PHYS-E0414 Advanced Quantum Mechanics

Final exam, December 11, 2019, 13.00-16.00

You should answer in English unless you have special permission to use another language. You are free to use the lecture notes, books, the exercises, electronic devices, etc. (No communication allowed.) Please write your name, student number, study program, course code, and the date in all of your papers. There are 3 problems in this exam set which consists of 2 pages.

Problem 1

Answer the following questions in your own words. No calculations are needed. Less than one page should suffice to answer all three questions:

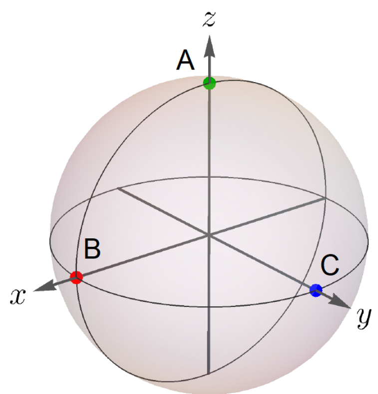
a) The *graviton* is expected to be a spin-2 particle. Is it a boson or a fermion? And what are the possible values of the z-component of its spin?

b) Explain what a Bell inequality tests.

c) Describe a quantum technology that makes use of entanglement.

Problem 2

Pure states of a spin-1/2 system correspond to points on a Bloch sphere as shown below.



a) The point A corresponds to the state $|\uparrow_z\rangle$. What states do the points B and C represent?

b) Evaluate the expectation value of the Hamiltonian $\hat{H}_0 = -\frac{\hbar\omega_0}{2}\hat{\sigma}_z$ with respect to each of the three states A, B, and C, where $\hat{\sigma}_z$ is the operator for the z-component of the spin.

c) If we measure the z-component of the spin, what are the probabilities of finding it in the state $|\uparrow_z\rangle$, if the initial state before the measurement was either *i*) A, *ii*) B, or *iii*) C?

d) The system now evolves according to the Hamiltonian $\hat{H}_0$ for the time span $\Delta t = 2\pi/\omega_0$. Describe how the states A, B, and C move on the Bloch sphere during this time evolution.

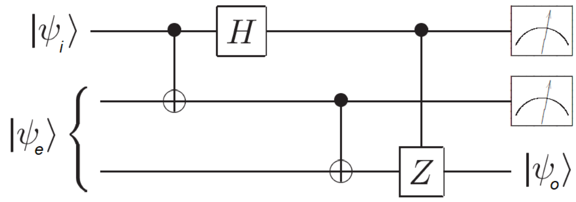
e) Show that the state $|\uparrow_z\rangle$ is an eigenstate of $\hat{H}_0$ and calculate corrections to its eigenenergy to first and second order in the perturbation $\hat{H}_1 = \lambda \frac{\hbar\omega_0}{2} \hat{\sigma}_x$, $\lambda \ll 1$. How is the eigenstate changed to first order in the perturbation?

f) If the (unperturbed) spin is put in contact with an environment at temperature T, it will relax to a thermal mixture given by the density matrix $\hat{\rho} = e^{-\hat{H}_0/(k_B T)}/\mathrm{tr}[e^{-\hat{H}_0/(k_B T)}]$. Explain how this mixed state can be represented by a point in the figure above.

g) Determine the purity $\mathrm{tr}[\hat{\rho}^2]$ of the thermal state in f) and discuss the limits of a low, $k_B T \ll \hbar\omega_0$, and a high temperature, $k_B T \gg \hbar\omega_0$.

Problem 3

Consider the 3-qubit circuit below, where $|\psi_i\rangle = \alpha|0\rangle + \beta|1\rangle$ with $|\alpha|^2 + |\beta|^2 = 1$ is the state of the first qubit, and $|\psi_e\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ is an entangled state of the other two. The Hadamard gate is denoted by H, and the Z-gate, similar to the CNOT gate, is *conditional* on the first qubit being $|1\rangle$. In that case, it acts as $|0\rangle \to |0\rangle$ and $|1\rangle \to -|1\rangle$ on the third qubit.



a) Show that the full, intermediate 3-qubit state after the Hadamard gate reads

$$\frac{1}{2}\Big[\alpha\big(|0\rangle_1 + |1\rangle_1\big)\big(|0\rangle_2|0\rangle_3 + |1\rangle_2|1\rangle_3\big) + \beta\big(|0\rangle_1 - |1\rangle_1\big)\big(|1\rangle_2|0\rangle_3 + |0\rangle_2|1\rangle_3\big)\Big]$$

b) Determine the full, intermediate 3-qubit state after the controlled Z-gate.

c) The circuit ends with a measurement in the basis $\{|0\rangle, |1\rangle\}$ of the first two qubits. What is the output state $|\psi_o\rangle$ for the third qubit?

d) Discuss how this circuit functions differently from a quantum teleportation circuit that uses classical communication.

We now consider a 3-qubit quantum gate that is represented by the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

in the basis $\{|000\rangle, |001\rangle, |010\rangle, |011\rangle, |100\rangle, |101\rangle, |110\rangle, |111\rangle\}$

e) Describe in your own words what this quantum gate does and suggest a suitable, graphical way of representing it in a circuit diagram.

End of exam set