process $N(t)$ is an irreducible Markov process with a finite state space

$$S_K = \{n = (n_1, \ldots, n_M) \mid n_1 + \ldots + n_M = K; \; n_i \in \{0,1,2,\ldots\}\}$$

and transition rates

$$q(\tilde{n} + e_i, \tilde{n} + e_j) = \mu_i p_{i,j}, \quad \tilde{n} \in S_{K-1}$$



- **Note:**
  The system is always stable and process $N(t)$ has a unique equilibrium distribution

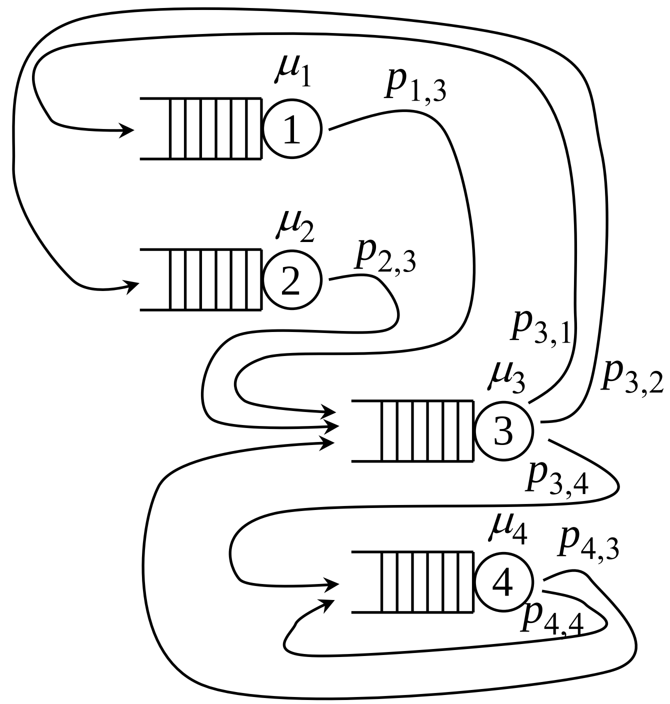
Flow conservation equations (1)

- Let θ_i denote the average rate at which customers leave queue i, (i.e., throughput)
- The θ_j clearly satisfy the following flow conservation equations (FCE) for any j:

$$\theta_j = \sum_{i=1}^{M} \theta_i p_{i,j}$$

- This is a linear system of equations with multiple solutions: for any constant c, these equations are solved by the vector

$$(c\theta_1, \ldots, c\theta_M)$$



Note that (FCE) equations do not depend on the service rates μ_i.

Flow conservation equations (2)

- Let

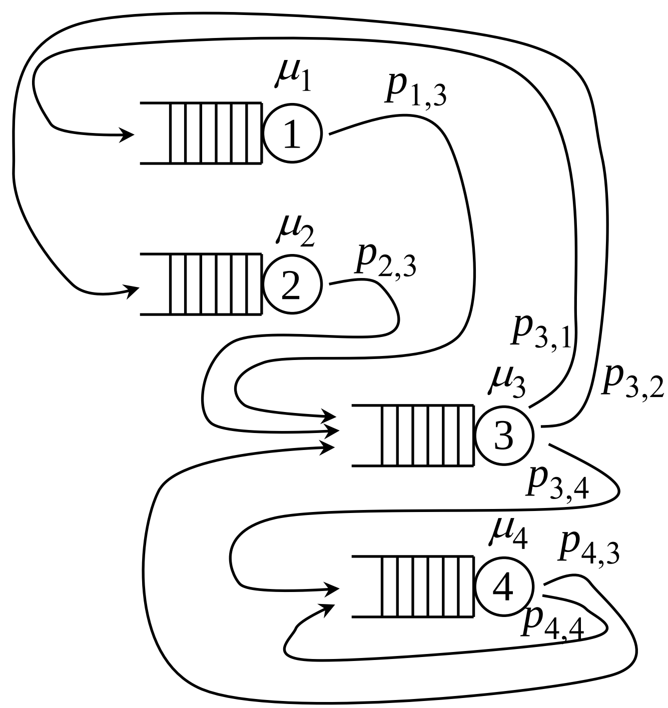
$$(\hat{\theta}_1, \ldots, \hat{\theta}_M)$$

be any non-zero solution of the flow conservation equations (FCE), and denote

$$\hat{\rho}_i := \frac{\hat{\theta}_i}{\mu_i} \quad \text{for all } i$$

- Note that there is $\hat{c}$ such that

$$\hat{\theta}_i = \hat{c}\,\theta_i \quad \text{for all } i$$



It follows that the values $\hat{\theta}_i$ do not depend on the service rates μ_i.

Gordon-Newell theorem

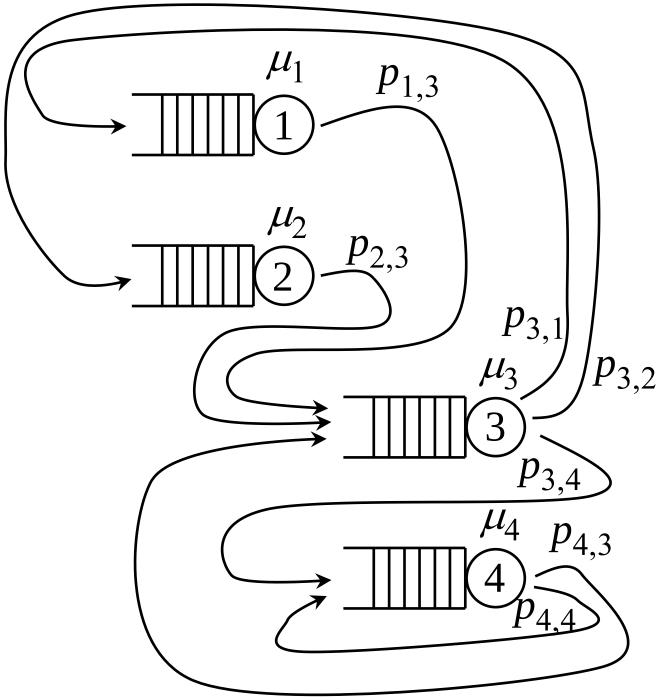
- **Theorem:**
  Consider a Gordon-Newell network. The steady-state distribution of process $N(t)$ is

$$P\{N=n\}=\frac{1}{\hat{G}_K}\prod_{i=1}^{M}\hat{\rho}_i^{n_i}$$

where

$$\hat{G}_K := \sum_{n'\in S_K}\prod_{i=1}^{M}\hat{\rho}_i^{n_i'}$$

- This result is known as the Gordon-Newell theorem



- **Note:**
  In steady state, the queues are not independent but the steady-state probability is still of product-form.

Arrival theorem (1)

- Let N^* denote the steady-state variable describing the state of the system seen by a customer entering any queue (with the entering customer excluded).
- Note that

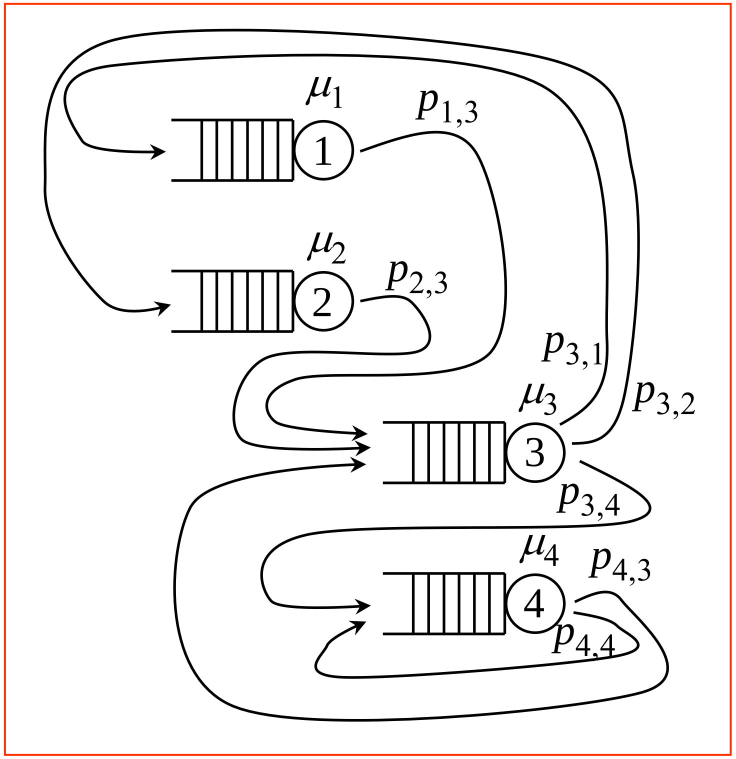
$$N^* \in S_{K-1}$$

- In addition, let $\tilde{N}$ denote the steady-state number of customers in the corresponding Gordon-Newell network where there are $K-1$ customers (instead of K),

$$\tilde{N} \in S_{K-1}$$

- Note that, for any i, we have

$$P\{N = \tilde{n} + e_i\}\hat{G}_K = P\{\tilde{N} = \tilde{n}\}\hat{\rho}_i\hat{G}_{K-1}$$

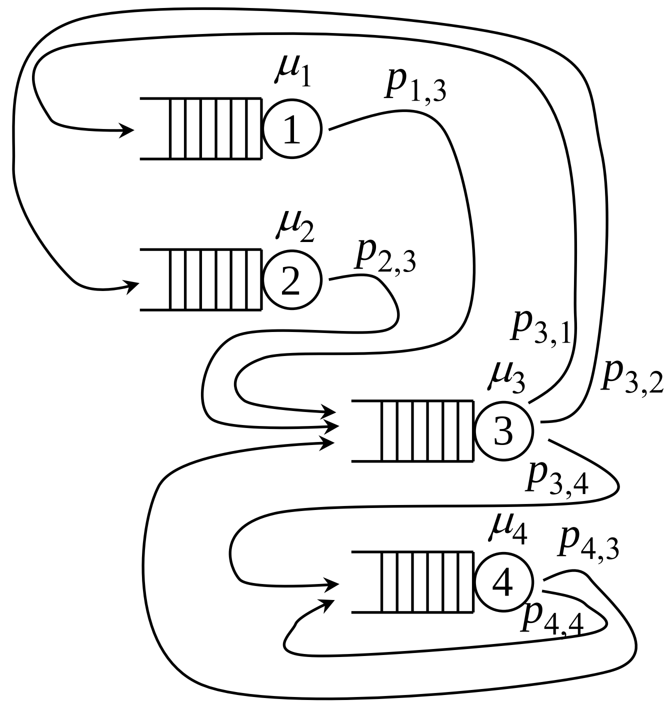


Arrival theorem (2)

- **Theorem:**
  Consider a Gordon-Newell network. The steady-state distribution seen by a customer entering any queue is the same as the steady-state distribution of the Markov process $\tilde{N}(t)$,

$$P\{N^* = \tilde{n}\} = P\{\tilde{N} = \tilde{n}\}$$

- This result is known as the arrival theorem for Gordon-Newell networks



Mean value analysis (MVA)

- Let $\bar{N}_i(k)$ denote the steady-state mean number of customers in queue i in a corresponding Gordon-Newell network where there are k customers.
- In addition, let $\bar{T}_i(k)$ denote the steady-state mean value of the time that a customer spends in queue i during one visit in such a network.
- Finally, let $\theta_i(k)$ denote the throughput of queue i in such a network.
- The following result gives a recursive method, known as Mean Value Analysis (MVA), to calculate these steady-state mean values

- **Theorem:**
For a Gordon-Newell network, we have the following recursive formulas for the steady-state mean values:

$$\bar{T}_i(k) = (1 + \bar{N}_i(k-1))\frac{1}{\mu_i}$$

$$\bar{N}_i(k) = \frac{k\hat{\theta}_i \bar{T}_i(k)}{\sum_{j=1}^{M} \hat{\theta}_j \bar{T}_j(k)}$$

$$\theta_i(k) = \frac{\bar{N}_i(k)}{\bar{T}_i(k)}$$

with initial value

$$\bar{N}_i(0) = 0$$

Summary

- Burke’s theorem
  - M/M/1, Poisson departure process, reversibility, tandem queue, independent M/M/1 queues, product-form steady-state distribution
- Open queueing networks
  - Jackson network, FCE, stability, Jackson’s theorem, independent M/M/1 queues, product-form steady-state distribution, SBE, arrival theorem
- Closed queueing networks
  - Gordon-Newell network, FCE, Gordon-Newell theorem, product-form steady-state distribution, SBE, arrival theorem, MVA