Queueing networks

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1 Burke's theorem

Consider an $M/M/1$ queue with arrival rate $\lambda > 0$ and mean service time $E[S] = 1/\mu$. Let $\rho := \lambda/\mu$ denote the load of the system. In addition, let $X(t)$ and $B(t)$ denote the queue length process and the departure process, respectively, at time t.

From the theory of single server queues we know that the average departure rate, i.e., the throughput θ equals the arrival rate λ whenever the system is stable ($\rho < 1$). *Burke's theorem* below gives even a much stronger result that characterizes the departure process from an M/M/1 queue.

Theorem 1

Consider a stationary M/M/1 queue with $\rho < 1$.

(i) The departure process $B(t)$ is a Poisson process with intensity λ.

(ii) For any t, the queue length $X(t)$ at time t is independent of the departure process $(B(s); s < t)$ prior to time t.

Proof This can be proved by a *reversibility* argument, see [6]. □

Consider now a tandem system of two queues where the departure process $B_1(t)$ of queue 1 is the arrival process $A_2(t)$ of queue 2. In addition, assume that the arrival process to queue 1 is a Poisson process with rate λ and service times in queue i are independent and exponentially distributed with mean $E[S_i] = 1/\mu_i$. As an immediate consequence of the first result in Burke's theorem, we obtain that both queues are M/M/1 queues. Moreover, from the second result, we can deduce that the queue lengths $N_1(t)$ and $N_2(t)$ in the two queues at time t are independent The same must be valid for the corresponding steady-state variables N_1 and N_2 assuming that both queues are stable. Thus, we have

$$P\{N_1 = n_1, N_2 = n_2\} = P\{N_1 = n_1\}P\{N_2 = n_2\} = (1-\rho_1)\rho_1^{n_1}(1-\rho_2)\rho_2^{n_2}$$

where $\rho_i := \lambda/\mu_i$. This can easily be generalized to a tandem system with any number of queues.

2 Open queueing networks

Consider a network of single server queues, each with an infinite number of customer places. Let M denote the total number of such queues. Service times in queue $i \in \{1, \dots, M\}$ are assumed to be independent and exponentially distributed with mean $E[S_i] = 1/\mu_i$. New customers arrive from outside according to a Poisson process with rate λ. An arriving customer is routed to server i with probability $p_{0,i}$, where

$$\sum_{i=1}^{M} p_{0,i} = 1.$$

Customers are served according to the FIFO service discipline. When the service in queue $i \in \{1, \dots, M\}$ is completed, the served customer is either routed to queue $j \in \{1, \ldots, M\}$, which happens with probability $p_{i,j}$, or it leaves the whole network, which happens with with probability $p_{i,0}$, where

$$\sum_{j=0}^{M} p_{i,j} = 1, \quad \sum_{i=0}^{M} p_{i,0} > 0.$$

All routing decisions are assumed to be independent. In addition, the routing probabilities are assumed to be such that each customer finally leaves the network. This kind of a queueing network is called a *Jackson network*.

Note that the network is *open* in the sense that there are new arrivals from outside and each customer leaves the system with probability 1. Thus, the total number of customers in the whole network is varying randomly.

The state of the whole network is described by vector

$$N(t) = (N_1(t), \ldots, N_M(t)),$$

where $N_i(t)$ refers to the number of customers in queue i at time t. The state space is clearly

$$\mathcal{S} = \{n = (n_1, \ldots, n_M); n_i \in \{0, 1, \ldots\}\}.$$

In addition, let e_i denote the unit vector to direction i in this space, $e_i = (n_1, \ldots, n_M)$ with $n_i = 1$ and $n_j = 0$ for $j \neq i$.

Due to the exponential assumptions made above, $N(t)$ is an irreducible Markov process with the following (positive) state transition rates for any $n \in \mathcal{S}$ and $i, j \in \{1, \ldots, M\}$:

$$\begin{aligned} q(n, n + e_i) &= \lambda p_{0,i}, \\ q(n + e_i, n + e_j) &= \mu_i p_{i,j} \\ q(n + e_i, n) &= \mu_i p_{i,0}. \end{aligned}$$

2.1 Flow conservation equations

Let θ_i denote the average rate at which customers leave queue i, i.e., its *throughput*. From the analysis of a single server queue, we know that queue i is stable if and only if $\theta_i < \mu_i$. Thus, the whole system will be stable if this is the case for all i. Equivalently, we can say that the system is stable if, for all $i \in \{1, \ldots, M\}$,

$$\rho_i < 1, \tag{1}$$

where we have defined

$$\rho_i := \frac{\theta_i}{\mu_i}.$$

For a stable system, the throughputs θ_j clearly satisfy the following *flow conservation equations* (FCE) for any $j \in \{1, \ldots, M\}$:

$$\theta_j = \lambda p_{0,j} + \sum_{i=1}^{M} \theta_i p_{i,j}. \tag{2}$$

Equations (2) constitute a linear system of equations, which have a unique solution $(\theta_1, \ldots, \theta_M)$.

Proposition 1

Consider a Jackson network. The system is stable if and only if the θ_i uniquely determined from (2) satisfy (1) for all $i \in \{1, \ldots, M\}$.

Note that the flow conservation equations (2) depend on the number of stations and their topology (via the routing probabilities $p_{i,j}$) but not at all on the service rates μ_i. It follows that the throughputs θ_i are independent of service rates μ_i in a stable system.

Note also that, by summing up all equations (2), we get

$$\sum_{j=1}^{M} \theta_j = \lambda + \sum_{i=1}^{M} \theta_i (1 - p_{i,0}),$$

which results in the following additional flow conservation equation:

$$\lambda = \sum_{i=1}^{M} \theta_i p_{i,0}. \tag{3}$$

So we see that in a stable system the arrival rate into the network equals the departure rate out of there, as it should be.

2.2 Jackson's theorem

Below we show that in the steady state the system behaves as M independent M/M/1 queues, which is known as *Jackson's theorem.*

Theorem 2

Consider a stable Jackson network. The steady-state distribution of process $N(t)$ is given by

$$P\{N = n\} = \prod_{i=1}^{M} (1 - \rho_i)\rho_i^{n_i}, \quad n \in \mathcal{S}.$$

Proof Let $n \in \mathcal{S}$ and denote

$$\pi(n) := \prod_{i=1}^{M} (1 - \rho_i)\rho_i^{n_i}.$$

For any $i \in \{1, \ldots, M\}$, we have recursion

$$\pi(n + e_i) = \pi(n)\rho_i. \tag{4}$$

Since $\pi(n)$ is a product of M geometric probabilities, the normalization condition (N) is clearly satisfied. It remains to prove that the global balance equations (GBE) are also satisfied for any $n \in \mathcal{S}$:

$$\sum_{n' \neq n} \pi(n)q(n, n') = \sum_{n' \neq n} \pi(n')q(n', n). \tag{5}$$

Let $n \in \mathcal{S}$ and $j \in \{1, \ldots, M\}$. Since $\theta_i = \rho_i \mu_i$ for all i, it follows from the flow conservation equation (2) that

$$\rho_j \mu_j (1 - p_{j,j}) = \lambda p_{0,j} + \sum_{i \neq j} \rho_i \mu_i p_{i,j}.$$

By multiplying both sides by $\pi(n)$ and applying recursion (4), we get

$$\pi(n + e_j) \mu_j (1 - p_{j,j}) = \pi(n) \lambda p_{0,j} + \sum_{i \neq j} \pi(n + e_i) \mu_i p_{i,j},$$

which is equivalent with

$$\pi(n + e_j) \left(q(n + e_j, n) + \sum_{i \neq j} q(n + e_j, n + e_i) \right) = \pi(n) q(n, n + e_j) + \sum_{i \neq j} \pi(n + e_i) q(n + e_i, n + e_j). \tag{6}$$

On the other hand, it follows from the flow conservation equation (3) that

$$\lambda = \sum_{i=1}^{M} \rho_i \mu_i p_{i,0}.$$

By again multiplying both sides by $\pi(n)$ and applying recursion (4), we get

$$\pi(n) \lambda = \sum_{i=1}^{M} \pi(n + e_i) \mu_i p_{i,0},$$

which is equivalent with

$$\pi(n) \sum_{i=1}^{M} q(n, n + e_i) = \sum_{i=1}^{M} \pi(n + e_i) q(n + e_i, n). \tag{7}$$

Equations (6) and (7), which together are called *station balance equations* (SBE), are thus true for any $n \in \mathcal{S}$. Note that equation (6) corresponds to transitions out of state $n+e_j$ and into that state generated by a customer that leaves or enters queue j, respectively. Similarly, equation (7) corresponds to transitions out of state n and into that state generated by a customer that enters or leaves the whole network, respectively. The global balance equations (5) follow from these station balance equations in a straightforward way by summing up the related SBE's. □

Corollary 1

Consider a stable Jackson network. For any $i \in \{1, \ldots, M\}$,

$$P\{N_i = n_i\} = (1 - \rho_i)\rho_i^{n_i}, \quad n_i \in \{0, 1, \ldots\},$$
$$E[N_i] = \frac{\rho_i}{1 - \rho_i}.$$

Let $X(t) = N_1(t) + \ldots + N_M(t)$ and X denote the total number of customers in the whole network at time t and the corresponding steady-state variable, respectively. In addition, let T denote the steady-state variable for the total time that a customer spends in the whole network.

Corollary 2

For a stable Jackson network,

$$E[X] = \sum_{i=1}^{M} \frac{\rho_i}{1 - \rho_i}, \quad E[T] = \sum_{i=1}^{M} \frac{\theta_i}{\lambda} \frac{1}{\mu_i - \theta_i}.$$

Note that θ_i / λ can be interpreted as the mean number of visits to queue i (during the time that a customer spends in the whole network).

2.3 Arrival theorem

Let N^* denote the steady-state variable describing the state of the system "seen" by a customer entering any queue (with the entering customer excluded). Below we show that the entering customer sees the system in equilibrium. Note that, for external arrivals, this can be justified by PASTA but not for internal movements from one queue to another. The result is known as the *arrival theorem* for Jackson networks.

Theorem 3

Consider a stable Jackson network. The steady-state distribution seen by a customer entering any queue is the same as the steady-state distribution of process $N(t)$, i.e.,

$$P\{N^* = n\} = P\{N = n\}, \quad n \in \mathcal{S}.$$

Proof Let $n \in \mathcal{S}$. Consider first a customer entering from outside to some fixed queue j with $p_{0,j} > 0$. Now

$$\begin{aligned} &P\{N^* = n \mid \text{arrival from outside to queue } j\} \\ &\quad = \frac{P\{N = n\}q(n, n + e_j)}{\sum_{n' \in \mathcal{S}} P\{N = n'\}q(n', n' + e_j)} \\ &\quad = \frac{P\{N = n\}\lambda p_{0,j}}{\sum_{n' \in \mathcal{S}} P\{N = n'\}\lambda p_{0,j}} \\ &\quad = \frac{P\{N = n\}}{\sum_{n' \in \mathcal{S}} P\{N = n'\}} = P\{N = n\}. \end{aligned}$$

If a customer moves from some fixed queue i to queue j with $p_{i,j} > 0$, we have

$$\begin{aligned} &P\{N^* = n \mid \text{move from queue } i \text{ to queue } j\} \\ &\quad = \frac{P\{N = n + e_i\}q(n + e_i, n + e_j)}{\sum_{n' \in \mathcal{S}} P\{N = n' + e_i\}q(n' + e_i, n' + e_j)} \\ &\quad = \frac{P\{N = n + e_i\}\mu_i p_{i,j}}{\sum_{n' \in \mathcal{S}} P\{N = n' + e_i\}\mu_i p_{i,j}} \\ &\quad = \frac{P\{N = n + e_i\}}{\sum_{n' \in \mathcal{S}} P\{N = n' + e_i\}} \\ &\quad = \frac{P\{N = n\}\rho_i}{\sum_{n' \in \mathcal{S}} P\{N = n'\}\rho_i} = P\{N = n\}, \end{aligned}$$

where the second last equality follows from (4). Thus, for any j, we have

$$P\{N^* = n \mid \text{customer enters queue } j\} = P\{N = n\},$$

which proves the claim. □

Another method to prove the arrival theorem is to (i) insert an additional single-server queue, say 0, with service rate μ_0, (ii) route all customers whose destination is queue j via this additional queue 0, and (iii) finally let $\mu_0 \to \infty$.

3 Closed queueing networks

From this on, we exclude the external arrivals and departures, which results in a system where K customers move in a *closed* network of M single-server queues, each with an infinite number of customer places. As for the Jackson networks, we assume that service times in each queue i are independent and exponentially distributed with mean $E[S_i] = 1/\mu_i$. Customers are served according to the FIFO service discipline. When the service in queue $i \in \{1, \ldots, M\}$ is completed, the served customer is routed to queue $j \in \{1, \ldots, M\}$ with probability $p_{i,j}$, where

$$\sum_{j=1}^{M} p_{i,j} = 1.$$

All routing decisions are assumed to be independent. In addition, the routing probabilities are assumed to be such that each customer finally visits each queue. This kind of a queueing network is called a *Gordon-Newell network.*

The state of the whole network is described by vector

$$N(t) = (N_1(t), \ldots, N_M(t)),$$

where $N_i(t)$ refers to the number of customers in queue i at time t. The state space is clearly

$$\mathcal{S}_K = \{n = (n_1, \ldots, n_M); n_1 + \ldots + n_M = K, n_i \in \{0, 1, \ldots\}\}.$$

In addition, let e_i denote the unit vector to direction i in this space, $e_i = (n_1, \ldots, n_M)$ with $n_i = 1$ and $n_j = 0$ for $j \neq i$.

Note that any state $n \in \mathcal{S}_K$ can be represented as

$$n = \tilde{n} + e_i,$$

where $\tilde{n} \in \mathcal{S}_{K-1}$ and $i \in \{1, \ldots, M\}$, and vice versa.

Due to the exponential assumptions made above, $N(t)$ is an irreducible Markov process with the following (positive) state transition rates for any $\tilde{n} \in \mathcal{S}_{K-1}$ and $i, j \in \{1, \ldots, M\}$:

$$q(\tilde{n} + e_i, \tilde{n} + e_j) = \mu_i p_{i,j}$$

3.1 Flow conservation equations

Let θ_i denote the average rate at which customers leave queue i, i.e., its *throughput*. Since the network is closed, the system is always stable. Thus, in any case, the throughputs θ_i satisfy the following *flow conservation equations* (FCE) for any $j \in \{1, \ldots, M\}$:

$$\theta_j = \sum_{i=1}^{M} \theta_i p_{i,j}. \tag{8}$$

Equations (8) constitute again a linear system of equations, but now they do not determine the θ_i uniquely. In addition to the real throughputs $(\theta_1, \ldots, \theta_M)$, equations (8) are solved by any vector $(c\theta_1, \ldots, c\theta_M)$.

Note also that the flow conservation equations (8) depend on the number of stations and their topology (via the routing probabilities $p_{i,j}$) but not at all on the service rates μ_i. It follows that the throughputs θ_i are independent of service rates μ_i.

3.2 Gordon-Newell theorem

Below we show that the steady-state distribution of the whole network is of *product-form*. The result is known as the *Gordon-Newell theorem*.

Theorem 4

Consider a Gordon-Newell network. Let $(\hat{\theta}_1, \ldots, \hat{\theta}_M)$ *be any non-zero solution of equations (8), and define, for any* $i \in \{1, \ldots, M\}$,

$$\hat{\rho}_i := \frac{\hat{\theta}_i}{\mu_i}.$$

The steady-state distribution of process $N(t)$ *is given by*

$$P\{N = n\} = \frac{1}{\hat{G}_K} \prod_{i=1}^{M} \hat{\rho}_i^{n_i}, \quad n \in \mathcal{S}_K,$$

where

$$\hat{G}_K := \sum_{n' \in \mathcal{S}_K} \prod_{i=1}^{M} \hat{\rho}_i^{n'_i}.$$

Proof Let $n \in \mathcal{S}_K$ and denote

$$\pi(n) := \frac{1}{\hat{G}_K} \prod_{i=1}^{M} \hat{\rho}_i^{n_i}.$$

Note that for any $\tilde{n} \in \mathcal{S}_{K-1}$ and $i, j \in \{1, \ldots, M\}$, we have

$$\frac{\pi(\tilde{n} + e_i)}{\hat{\rho}_i} = \frac{\pi(\tilde{n} + e_j)}{\hat{\rho}_j}. \tag{9}$$

Probabilities $\pi(n)$ clearly sum up to 1 so that the normalization condition (N) is satisfied. It remains to prove that the global balance equations (GBE) are also satisfied for any $n \in \mathcal{S}_K$:

$$\sum_{n' \neq n} \pi(n) q(n, n') = \sum_{n' \neq n} \pi(n') q(n', n). \tag{10}$$

Let $\tilde{n} \in \mathcal{S}_{K-1}$ and $j \in \{1, \ldots, M\}$. Since $\hat{\theta}_i = \hat{\rho}_i \mu_i$ for all i, it follows from the flow conservation equation (8) that

$$\hat{\rho}_j \mu_j (1 - p_{j,j}) = \sum_{i \neq j} \hat{\rho}_i \mu_i p_{i,j}.$$

By multiplying both sides by $\pi(\tilde{n} + e_j)/\hat{\rho}_j$ and applying formula (9), we get

$$\pi(\tilde{n} + e_j)\mu_j (1 - p_{j,j}) = \sum_{i \neq j} \pi(\tilde{n} + e_i)\mu_i p_{i,j},$$

which is equivalent with

$$\pi(\tilde{n} + e_j) \sum_{i \neq j} q(\tilde{n} + e_j, \tilde{n} + e_i) = \sum_{i \neq j} \pi(\tilde{n} + e_i) q(\tilde{n} + e_i, \tilde{n} + e_j). \tag{11}$$

Equations (11) are called *station balance equations* (SBE). For fixed $\tilde{n}$ and j, it corresponds to transitions out of state $\tilde{n} + e_j$ and into that state generated by a customer that leaves or enters queue j, respectively. The global balance equations (10) follow from these station balance equations in a straightforward way by summing up the related SBE's. □

3.3 Arrival theorem

Let N^* denote the steady-state variable describing the state of the system "seen" by a customer entering any queue (with the entering customer excluded). Note that $N^* \in \mathcal{S}_{K-1}$. In addition, let $\tilde{N}$ denote the steady-state number of customers in the corresponding Gordon-Newell network where there are $K - 1$ customers (instead of K). It follows from the Gordon-Newell theorem that

$$P\{\tilde{N} = \tilde{n}\} = \frac{1}{\hat{G}_{K-1}} \prod_{i=1}^{M} \hat{\rho}_i^{\tilde{n}_i}, \quad \tilde{n} \in \mathcal{S}_{K-1},$$

where

$$\hat{G}_{K-1} := \sum_{\tilde{n}' \in \mathcal{S}_{K-1}} \prod_{i=1}^{M} \hat{\rho}_i^{\tilde{n}'_i}.$$

Thus, for any $i \in \{1, \ldots, M\}$,

$$P\{N = \tilde{n} + e_i\} = P\{\tilde{N} = \tilde{n}\}\hat{\rho}_i \frac{\hat{G}_{K-1}}{\hat{G}_K}. \tag{12}$$

Below we show that the entering customer sees the corresponding system, where there are $K-1$ customers, in equilibrium. The result is known as the *arrival theorem* for Gordon-Newell networks.

Theorem 5

Consider a Gordon-Newell network. The steady-state distribution seen by a customer entering any queue is the same as the steady-state distribution of process $\tilde{N}(t)$*, i.e.,*

$$P\{N^* = \tilde{n}\} = P\{\tilde{N} = \tilde{n}\}, \quad \tilde{n} \in \mathcal{S}_{K-1}.$$

Proof Let $\tilde{n} \in \mathcal{S}_{K-1}$. Consider a customer moving from some fixed queue i to another fixed queue j with $p_{i,j} > 0$. Now

$$\begin{aligned}
&P\{N^* = \tilde{n} \mid \text{move from queue } i \text{ to queue } j\} \\
&\quad = \frac{P\{N = \tilde{n} + e_i\} q(\tilde{n} + e_i, \tilde{n} + e_j)}{\sum_{\tilde{n}' \in \mathcal{S}_{K-1}} P\{N = \tilde{n}' + e_i\} q(\tilde{n}' + e_i, \tilde{n}' + e_j)} \\
&\quad = \frac{P\{N = \tilde{n} + e_i\} \mu_i p_{i,j}}{\sum_{\tilde{n}' \in \mathcal{S}_{K-1}} P\{N = \tilde{n}' + e_i\} \mu_i p_{i,j}} \\
&\quad = \frac{P\{N = \tilde{n} + e_i\}}{\sum_{\tilde{n}' \in \mathcal{S}_{K-1}} P\{N = \tilde{n}' + e_i\}} \\
&\quad = \frac{P\{\tilde{N} = \tilde{n}\}\hat{\rho}_i \frac{\hat{G}_{K-1}}{\hat{G}_K}}{\sum_{\tilde{n}' \in \mathcal{S}_{K-1}} P\{\tilde{N} = \tilde{n}'\}\hat{\rho}_i \frac{\hat{G}_{K-1}}{\hat{G}_K}} = P\{\tilde{N} = \tilde{n}\},
\end{aligned}$$

where the second last equality follows from (12). Thus, for any j, we have

$$P\{N^* = \tilde{n} \mid \text{customer enters queue } j\} = P\{\tilde{N} = \tilde{n}\},$$

which proves the claim. □

As in the case of Jackson networks, another method to prove the arrival theorem is to (i) insert an additional single-server queue, say 0, with service rate μ_0, (ii) route all customers whose destination is queue j via this additional queue 0, and (iii) finally let $\mu_0 \to \infty$.

3.4 Mean value analysis

The arrival theorem gives us a tool to develop a recursive method to calculate the steady-state mean values for the number of customers in each queue as well as the time spent in each queue separately. This method is know as *Mean Value Analysis* (MVA) of Gordon-Newell networks.

Let $\bar{N}_i(k)$ and $\bar{T}_i(k)$ denote the steady-state mean value of the number of customers in queue i and the time that a customer spends in queue i during one visit, respectively, in a corresponding Gordon-Newell network where there are k customers. In addition, let $\theta_i(k)$ denote the throughput of queue i in such a network.

Theorem 6

Consider a Gordon-Newell network. The steady-state mean values satisfy the following recursion: $\bar{N}_i(0) = 0$ *and, for any* $k \in \{1, 2, \ldots\}$,

$$\bar{T}_i(k) = (1 + \bar{N}_i(k-1))\frac{1}{\mu_i}, \tag{13}$$

$$\bar{N}_i(k) = \frac{k\hat{\theta}_i \bar{T}_i(k)}{\sum_{j=1}^M \hat{\theta}_j \bar{T}_j(k)}, \tag{14}$$

$$\theta_i(k) = \frac{\bar{N}_i(k)}{\bar{T}_i(k)}, \tag{15}$$

where $(\hat{\theta}_1, \ldots, \hat{\theta}_M)$ *is any solution of equations (8).*

Proof Clearly $\bar{N}_i(0) = 0$. Let $k \in \{1, 2, \ldots\}$. Let $\bar{N}_i^*(k)$ denote the steady-state mean value of the number of customers in queue i seen by a customer entering that queue in a corresponding Gordon-Newell network where there are k customers. It follows from the FIFO service discipline and exponential service times that

$$\bar{T}_i(k) = (1 + \bar{N}_i^*(k))\frac{1}{\mu_i}.$$

By the arrival theorem (Theorem 5), we have $\bar{N}_i^*(k) = \bar{N}_i(k-1)$, which results in formula (13). Formula (15) follows directly from Little's formula applied to queue i,

$$\bar{N}_i(k) = \theta_i(k)\bar{T}_i(k).$$

Finally, since $\sum_{j=1}^M \bar{N}_j(k) = k$, we have

$$\bar{N}_i(k) = \frac{k\bar{N}_i(k)}{\sum_{j=1}^M \bar{N}_j(k)}.$$

By further applying Little's formula, we get

$$\bar{N}_i(k) = \frac{k\theta_i(k)\bar{T}_i(k)}{\sum_{j=1}^M \theta_j(k)\bar{T}_j(k)}.$$

Formula (14) follows now from the observation that for any solution $(\hat{\theta}_1, \ldots, \hat{\theta}_M)$ of equations (8) there is $\hat{c}(k)$ such that, for any j,

$$\hat{\theta}_j = \hat{c}(k)\theta_j(k),$$

which completes the proof. □

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