PHYS-E0414 Advanced Quantum Mechanics

Final exam, December 09, 2020, 13.00-16.00

You should answer in English unless you have special permission to use another language. You are free to use the lecture notes, books, the exercises, electronic devices, etc. (No communication allowed.) Please write your name, student number, study program, course code, and the date in all of your papers. There are 3 problems in this exam set which consists of 2 pages.

Exercise 1

Answer the following questions in your own words. No calculations are needed. Less than one page should suffice to answer all three questions:

a) What is the difference between orbital and spin angular momentum?

b) How do two pure spin states on the equator of the Bloch sphere differ?

c) What are the issues that currently prevent quantum computers from working?

Exercise 2

Consider a harmonic oscillator coupled to a two-level system described by the Hamiltonian

$$\hat{H} = \hbar\omega_0(\hat{a}^\dagger\hat{a} + 1/2) + \hbar\Omega\hat{\sigma}^\dagger\hat{\sigma} + \hbar g\left(\hat{a}\hat{\sigma}^\dagger + \hat{a}^\dagger\hat{\sigma}\right), \tag{1}$$

where $\hat{a}^\dagger$ and $\hat{a}$ are the ladder operators of the oscillator, and $\hat{\sigma} = (\hat{\sigma}_x + i\hat{\sigma}_y)/2$ acts on the two-level system with $\hat{\sigma}_x$, $\hat{\sigma}_y$, and $\hat{\sigma}_z$ being the usual Pauli matrices. Physically, the harmonic oscillator can be an optical cavity, and each state of the oscillator, $|n\rangle$, then corresponds to having a certain number $n = 0, 1, 2, ...$ of photons in the cavity. The two-level system may be an atom with a ground state that we denote by $|g\rangle$ and an excited state that we denote by $|e\rangle$.

a) Determine the matrix representations of $\hat{\sigma}$ and $\hat{\sigma}^\dagger$ in the basis $\{|g\rangle, |e\rangle\}$ (with $\hat{\sigma}_y|g\rangle = i|e\rangle$ and $\hat{\sigma}_y|e\rangle = -i|g\rangle$). How do $\hat{\sigma}$ and $\hat{\sigma}^\dagger$ act on the ground and excited state of the atom?

b) Describe the physical meaning of each of the three terms in the Hamiltonian above. Specify in particular the physical meaning of ω_0, Ω, and g.

In the following, we set $\omega_0 = \Omega$.

c) Show that the Hamiltonian may be written as $\hat{H} = \hbar\omega_0\left(\hat{\gamma}^2 + \frac{1}{2}\right) + \hbar g\hat{\gamma}$, where $\hat{\gamma} = \hat{a}\hat{\sigma}^\dagger + \hat{a}^\dagger\hat{\sigma}$.

d) Determine the eigenvalues and eigenstates of $\hat{H}$.
Hint: It may help first to consider how $\hat{\gamma}$ acts on the states $|n+1\rangle|g\rangle$ and $|n\rangle|e\rangle$.

e) Determine the state $|\Psi(\tau)\rangle$ of the combined atom-cavity system at the time $t = \tau$, if the initial state of the system at the time $t = 0$ is $|\Psi(0)\rangle = |n\rangle|e\rangle$.

f) Find the reduced density matrix of the atom in e) for $t = \tau$. Determine the times, when the atom is in a pure state.

Now, the energy of the excited atom is changed from $\hbar\Omega$ to $\hbar(\Omega + \delta\Omega)$, $\delta\Omega \ll \Omega$ (with $\omega_0 = \Omega$).

g) Using perturbation theory, find the corrections to the eigenenergy of the states $|\Phi_n^{(\pm)}\rangle = (|n\rangle|e\rangle \pm |n+1\rangle|g\rangle)/\sqrt{2}$ for $n \geq 0$ to first order in $\delta\Omega$.

Exercise 3

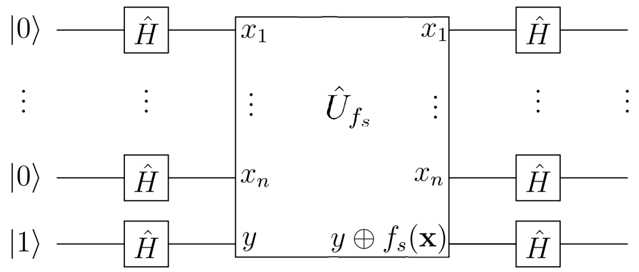
Consider a secret string of n bits, $\mathbf{s} = (s_1, \ldots, s_n)$, encoded in a function $f_{\mathbf{s}}(\mathbf{x})$ that for a given input $\mathbf{x} = (x_1, \ldots, x_n)$ returns 0, if the inner product $\mathbf{x} \cdot \mathbf{s} \equiv x_1 s_1 + \cdots + x_n s_n$ is even (including zero), and 1, if the inner product is odd. Classically, one can determine $\mathbf{s}$ by evaluating the function n times with the inputs $\mathbf{x_1} = (1, 0, \ldots, 0)$, $\mathbf{x_2} = (0, 1, \ldots, 0)$,. . ., $\mathbf{x_n} = (0, 0, \ldots, 1)$.

Let us see, if we can determine $\mathbf{s}$ with less calls of $f_{\mathbf{s}}(\mathbf{x})$ using the principles of quantum mechanics. To this end, we assume that the function is implemented quantum mechanically as

$$\hat{U}_{f_s}|\mathbf{x}\rangle|y\rangle = |\mathbf{x}\rangle|y \oplus f_s(\mathbf{x})\rangle, \quad y = 0, 1,$$

where $|\mathbf{x}\rangle = |x_1\rangle|x_2\rangle \cdots |x_n\rangle$, and $0 \oplus 0 = 0$, $0 \oplus 1 = 1$, $1 \oplus 0 = 1$, but $1 \oplus 1 = 0$.

We now analyze the circuit below with the input $|0\rangle \cdots |0\rangle|1\rangle$, and $\hat{H}$ denotes a Hadamard gate:



a) First, show that n Hadamard gates act as $\hat{H}^{\otimes n}|\mathbf{x}\rangle = \frac{1}{\sqrt{2}^n}\sum_{\mathbf{z}}(-1)^{\mathbf{x}\cdot\mathbf{z}}|\mathbf{z}\rangle$, where the sum runs over all possible strings. *Hint:* Start with a single qubit, for which $\hat{H}|x\rangle = \frac{|0\rangle + (-1)^x|1\rangle}{\sqrt{2}}$.

b) Next, show that the operator $\hat{U}_{f_s}$ acts as $\hat{U}_{f_s}|\mathbf{x}\rangle|-\rangle = (-1)^{f_s(\mathbf{x})}|\mathbf{x}\rangle|-\rangle$, where $|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$.

c) For the circuit above, show that the state after the operator $\hat{U}_{f_s}$ reads $\frac{1}{\sqrt{2}^n}\sum_{\mathbf{x}}(-1)^{f_s(\mathbf{x})}|\mathbf{x}\rangle|-\rangle$.

d) Show that $\hat{H}^{\otimes n}|\mathbf{s}\rangle = \frac{1}{\sqrt{2}^n}\sum_{\mathbf{x}}(-1)^{f_s(\mathbf{x})}|\mathbf{x}\rangle$ and $(\hat{H}^{\otimes n})^2 = \hat{1}$ and use this to determine the state at the output of the circuit. Comment on the result.

e) Show that $\hat{U}_{f_s} = \prod_{i=1}^{n}(\hat{C}_{i,n+1})^{s_i}$, where $\hat{C}_{i,j}$ is a CNOT gate that flips qubit j, if qubit i is in the state $|1\rangle$. Draw a circuit diagram that implements $\hat{U}_{f_s}$ for $\mathbf{s} = (1, 0, 1)$.

End of exam set