Lecture 7: Quantitative Spatial Equilibrium Models

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Introduction

Quantitative Spatial Equilibrium Models

- Traditional regional and urban models highly stylized
  - Great for novel insights
  - Very hard to take to data
- A new generation of regional and urban QSE models can be much more readily taken to data
- The most common of these feature unique equilibria and allow model inversion to recover structural parameters
  - Quite often exactly identified
  - Overidentification tests to strengthen confidence in robustness
- With these in hand, one can conduct counterfactual policy experiments
- However, because QSE models often include many mechanisms and forces (in order to fit the data), many times it will be harder to analytically separate the different forces at play.

Road Map for Today

- Intro to Redding and Sturm (2008)
- QSE model of multiple regions (based on Redding and Sturm 2008)
- Intro to Ahlfeldt, Redding, Sturm and Wolf (2015)
- QSE model of a city (based on Ahlfeldt et al. 2015)

Intro to Redding and Sturm (2008): The Costs of Remoteness: Evidence from German Division and Reunification

Redding and Sturm (2008): Summary of Paper

- **Main Research Question:** How much does market access impact economic development of regions?
- Use division of Germany after WWII to estimate effect of loss of mkt access on size of regions. Map
- Intuition: division has 3 (immediate) effects on cities in West Germany
  1. Consumers in all West German cities lose access to tradeable varieties from the East $\Rightarrow \uparrow$ consumers' cost of living $\Rightarrow \downarrow$ real wages (**cost of living effect**)

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    3. Western firms face lowered competition, since they don't have to compete with Eastern firms, this pushes wages upwards (**market crowding effect**)
- 1 + 2 dominate, so wages go down. In cities closer to the border, wages go down more because the loss of mkt access was higher (lower distance $\Rightarrow$ higher trade share with East).

Redding and Sturm (2008): Reduced Form Results

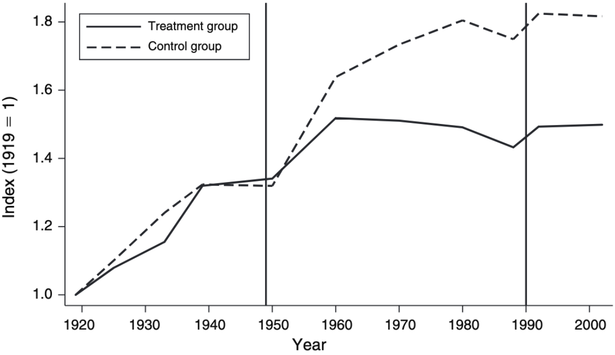

FIGURE 3. INDICES OF TREATMENT AND CONTROL CITY POPULATION

Figure 1: Impact of division of Germany on Western cities. Population expressed as index relative to 1919 population. Source: Redding and Sturm (2008).

Redding and Sturm (2008): QSE Model

- They develop one of the first QSE models.
- model formalizes role of mkt access in shaping the distribution of population across space
- Calibrate the model to city-level data for Germany in 1939 and simulate the impact of the postwar division on the equilibrium distribution of population across West German cities. Simulation vs Estimated Results
- In the model, the **only** effects on the distribution of population come from reduction in mkt access.
- So, the fact that the model matches the reduced form results provide convincing evidence of the mechanism.

A QSE Model of Regions

A Quantitative Spatial Model of Regions

- Multiregion version of Helpman (1998).
- Very similar to Redding and Sturm (2008).
- Can be used to study determinants of the spatial distribution of economic activity across a set of regions.
- Regions will be connected by goods trade and factor mobility.
- Will provide the basic building blocks for most regional economics QSE models.
- Can be extended to include amenities, endogenous supply of land, multiple industries, input-output linkages, commercial land use, heterogenous labor, etc.

Setup

- Economy consisting of a set N of regions (indexed by n).
- $H_i \rightarrow$ exogenous supply of land in region i.
- $\overline{L} \rightarrow$ number of workers in whole economy.
- We assume perfect geographic mobility of workers $\Rightarrow$real wages are equalized across regions.
- Regions connected by bilateral transport network.
- Goods shipped subject to bilateral iceberg cost: $d_{ni} = d_{in} > 1$ units must be shipped from region i so that one unit arrives to region n $\neq$i, and $d_{nn} = 1$.
- Is $d_{ni} = d_{in}$ realistic?

Consumer Preferences

Cobb-Douglas references over goods consumption C_n and residential land use h_n:

$$U_n = \left(\frac{C_n}{\alpha}\right)^{\alpha} \left(\frac{h_n}{1-\alpha}\right)^{1-\alpha}, \ 0 < \alpha < 1.$$

The goods consumption index $C_n$is CES bundle of $c_{ni}(j)$, where $c_{ni}(j)$ is the consumption in n of the variety$j$ produced in i:

$$C_n = \left[\sum_{i \in N} \int_0^{M_i} c_{ni}(j)^{\rho} \ dj\right]^{\frac{1}{\rho}}.$$

Production 1

Varieties of the consumption good are produced under monopolistic competition and increasing returns to scale. Total amount of labor, $l_i(j)$ required to produce $x_i(j)$ units of variety j in location i:

$$l_i(j) = F + \frac{x_i(j)}{A_i}.$$

Profit max. and zero profit condition imply prices are constant markup over marginal cost:

$$p_{ni}(j) = \left(\frac{\sigma}{\sigma - 1}\right)\left(d_{ni}\frac{w_i}{A_i}\right),$$

and equilibrium output of each variety depends on fixed cost (F), location productivity (A_i) and elasticity of substitution (σ):

$$x_i(j) = \bar{x}_i = A_i(\sigma - 1)F.$$

Production 2

Using labor requirement equation, we find that equilibrium employment for each variety is the same in all regions:

$$l_i(j) = \bar{l} = \sigma F.$$

Given this constant equilibrium employment for each variety, labor market clearing implies total measure of varieties supplied by each location (M_i) is proportional to endogenous supply of workers in that location (L_i):

$$M_i = \frac{L_i}{\sigma F}.$$

Price Indices and Expenditure Shares

From CES expenditure function we know that the share of location n's expenditure of goods produced in location i is:

$$\pi_{ni} = \frac{M_i {p_{ni}}^{1-\sigma}}{\sum_{k \in N} M_k {p_{nk}}^{1-\sigma}}.$$

Using

$$p_{ni} = \left(\frac{\sigma}{\sigma - 1}\right) d_{ni} \frac{w_i}{A_i}, \text{ and that } M_i = \frac{L_i}{\sigma F},$$

we get

$$\pi_{ni} = \frac{L_i \left(d_{ni} \frac{w_i}{A_i}\right)^{1-\sigma}}{\sum_{k \in N} L_k \left(d_{nk} \frac{w_k}{A_k}\right)^{1-\sigma}}.$$

Gravity Equation

Looking at

$$\pi_{ni} = \frac{L_i \left(d_{ni} \frac{w_i}{A_i} \right)^{1-\sigma}}{\sum_{k \in N} L_k \left(d_{nk} \frac{w_k}{A_k} \right)^{1-\sigma}},$$

we see that bilateral trade between n and i implies a gravity equation: it depends on bilateral transport cost (d_{ni}), multilateral transport costs ($d_{nk}$for all regions k), as well as the size of region i (L_i) relative to all other regions ($L_k$for all k).

More on Price Index

Income and Population Mobility 1 - Income

We assume land in each region is distributed lump sum to all workers residing in that location. So per capita income in each location (v_n) equals labor income plus per capita expenditure on residential land:

$$v_n = \overbrace{w_n}^{\text{labor income}} + \underbrace{(1-\alpha)v_n}_{\text{land expenditure}} .$$

Which implies that total income in region i is

$$v_n L_n = \frac{w_n L_n}{\alpha}.$$

Income and Population Mobility 2 - Land Mkt Clearing

From FOC of consumer problem, aggregate expenditure in land in n will be $(1-\alpha)v_n L_n$. Land market clearing implies that supply of land (H_n) equals demand for land, which implies:

$$H_n r_n = (1-\alpha)v_n L_n \;\Rightarrow r_n = \frac{(1-\alpha)v_n L_n}{H_n}.$$

Using equation for total income in region n,

$$r_n = \frac{1-\alpha}{\alpha}\frac{w_n L_n}{H_n}.$$

Free mobility Condition

Free population mobility across regions implies a worker's utility must be the same in all regions with non-zero population, so

$$V_n = \frac{v_n}{P_n^{\alpha} r_n^{1-\alpha}} = \bar{V}.$$

Substituting in for v_n,P_n,and $r_n$we get

$$\bar{V} = \frac{A_n^{\alpha} H_n^{1-\alpha} \pi_{nn}^{-\alpha/(\sigma-1)} L_n^{-\frac{\sigma(1-\alpha)-1}{\sigma-1}}}{\alpha \left(\frac{\alpha}{\alpha-1}\right)^{\sigma} \left(\frac{1}{\sigma F}\right)^{\frac{\alpha}{1-\sigma}} \left(\frac{1-\alpha}{\alpha}\right)^{1-\alpha}}.$$

Population Share

Solving for $L_n$from the free mobility condition we can express the population share of each location ($\lambda_n \equiv L_n/\bar{L}$) as

$$\lambda_n = \frac{\left[A_n^{\alpha} H_n^{1-\alpha} \pi_{nn}{}^{-\alpha}\big/(\sigma-1)\right]^{\frac{\sigma-1}{\sigma(1-\alpha)-1}}}{\sum_{k\in N}\left[A_k^{\alpha} H_k^{1-\alpha} \pi_{kk}{}^{-\alpha}\big/(\sigma-1)\right]^{\frac{\sigma-1}{\sigma(1-\alpha)-1}}}.$$

So population share in each location depends on its productivity A_n, the supply of land H_n, and the domestic trade share π_{nn}, relative to those of all other locations.

General Equilibrium

Combining trade share equation:

$$\pi_{ni} = \frac{L_i \left(d_{ni} \frac{w_i}{A_i} \right)^{1-\sigma}}{\sum_{k \in N} L_k \left(d_{nk} \frac{w_k}{A_k} \right)^{1-\sigma}},$$

population mobility condition:

$$V_n = \frac{v_n}{P_n^{\alpha} r_n^{1-\alpha}} = \bar{V},$$

and price index:

$$P_{ni} = \left(\frac{\sigma}{\sigma - 1} \right) d_{ni} \frac{w_i}{A_i},$$

and assuming symmetric trade costs ($d_{ni} = d_{in}$), we transform these three sets of equations into a system of N equations and N unknowns that can be solved numerically (Allen and Arkolakis 2014).

General Equilibrium - Allen and Arkolakis (2014)

- Allen and Arkolakis (2014) show that as long as parameter restriction holds, there exists a unique L_n that solves N equations given parameters $\{H_n, A_n, d_{ni}\}$. Equations
- Parameter restriction required implies **congestion forces always dominate agglomeration forces**.
- In this model, we need $\sigma(1-\alpha) > 1$.
- Intuition:
  - As population concentrates in one location, the measure of varieties expands, which makes the location more attractive (with positive trade costs) $\rightarrow$Agglomeration force.
  - As population concentrates, land prices increase $\rightarrow$Dispersion force.
  - High elasticity of substitution (σ) $\Rightarrow$low agglomeration force.
  - Higher share of land $(1-\alpha)$ $\Rightarrow$stronger dispersion force.
  - So high enough $\sigma(1-\alpha)$ implies dispersion/congestion force dominates agglomeration force.

General Equilibrium - Unique Equilibrium

- All QSE models impose parameter restrictions so that the equilibrium is unique.
- This allows for simple conterfactual analysis: if a policy intervention could lead to multiple equilibrium distributions of population and economics activity, we would have to develop equilibrium selection criteria and it wouldn't be clear which counterfactual we should compare against.
- It also greatly simplifies the numerical equilibrium solution when a closed form solution does not exist.
- Because of this, most QSE models are not suited to study problems where multiple equilibria might be relevant.

Model Inversion

- Quantitative spatial equilibrium models are written so as to perfectly rationalize observed data (i. e. they are exactly identified).
- Suppose researcher has already estimated key parameters (α and σ) and measured trade costs d_{ni}.
- Researcher also observes the population vector $\{L_n\}$ and nominal wages $\{w_n\}$.
- Inverting model: using equilibrium conditions, parameter values and observed data on $\{L_n, w_n\}$ to solve for unobserved $\{H_n, A_n\}$.
- Note: invertibility requires an injective relation from $\{L_n, w_n\}$ to $\{H_n, A_n\}$ given parameters. Uniqueness of equilibrium will guarantee this.
- Model inversion allows us to decompose observed variation in endogenous variables (population and wages) into the contribution of different exogenous determinants (trade costs, productivity, quality-adjusted supply of land).

Equations

Counterfactuals

- QSE models are often used to study counterfactuals for the effects of public policy interventions (e. g. transport improvements).
- Using observed values of endogenous variables in initial equilibrium, we can calculate relative changes that should take place in the counterfactual equilibrium without solving for unobserved location characteristics.
- Let $\hat{x} = x'/x$, where x' is the value of a variable in counterfactual equilibrium and x is the value in the initial equilibrium (pre policy).
- Using equilibrium conditions, we can find a system of equations that relate observed wages, trade shares, and population shares in the initial equilibrium $\{w_n \pi_{ni}, \lambda_n\}$ to counterfactual changes in these variables, $\{\hat{w}_n \hat{\pi}_{ni}, \hat{\lambda}_n\}$, given changes in transport costs ($\{\hat{d}_{ni}\}$) and parameters ($\{\alpha, \sigma\}$).

Welfare

- For broad class of quantitative spatial equilibrium models, welfare effects of public policy interventions that change trade costs can be calculated using only empirically observable sufficient statistics.
- Consider a reduction in trade costs between initial eq. (indexed by 0) and new eq. (indexed by 1).
- Because of perfect pop. mobility, reduction in trade costs implies reallocation of population until real wages are equalized.
- Using population mobility condition, the changes in domestic expenditure shares (π_{nn}) and population shares (λ_n) are sufficient statistics for welfare impact of transport improvements:

$$\frac{\bar{V}_1}{\bar{V}_0} = \left(\frac{\pi_{nn}^0}{\pi_{nn}^1}\right)^{\frac{\alpha}{\sigma-1}} \left(\frac{\lambda_n^0}{\lambda_n^1}\right)^{\frac{\sigma(1-\alpha)-1}{\sigma-1}}.$$

Ahlfeldt, Redding, Sturm and Wolf (2015): The Economics of Density: Evidence From the Berlin Wall

Ahlfeldt et al. (2015): Summary

- Develops a QSE model of a city that features agglomeration and dispersion forces.
- Uses division and reunification of Berlin to estimate extent of agglomeration and dispersion forces.
- Model can account both qualitatively and quantitatively for the observed changes in city structure.
- Structure of model allows for counterfactual analysis of policies that affect city structure (e.g. transportation, housing policies, etc.)
- Extensions of this model have been used many times to study all kinds of policies.

Ahlfeldt et al. (2015): Motivating Maps

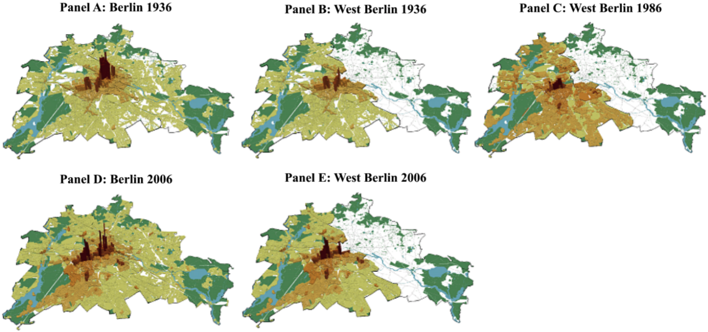

FIGURE 2.—The evolution of land prices in Berlin over time.

A Quantitative Urban Model

A Quantitative Urban Model

- Canonical model of a city based on Lucas and Rossi-Hansberg (2002) and Ahlfeldt et al. (2015).
- This will be a model of a city with a large number of discrete locations.
- Will allow for differences in production fundamentals, residential fundamentals, and transport costs across locations.
- Model can be used to quantify the role of productivity and amenities in determining the internal structure of cities.
- Used to evaluate counterfactual policy questions (e. g. impact of transport infrastructure improvements in a city).

Tractability of the Model

- Such a model would normally be highly non-tractable
- A worker living at the same distance from two firms would, for example, work for the firm that offers epsilon higher wages.
- Model assumes heterogeneity in workers' commuting choices, modeled following Eaton and Kortum (2002), to deal with this problem.
- These shocks imply an upward sloping labor supply function in each location.
- These idiosyncratic shocks also create a gravity structure for commuting flows which fits the data very well.

Setup

- City embedded in a larger economy. Utility outside city is reservation utility $\bar{U}$.
- City consists of S discrete blocks indexed by n or i.
- H_i $\rightarrow$Supply of floor space in block i.
- Single final good, costlessly traded, chosen as numeraire: $p_i = 1$.
- Markets are perfectly competitive.

Model Setup Continued

- Workers are in eq. indifferent between moving to the city and receiving $\bar{U}$.
- Conditional on moving to the city, workers optimally choose a location of residence and a location of employment.
- These choices will depend on the idiosyncratic taste shock.
- Workers face commuting costs which depend on the transport infrastructure connection any two locations in the city.
- Productivity in each location depends on fundamentals (a_i) and spillovers (Y_i).
- Amenities in each location depend on fundamentals (b_i) and spillovers (Ω_i).

Consumption

- Preferences are Cobb-Douglas over consumption good and floor space. Utility for worker ω residing in location i and working in location j is:

$$U_{ij\omega} = \frac{B_i z_{ij\omega}}{d_{ij}} \left(\frac{c_{ij}}{\beta}\right)^{\beta} \left(\frac{l_{ij}}{1-\beta}\right)^{1-\beta} \quad , \quad 0 < \beta < 1$$

  - c_{ij}: consumption of final good
  - l_{ij}: residential floor space
  - B_i: residential amenity
  - d_{ij}: commuting costs
  - $z_{ij\omega}$: idiosyncratic shock that captures idiosyncratic reasons for a worker to live in i and work in j.

Distribution of Idiosyncratic Shocks

- The idiosyncratic shock to worker productivity is drawn from a Frechet distribution:

$$F(z_{ij\omega}) = \exp(-T_i E_j z_{ij\omega}^{\epsilon}), \quad T_i, E_j > 0, /\epsilon > 1.$$

- The shape of ϵ is inversely related to the variance of the utility shock.
- The scale of the utility shock varies across location with both an origin (T_i) and a destination (E_j) component.

Indirect Utility Function

- This setup gives rise to the following indirect utility:

$$U_{ij\omega} = \frac{w_j B_i z_{ij\omega}}{d_{ij} P_i^{\beta} Q_i^{1-\beta}}$$

- Living in i and working in j is more attractive for a worker ω if:
  - The workplace pays well (higher w_j),
  - the residence location is pleasant (high B_i), offers cheap housing (low Q_i),
  - or the commute costs are lower (lower d_{ij}).
  - Workers are also drawn to higher utility shocks $z_{ij\omega}$.

Commuting Decisions

- Probability that worker chooses to live in i and work in j:

$$\lambda_{ij} = \frac{T_i E_j \left(d_{ij} Q_i^{1-\beta} \right) (B_i w_j)^{\epsilon}}{\sum_r \sum_s T_r E_s \left(d_{rs} Q_r^{1-\beta} \right) (B_r w_s)^{\epsilon}} = \frac{\Phi_{ij}}{\Phi}$$

- Conditional on living in i, the probability that a worker commutes to location j follows a gravity equation:

$$\lambda_{ij|i} = \frac{E_j \left(w_j / d_{ij} \right)^{\epsilon}}{\sum_s E_s \left(w_s / d_{is} \right)^{\epsilon}}.$$

Commuting Market Clearing

- Workplace employment in j equals the sum across all i of residence employment times the probability of commuting from i to j:

$$L_{Mj} = \sum_{s} \frac{E_j \left(w_j / d_{ij} \right)^{\epsilon}}{\sum_s E_s \left(w_s / d_{is} \right)^{\epsilon}} L_{Ri}.$$

- If we observe workplace employment in each employment location j and residence employment in each residence location i, as well as bilateral commute costs d_{ij},
- then we can solve for the unique (up to a scalar) vector of wages for which the observed values of workplace and residence choices are an equilibrium of the model, given the parameter values.

Production

Final good produced from labor and commercial floor space using a Cobb-Douglas production function with unit cost function:

$$1 = \frac{1}{A_i} w_i^{\alpha} q_i^{1-\alpha},\ 0 < \alpha < 1,$$

- w_i $\rightarrow$wages
- q_i $\rightarrow$price of commercial floor space
- A_i $\rightarrow$productivity in each location.

General Equilibrium

- As with previous model, we can use Allen and Arkolakis (2014) to show that there exists a unique vector of prices (floor space prices and wages), and distribution of population (residential and workplace), such that, for a given set of parameters and commute costs, markets clear, residents maximize utility, and firms maximize profits.
- However, this is only the case when there are no residential and production externalities. The interesting model features externalities, which may cause multiple eq.
- ARSW show that for the estimated parameter values, the eq. is unique.

Model Inversion

- Model inversion will be similar to Redding and Sturm (2008).
- **Key difference:** In Redding and Sturm (2008), wages (w_i) were observed. Here, we only see residential population (L_i), workplace population (L_j), commuting times, and floor space prices (Q_i).
- Wages are "calculated" from Commuting Market Clearing condition so as to perfectly close the model ("model-implied wages").
- Intuition: wages are consistent with population distribution (workplace and residential), conditional on commuting costs.

Counterfactuals

- In the same way as in Redding and Sturm (2008), once the model has been inverted, and unobserved agglomeration and dispersion forces are estimated, we can simulate counterfactual scenarios.
- Example: change commute costs $\{d_{ij}\}$ and solve for population distribution and prices under new costs.

Extensions of the Basic Frameworks

- ARSW has been extended a million different ways. Some examples:
- Tsivanidis (2020): Adds multiple industries and two income levels to study effect of BRT lines in Bogota.
- Monte, Redding, Rossi-Hansberg (2018 AER): combines elements of ARSW and RS into a model with regions and commuting within cities.
- Tombe and Zhu (2019 AER), Heblich, S., Redding, S. J., & Sturm, D. M. (2020 QJE), Faber, B., & Gaubert, C. (2019 AER), Delventhal, M., & Parkhomenko, A. (2020), Zárate (2020), etc.

Next Class

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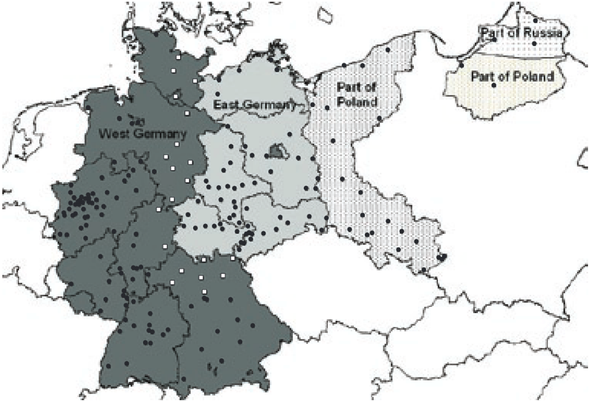
Redding and Sturm (2008) - Map



Map 1. The Division of Germany after the Second World War

Back

Redding and Sturm (2008) - Model Simulation vs Reduced Form Estimation

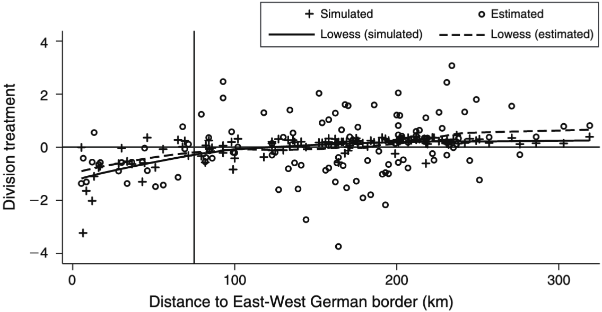


FIGURE 6. SIMULATED AND ESTIMATED DIVISION TREATMENTS

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Price Index

Using

$$p_{ni}(j) = \left(\frac{\sigma}{\sigma - 1}\right) d_{ni} \frac{w_i}{A_i}, \text{ and that } M_i = \frac{L_i}{\sigma F},$$

we can write the CES price index:

$$P_n = \left[\sum_{i \in N} \int_0^{M_i} p_{ni}(j)^{1-\sigma} dj\right]^{\frac{1}{1-\sigma}}$$

as

$$P_n = \left(\frac{1}{\sigma F}\right)^{\frac{1}{1-\sigma}} \frac{\sigma}{\sigma - 1} \left[\sum_{i \in N} L_i \left(d_{ni} \frac{w_i}{A_i}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}}.$$

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Price Indices and Expenditure Shares 4

Combining

$$P_n = \left(\frac{1}{\sigma F}\right)^{\frac{1}{1-\sigma}} \frac{\sigma}{\sigma - 1} \left[\sum_{i \in N} L_i \left(d_{ni} \frac{w_i}{A_i}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}},$$

and

$$\pi_{ni} = \frac{L_i \left(d_{ni} \frac{w_i}{A_i}\right)^{1-\sigma}}{\sum_{k \in N} L_k \left(d_{nk} \frac{w_k}{A_k}\right)^{1-\sigma}},$$

we can express each location's price index in terms of its trade share with itself:

$$P_n = \frac{\sigma}{\sigma - 1} \left(\frac{L_n}{\sigma F \pi_{nn}}\right)^{\frac{1}{1-\sigma}} \frac{w_n}{A_n}.$$

Population Share - Derivation

Solving for $L_n$from the free mobility condition we get

$$L_n = \left[\frac{A_n^{\alpha} H_n^{1-\alpha} \pi_{nn}{}^{-\alpha/\sigma-1)}}{\alpha \left(\frac{\alpha}{\alpha-1}\right)^{\sigma} \left(\frac{1}{\sigma F}\right)^{\frac{\alpha}{1-\sigma}} \left(\frac{1-\alpha}{\alpha}\right)^{1-\alpha} \bar{V}} \right]^{\frac{\sigma-1}{\sigma(1-\alpha)-1}} .$$

Since $\bar{L} = \sum_{k \in N} L_k$,we can write the population share of each location ($\lambda_n \equiv L_n / \bar{L}$) as

$$\lambda_n = \frac{\left[A_n^{\alpha} H_n^{1-\alpha} \pi_{nn}{}^{-\alpha/\sigma-1)}\right]^{\frac{\sigma-1}{\sigma(1-\alpha)-1}}}{\sum_{k \in N} \left[A_k^{\alpha} H_k^{1-\alpha} \pi_{kk}{}^{-\alpha/\sigma-1)}\right]^{\frac{\sigma-1}{\sigma(1-\alpha)-1}}}.$$

General Equilibrium - Allen and Arkolakis (2014)

Applying Allen and Arkolakis (2014), the equilibrium conditions can be reduced to a set of N equations:

$$L_n^{\tilde{\sigma}\gamma_1} A_n^{-\frac{(\sigma-1)(\sigma-1)}{2\sigma-1}} H_n^{-\frac{\sigma(\sigma-1)(1-\alpha)}{\alpha(2\sigma-1)}} =$$

$$= \bar{W}^{1-\sigma} \sum_{i \in N} \frac{1}{\sigma F} \left(\frac{\sigma}{1-\sigma} d_{ni} \right)^{1-\sigma} \left(L_i^{\tilde{\sigma}\gamma_1} \right)^{\frac{\gamma_2}{\gamma_1}} A_i^{\frac{\sigma(\sigma-1)}{2\sigma-1}} H_i^{\frac{(\sigma-1)(\sigma-1)(1-\alpha)}{\alpha(2\sigma-1)}}.$$

Where $\bar{W}$ is determined by labor market clearing condition $\left(\sum_{n \in N} L_n = \bar{L}\right)$,and

$$\tilde{\sigma} \equiv \frac{\sigma-1}{2\sigma-1},\ \gamma_1 \equiv \frac{\sigma(1-\alpha)}{\alpha},\ \gamma_2 \equiv 1 + \frac{\sigma}{\sigma-1} - \frac{(\sigma-1)(1-\sigma)}{\alpha}.$$

Wages are implicitly determined by

$$w_n^{1-2\sigma} A_n^{\sigma-1} L_n^{(\sigma-1)\frac{1-\alpha}{\alpha}} H_n^{-(\sigma-1)\frac{1-\alpha}{\alpha}} = \xi.$$

Where ξ is a scalar that normalizes wages. Back Back 2