Math Camp - Final Exam

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The exam is 4 hours and has a total of 120 points. Please answer as many questions as you can. Answer shortly but justify your answers and explain accurately what you are doing. If you are confused about some question statement, please explain clearly what you assume when answering. Point totals reflect the difficulty of the problem and give a rough estimate for how long the question should take.

1. Consider the following maximization problem

$$\max_{x\in\mathbb{R}^6_+} \sum_{i=1}^{6} -x_i \log(6x_i)$$
$$\text{s.t. } x_1 \geq 2x_2$$
$$\sum_{i=1}^{6} x_i \geq 1$$

   (a) *(10 points)* Are the KKT conditions necessary for a maximum? Sufficient? Why?

   (b) *(10 points)* Write down the KKT conditions, ignoring the non-negativity constraints (i.e. feel free to assume that all non-negativity constraints have 0 multipliers).

   (c) *(10 points)* Show that the KKT conditions imply that for all $i, j \notin \{1, 2\}$, $x_i = x_j$.

   (d) *(15 points)* Argue using the KKT conditions that $x_1 = 2x_2$ at a maximum.

2. A firm is selling two goods, (x, y), to a consumer. To entice the consumer to purchase more, the firm offers a discount on the price of each good based on the amount of the other good the consumer purchases. The consumer solves the following problem

$$\max_{x,y\geq 0} \sqrt{x} + \sqrt{y}$$
$$\text{s.t. } p(y)x + q(x)y \leq m$$

   where $p : \mathbb{R}_+ \to \mathbb{R}_+$ and $q : \mathbb{R}_+ \to \mathbb{R}_+$ are decreasing, twice continuously differentiable functions.

   (a) *(10 points)* Show that if $p(y), q(x) \geq K$ for some $K > 0$ for all $x, y \geq 0$, then a maximum exists.

   (b) *(15 points)* What condition must the first and second derivatives of $p(y)$ and $q(x)$ satisfy to make $p(y)x + q(x)y - m$ a convex function? If this condition holds, how do you know the maximum is unique?

(c) *(15 points)* Suppose that $p(x) = q(x) = 1 + e^{-\alpha x}$ for some $\alpha > 0$, so the problem becomes

$$V(\alpha, m) = \max_{x,y \geq 0} \sqrt{x} + \sqrt{y}$$
$$\text{s.t. } x\left(1 + e^{-\alpha y}\right) + y\left(1 + e^{-\alpha x}\right) \leq m$$

and let $x(\alpha, m)$ and $y(\alpha, m)$ be the corresponding arg maxes. Using the envelope theorem, express $\frac{\partial V}{\partial \alpha} / \frac{\partial V}{\partial m}$ in terms of $x(\alpha, m)$ and $y(\alpha, m)$.

3. Suppose X and Y are distributed

$$f_{x,y} = \begin{cases} \frac{4x^2}{y^2}\left(1 - x\left(\frac{1-y}{y}\right)\right) & \text{if } x \in (0,1), y \in (x/(x+1), 1) \\ 0 \text{ o.w.} \end{cases}$$

Consider the random variables U, V, with $U = X$, $V = X\left(\frac{1-Y}{Y}\right)$.

(a) *(10 points)* Show that (U, V) has support $(0,1)^2$.

(b) *(10 points)* What is the joint pdf of U, V.

(c) *(15 points)* Are U and V independent?