PHYS-E0414 Advanced Quantum Mechanics

Final exam, December 15, 2021, 13.00-16.00

You should answer in English unless you have special permission to use another language. You are free to use the lecture notes, books, the exercises, electronic devices, etc. (No communication allowed.) Please write your name, student number, study program, course code, and the date in all of your papers. There are 4 problems in this exam set which consists of 2 pages.

Exercise 1

Answer the following questions in your own words. No calculations are needed. Less than one page should suffice to answer all three questions:

a) What distinguishes separable and entangled quantum states?

b) Discuss the main differences between fermions and bosons.

c) Consider two electrons in the spin singlet state $|\Psi\rangle = \left(|\uparrow_{\hat{\mathbf{z}}}\rangle_A|\downarrow_{\hat{\mathbf{z}}}\rangle_B - |\downarrow_{\hat{\mathbf{z}}}\rangle_A|\uparrow_{\hat{\mathbf{z}}}\rangle_B\right)/\sqrt{2}$. What is the probability of finding the spin at A pointing up along the direction $\hat{\mathbf{y}}$?

Exercise 2

Consider two quantum harmonic oscillators given by the Hamiltonian $\hat{H} = \hat{H}_0 + \hat{V}$, where

$$\hat{H}_0 = \hbar\omega_0(\hat{a}_1^\dagger\hat{a}_1 + 1/2) + \hbar\omega_0(\hat{a}_2^\dagger\hat{a}_2 + 1/2)$$

describes the uncoupled oscillators (both with frequency ω_0), and the coupling reads

$$\hat{V} = \hbar\alpha(\hat{a}_1^\dagger\hat{a}_2^\dagger + \hat{a}_1\hat{a}_2),$$

where $\hat{a}_j^\dagger$ and $\hat{a}_j$, $j = 1, 2$, are the ladder operators, and the coupling strength α is real.

a) Consider the operator $\hat{S} = \alpha(\hat{a}_1^\dagger\hat{a}_2^\dagger - \hat{a}_1\hat{a}_2)/(2\omega_0)$ and show that $[\hat{S}, \hat{V}] = -(\alpha/\omega_0)^2\hat{H}_0$.

b) Using that $[\hat{S}, \hat{H}_0] = -\hat{V}$ and $e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + [\hat{A}, \hat{B}] + [\hat{A}, [\hat{A}, \hat{B}]]/2 + \ldots$, evaluate the unitary transformation $e^{\hat{S}}\hat{H}e^{-\hat{S}}$ up to second order in α.

c) Calculate the groundstate energy of $\hat{H}$ up to second order in α using perturbation theory in $\hat{V}$ and compare with the result in b).

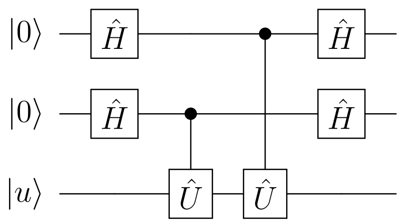
d) Show that the groundstate up to first order in α reads $|\Psi\rangle = |00\rangle - \alpha/(2\omega_0)|11\rangle$, where $|00\rangle$ is the groundstate of $\hat{H}_0$ and $|11\rangle = \hat{a}_1^\dagger\hat{a}_2^\dagger|00\rangle$.

Exercise 3

Consider a unitary operator $\hat{U}$ together with the eigenvalue problem $\hat{U}|u\rangle = u|u\rangle$.

a) Show that the eigenvalue u can be expressed as $u = e^{i\phi}$ for some $0 \leq \phi < 2\pi$.

Consider the quantum circuit below with two controlled-$\hat{U}$ gates that apply $\hat{U}$ to the third qubit, if the control qubit (marked with a dot) is $|1\rangle$, and Hadamard gates are denoted by $\hat{H}$:



b) Determine the state of the system through each step of the circuit with the input $|0\rangle|0\rangle|u\rangle$.

c) Show that the probability to observe the state $|0\rangle$ in the first output is $P = \cos^2(\phi/2)$.

d) What is the probabibitity to observe the first two qubits in the state $|1\rangle|1\rangle$?

Exercise 4

Consider the spin of an electron, whose dynamics is governed by the Lindblad equation

$$\frac{d}{dt}\hat{\rho} = -\frac{i}{\hbar}\left[\hat{H}, \hat{\rho}\right] + \frac{\gamma}{2}\left(\hat{\sigma}_-\hat{\rho}\hat{\sigma}_-^\dagger - (\hat{\sigma}_-^\dagger\hat{\sigma}_-\hat{\rho} + \hat{\rho}\hat{\sigma}_-^\dagger\hat{\sigma}_-)/2\right).$$

Here $\hat{\rho}$ is the density matrix of the spin, $\hat{H} = \varepsilon\hat{\sigma}_z/2$ is the Hamiltonian, and we have defined $\hat{\sigma}_- = (\hat{\sigma}_x - i\hat{\sigma}_y)/2$. The parameter γ denotes the strength of the coupling to the environment.

Below, it may be useful to employ the following matrix representations of the operators:

$$\hat{\rho} = \begin{bmatrix} p & c^* \\ c & 1-p \end{bmatrix}, \quad \hat{H} = \frac{\varepsilon}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \hat{\sigma}_- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \hat{\sigma}_-^\dagger\hat{\sigma}_- = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

where p is a real number and c is complex.

a) Determine the stationary state of the spin, defined by $\frac{d}{dt}\hat{\rho} = 0$.

b) Is the stationary state mixed or pure?

End of exam set