Aalto University
School of Electrical
Engineering

Lecture 9: Space-Vector Models

ELEC-E8405 Electric Drives

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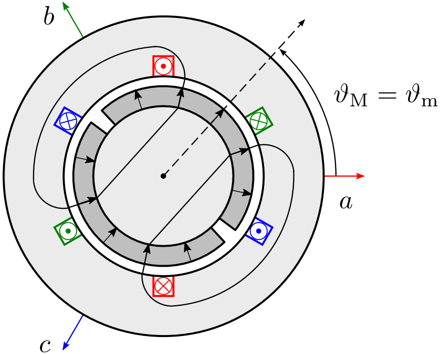
Learning Outcomes

After this lecture and exercises you will be able to:

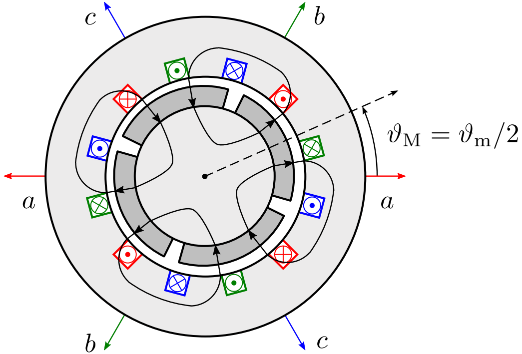
- Include the number of pole pairs in the machine models
- Transform phase variables to a space vector (and vice versa)
- Transform space vectors to different coordinates
- Express the space-vector model of the synchronous machine in rotor coordinates
- Calculate steady-state operating points of the synchronous machine

Outline

Number of Pole Pairs n_p



2 poles ($n_\mathrm{p} = 1$)



4 poles ($n_\mathrm{p} = 2$)

Electrical angular speed $\omega_\mathrm{m} = n_\mathrm{p}\omega_\mathrm{M}$ and electrical angle $\vartheta_\mathrm{m} = n_\mathrm{p}\vartheta_\mathrm{M}$

Synchronous Rotor Speeds

- Stator (supply) frequency f (Hz)
- Electrical angular speed (rad/s)

$$\omega_\mathrm{m} = 2\pi f$$

- Rotor angular speed (rad/s)

$$\omega_\mathrm{M} = \frac{\omega_\mathrm{m}}{n_\mathrm{p}}$$

- Rotor speed (r/min)

$$n = \frac{f}{n_\mathrm{p}} \frac{60\ \mathrm{s}}{\mathrm{min}}$$

Speeds for $f = 50$ Hz

No of pole pairs n_p	Speed n (r/min)
1	3000
2	1500
3	1000
4	750
5	600
6	500

Note that in converter-fed motor drives, the rated supply frequency of the motor does not need to be 50 Hz.

1-Phase Machine

- Phase voltage

$$u_\mathrm{a} = R i_\mathrm{a} + \frac{\mathrm{d}\psi_\mathrm{a}}{\mathrm{d}t}$$

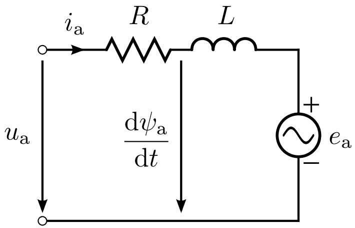
- Phase flux linkage

$$\psi_\mathrm{a} = L i_\mathrm{a} + \psi_\mathrm{fa}$$

where $\psi_\mathrm{fa} = \psi_\mathrm{f}\cos(\vartheta_\mathrm{m})$

- Back-emf

$$e_\mathrm{a} = \frac{\mathrm{d}\psi_\mathrm{fa}}{\mathrm{d}t} = -\omega_\mathrm{m}\psi_\mathrm{f}\sin(\vartheta_\mathrm{m})$$



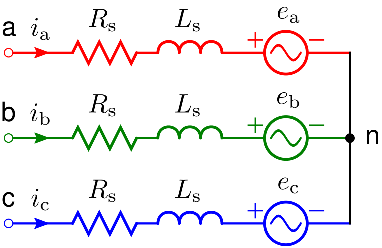
- Mechanical power

$$p_\mathrm{M} = e_\mathrm{a} i_\mathrm{a} = \tau_\mathrm{M}\omega_\mathrm{m}/n_\mathrm{p}$$

- Torque

$$\tau_\mathrm{M} = -n_\mathrm{p}\, i_\mathrm{a}\psi_\mathrm{f}\sin(\vartheta_\mathrm{m})$$

Synchronous Machine: Phase-Variable Model



$$u_\mathrm{a} = R_\mathrm{s} i_\mathrm{a} + \frac{\mathrm{d}\psi_\mathrm{a}}{\mathrm{d}t}$$

$$u_\mathrm{b} = R_\mathrm{s} i_\mathrm{b} + \frac{\mathrm{d}\psi_\mathrm{b}}{\mathrm{d}t}$$

$$u_\mathrm{c} = R_\mathrm{s} i_\mathrm{c} + \frac{\mathrm{d}\psi_\mathrm{c}}{\mathrm{d}t}$$

$$\psi_\mathrm{a} = L_\mathrm{s} i_\mathrm{a} + \psi_\mathrm{f}\cos(\vartheta_\mathrm{m})$$

$$\psi_\mathrm{b} = L_\mathrm{s} i_\mathrm{b} + \psi_\mathrm{f}\cos(\vartheta_\mathrm{m} - 2\pi/3)$$

$$\psi_\mathrm{c} = L_\mathrm{s} i_\mathrm{c} + \psi_\mathrm{f}\cos(\vartheta_\mathrm{m} - 4\pi/3)$$

$$e_\mathrm{a} = -\omega_\mathrm{m}\psi_\mathrm{f}\sin(\vartheta_\mathrm{m})$$

$$e_\mathrm{b} = -\omega_\mathrm{m}\psi_\mathrm{f}\sin(\vartheta_\mathrm{m} - 2\pi/3)$$

$$e_\mathrm{c} = -\omega_\mathrm{m}\psi_\mathrm{f}\sin(\vartheta_\mathrm{m} - 4\pi/3)$$

$$\tau_\mathrm{M} = -n_\mathrm{p}\psi_\mathrm{f}\left[i_\mathrm{a}\sin(\vartheta_\mathrm{m}) + i_\mathrm{b}\sin(\vartheta_\mathrm{m} - 2\pi/3) + i_\mathrm{c}\sin(\vartheta_\mathrm{m} - 4\pi/3)\right]$$

Outline

Why Space Vectors?

1. Complex phasor models
   - Simple to use but limited to steady-state conditions
2. Phase-variable models
   - Valid both in transient and steady states
   - Too complicated
3. Space-vector models
   - Phase-variable models can be directly transformed to space-vector models
   - Compact representation, insightful physical interpretations
   - Commonly applied to analysis, modelling, and control of 3-phase systems

About Complex Numbers

- Complex number

$$\boldsymbol{z} = x + \mathrm{j}y$$

- Complex conjugate of $\boldsymbol{z}$

$$\boldsymbol{z}^* = x - \mathrm{j}y$$

- Magnitude of $\boldsymbol{z}$

$$z = |\boldsymbol{z}| = \sqrt{x^2 + y^2}$$

- Euler's formula

$$\mathrm{e}^{\mathrm{j}\vartheta} = \cos\vartheta + \mathrm{j}\sin\vartheta$$

- Rotating the position vector by 90°

$$\mathrm{j}\boldsymbol{z} = \mathrm{j}(x + \mathrm{j}y) = -y + \mathrm{j}x$$

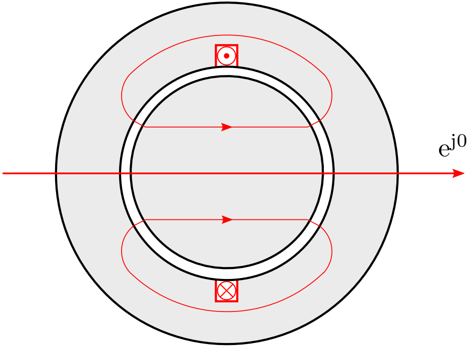
- Dot product

$$\begin{aligned} \mathrm{Re}\{\boldsymbol{z}_1\boldsymbol{z}_2^*\} &= \mathrm{Re}\{(x_1 + \mathrm{j}y_1)(x_2 - \mathrm{j}y_2)\} \\ &= x_1x_2 + y_1y_2 \end{aligned}$$

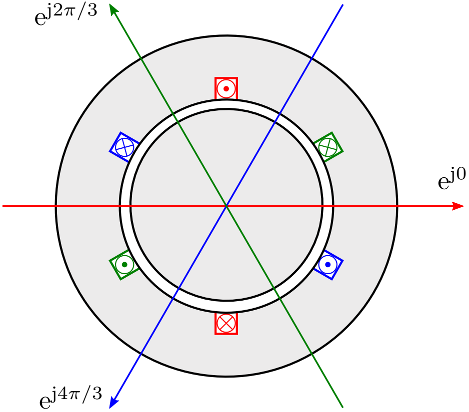
- Cross product

$$\begin{aligned} \mathrm{Im}\{\boldsymbol{z}_1\boldsymbol{z}_2^*\} &= \mathrm{Im}\{(x_1 + \mathrm{j}y_1)(x_2 - \mathrm{j}y_2\} \\ &= y_1x_2 - y_2x_1 \end{aligned}$$

Magnetic Axes in the Complex Plane



Phase a



All 3 phases

Windings are sinusoidally distributed along the air gap

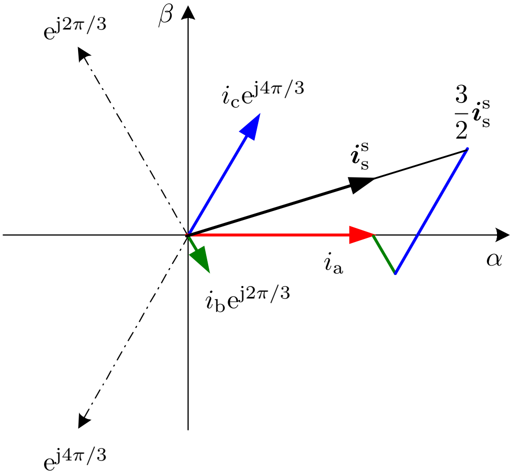
Space-Vector Transformation

- Space vector is a complex variable (signal)

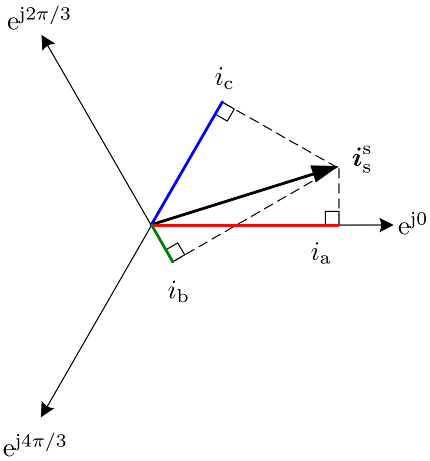
$$\boxed{\boldsymbol{i}_\mathrm{s}^\mathrm{s} = \frac{2}{3}\left(i_\mathrm{a} + i_\mathrm{b}\mathrm{e}^{\mathrm{j}2\pi/3} + i_\mathrm{c}\mathrm{e}^{\mathrm{j}4\pi/3}\right)}$$

  where i_a, i_b, and i_c are arbitrarily varying instantaneous phase variables
- Superscript s marks stator coordinates
- Same transformation applies for voltages and flux linkages
- Space vector does not include the zero-sequence component (not a problem since the stator winding is delta-connected or the star point is not connected)

Peak-value scaling of space vectors will be used in this course. Furthermore, we will use the subscript s to refer to the stator quantities, e.g., the stator current vector $\boldsymbol{i}_\mathrm{s}$ and the stator voltage vector $\boldsymbol{u}_\mathrm{s}$, since this is a very common convention in the literature.



$$\boldsymbol{i}_\mathrm{s}^\mathrm{s} = \frac{2}{3}\left(i_\mathrm{a} + i_\mathrm{b}\mathrm{e}^{\mathrm{j}2\pi/3} + i_\mathrm{c}\mathrm{e}^{\mathrm{j}4\pi/3}\right)$$



$$i_\mathrm{a} = \mathrm{Re}\left\{\boldsymbol{i}_\mathrm{s}^\mathrm{s}\right\}$$

$$i_\mathrm{b} = \mathrm{Re}\left\{\mathrm{e}^{-\mathrm{j}2\pi/3}\boldsymbol{i}_\mathrm{s}^\mathrm{s}\right\}$$

$$i_\mathrm{c} = \mathrm{Re}\left\{\mathrm{e}^{-\mathrm{j}4\pi/3}\boldsymbol{i}_\mathrm{s}^\mathrm{s}\right\}$$

Examples: Space Vectors Rotate in Steady State

- Positive sequence

$$i_{\mathrm{a}} = \sqrt{2}I_+ \cos(\omega_{\mathrm{m}}t + \phi_+)$$
$$i_{\mathrm{b}} = \sqrt{2}I_+ \cos(\omega_{\mathrm{m}}t - 2\pi/3 + \phi_+)$$
$$i_{\mathrm{c}} = \sqrt{2}I_+ \cos(\omega_{\mathrm{m}}t - 4\pi/3 + \phi_+)$$

- Space vector

$$\boldsymbol{i}_{\mathrm{s}}^{\mathrm{s}} = \sqrt{2}I_+\, \mathrm{e}^{\mathrm{j}(\omega_{\mathrm{m}}t + \phi_+)}$$

- Negative sequence

$$i_{\mathrm{a}} = \sqrt{2}I_- \cos(\omega_{\mathrm{m}}t + \phi_-)$$
$$i_{\mathrm{b}} = \sqrt{2}I_- \cos(\omega_{\mathrm{m}}t - 4\pi/3 + \phi_-)$$
$$i_{\mathrm{c}} = \sqrt{2}I_- \cos(\omega_{\mathrm{m}}t - 2\pi/3 + \phi_-)$$

- Space vector

$$\boldsymbol{i}_{\mathrm{s}}^{\mathrm{s}} = \sqrt{2}I_-\, \mathrm{e}^{-\mathrm{j}(\omega_{\mathrm{m}}t + \phi_-)}$$

- Non-sinusoidal periodic waveform

$$\boldsymbol{i}_{\mathrm{s}}^{\mathrm{s}} = \sqrt{2}I_1\, \mathrm{e}^{\mathrm{j}(\omega_{\mathrm{m}}t + \phi_1)} + \sqrt{2}I_5\, \mathrm{e}^{-\mathrm{j}(5\omega_{\mathrm{m}}t + \phi_5)} + \sqrt{2}I_7\, \mathrm{e}^{\mathrm{j}(7\omega_{\mathrm{m}}t + \phi_7)} \ldots$$

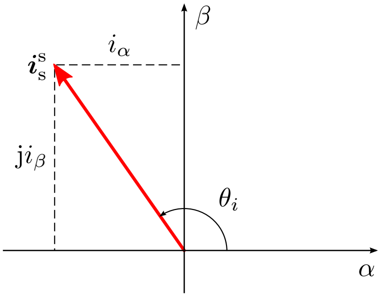
Representation in Component and Polar Forms

- Component form

$$\boldsymbol{i}_\mathrm{s}^\mathrm{s} = i_\alpha + \mathrm{j} i_\beta$$

- Polar form

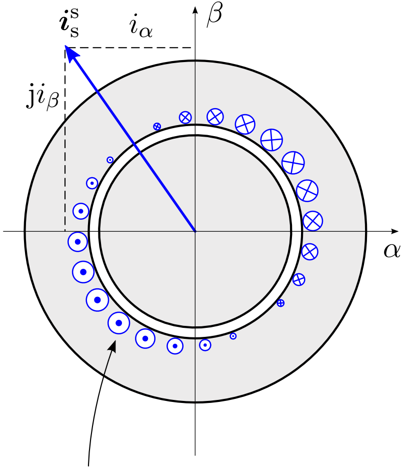
$$\begin{aligned}\boldsymbol{i}_\mathrm{s}^\mathrm{s} &= i_\mathrm{s} \mathrm{e}^{\mathrm{j}\theta_i} \\ &= \underbrace{i_\mathrm{s} \cos(\theta_i)}_{i_\alpha} + \mathrm{j}\, \underbrace{i_\mathrm{s} \sin(\theta_i)}_{i_\beta}\end{aligned}$$



- Generally, both the magnitude i_s and the angle θ_i may vary arbitrarily in time
- Positive sequence in steady state: $i_\mathrm{s} = \sqrt{2} I$ is constant and $\theta_i = \omega_\mathrm{m} t + \phi$

Physical Interpretation: Sinusoidal Distribution in Space

- 3-phase winding creates the current and the mmf, which are sinusoidally distributed along the air gap
- Space vector represents the instantaneous magnitude and angle of the sinusoidal distribution in space
- Magnitude and the angle can vary freely in time



Rotating current distribution produced by the 3-phase stator winding

Outline

Space-Vector Model of the Synchronous Machine

- Stator voltage

$$\boldsymbol{u}_\mathrm{s}^\mathrm{s} = R_\mathrm{s}\boldsymbol{i}_\mathrm{s}^\mathrm{s} + \frac{\mathrm{d}\boldsymbol{\psi}_\mathrm{s}^\mathrm{s}}{\mathrm{d}t}$$

- Stator flux linkage

$$\boldsymbol{\psi}_\mathrm{s}^\mathrm{s} = L_\mathrm{s}\boldsymbol{i}_\mathrm{s}^\mathrm{s} + \psi_\mathrm{f}\mathrm{e}^{\mathrm{j}\vartheta_\mathrm{m}}$$

- Torque can be expressed in various forms
- Following form is convenient since it holds for other AC machines as well

$$\tau_\mathrm{M} = \frac{3n_\mathrm{p}}{2}\,\mathrm{Im}\left\{\boldsymbol{i}_\mathrm{s}^\mathrm{s}\boldsymbol{\psi}_\mathrm{s}^{\mathrm{s}*}\right\}$$

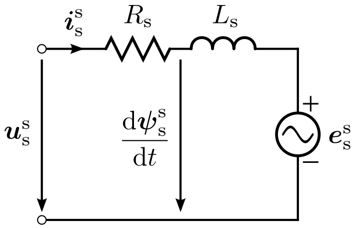
Derive these voltage and flux linkage equations starting from the phase-variable model and the definition of the space vector. Also show that the space-vector and phase-variable formulations for the torque are equal.

Space-Vector Equivalent Circuit

- Stator voltage can be rewritten as

$$\boldsymbol{u}_\mathrm{s}^\mathrm{s} = R_\mathrm{s}\boldsymbol{i}_\mathrm{s}^\mathrm{s} + L_\mathrm{s}\frac{\mathrm{d}\boldsymbol{i}_\mathrm{s}^\mathrm{s}}{\mathrm{d}t} + \boldsymbol{e}_\mathrm{s}^\mathrm{s}$$

- Back-emf $\boldsymbol{e}_\mathrm{s}^\mathrm{s} = \mathrm{j}\omega_\mathrm{m}\psi_\mathrm{f}\mathrm{e}^{\mathrm{j}\vartheta_\mathrm{m}}$ is proportional to the speed



Torque

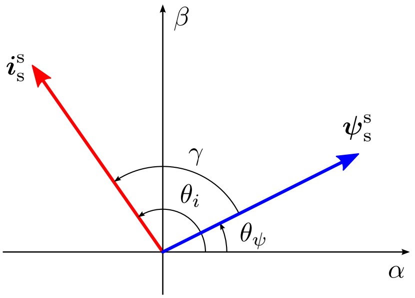
- Vectors in the polar form

$$\boldsymbol{i}_\mathrm{s}^\mathrm{s} = i_\mathrm{s}\mathrm{e}^{\mathrm{j}\theta_i} \qquad \boldsymbol{\psi}_\mathrm{s}^\mathrm{s} = \psi_\mathrm{s}\mathrm{e}^{\mathrm{j}\theta_\psi}$$

- Instantaneous torque

$$\tau_\mathrm{M} = \frac{3n_\mathrm{p}}{2}\,\mathrm{Im}\left\{\boldsymbol{i}_\mathrm{s}^\mathrm{s}\boldsymbol{\psi}_\mathrm{s}^{\mathrm{s}*}\right\} = \frac{3n_\mathrm{p}}{2} i_\mathrm{s}\psi_\mathrm{s}\sin(\gamma)$$

  where $\gamma = \theta_i - \theta_\psi$
- Nonzero γ is needed for torque production



Power

- Vectors in the component and polar forms

$$\boldsymbol{u}_\mathrm{s}^\mathrm{s} = u_\alpha + \mathrm{j}u_\beta = u_\mathrm{s}\mathrm{e}^{\mathrm{j}\theta_u} \qquad \boldsymbol{i}_\mathrm{s}^\mathrm{s} = i_\alpha + \mathrm{j}i_\beta = i_\mathrm{s}\mathrm{e}^{\mathrm{j}\theta_i}$$

- Instantaneous power fed to the stator

$$\begin{aligned} p_\mathrm{s} &= \frac{3}{2}\,\mathrm{Re}\left\{\boldsymbol{u}_\mathrm{s}^\mathrm{s}\boldsymbol{i}_\mathrm{s}^{\mathrm{s}*}\right\} \\ &= \frac{3}{2}\left(u_\alpha i_\alpha + u_\beta i_\beta\right) \\ &= \frac{3}{2}u_\mathrm{s}i_\mathrm{s}\cos(\varphi) \end{aligned}$$

where $\varphi = \theta_u - \theta_i$

The power calculated using the space vectors naturally agrees with the power $p_\mathrm{s} = u_\mathrm{a}i_\mathrm{a} + u_\mathrm{b}i_\mathrm{b} + u_\mathrm{c}i_\mathrm{c}$ calculated from the phase variables. Furthermore, in steady state, the rms-valued expression $P_\mathrm{s} = 3U_\mathrm{s}I_\mathrm{s}\cos(\varphi)$ is obtained, since $u_\mathrm{s} = \sqrt{2}U_\mathrm{s}$ and $i_\mathrm{s} = \sqrt{2}U_\mathrm{s}$ hold.

Outline

Example: Stopping the Rotating Vector

- Positive-sequence space vector in stator coordinates

$$\boldsymbol{i}_\mathrm{s}^\mathrm{s} = \sqrt{2}I\,\mathrm{e}^{\mathrm{j}(\omega_\mathrm{m}t+\phi)}$$

- Rotating vector can be stopped by the transformation

$$\boldsymbol{i}_\mathrm{s} = \boldsymbol{i}_\mathrm{s}^\mathrm{s}\,\mathrm{e}^{-\mathrm{j}\omega_\mathrm{m}t} = \sqrt{2}I\,\mathrm{e}^{\mathrm{j}\phi}$$

- In other words, we observe the vector now in a coordinate system rotating at ω_m
- In rotating coordinates, the vector is denoted without a superscript and the components are marked with the subscripts d and q

$$\boldsymbol{i}_\mathrm{s} = i_\mathrm{d} + \mathrm{j}i_\mathrm{q}$$

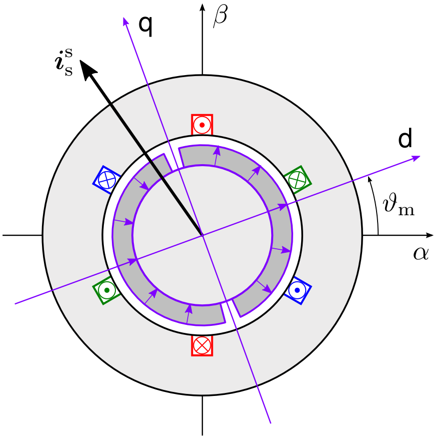
Coordinate Transformation

- Previous example assumed the rotor speed ω_m to be constant
- General dq transformation and its inverse are

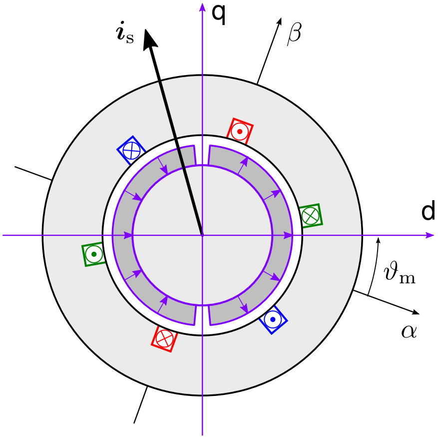
$$\boldsymbol{i}_\mathrm{s} = \boldsymbol{i}_\mathrm{s}^\mathrm{s} \mathrm{e}^{-\mathrm{j}\vartheta_\mathrm{m}} \qquad \text{dq transformation}$$
$$\boldsymbol{i}_\mathrm{s}^\mathrm{s} = \boldsymbol{i}_\mathrm{s} \mathrm{e}^{\mathrm{j}\vartheta_\mathrm{m}} \qquad \alpha\beta \text{ transformation}$$

where the rotor angle is

$$\vartheta_\mathrm{m} = \int \omega_\mathrm{m} \mathrm{d}t$$

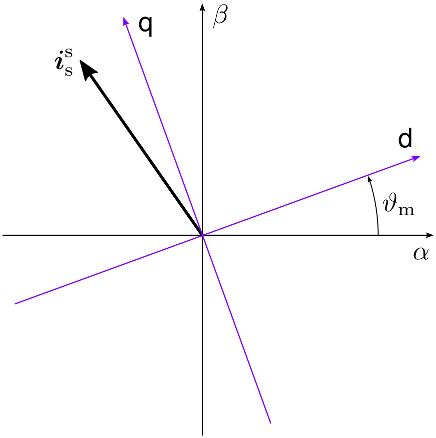


Stator coordinates ($\alpha\beta$)

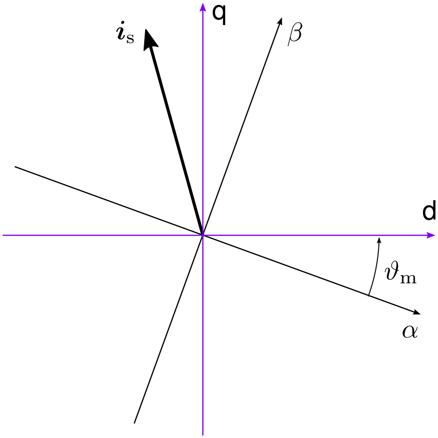


Rotor coordinates (dq)

$$\boldsymbol{i}_\mathrm{s} = \boldsymbol{i}_\mathrm{s}^\mathrm{s}\,\mathrm{e}^{-\mathrm{j}\vartheta_\mathrm{m}}$$



Stator coordinates ($\alpha\beta$)



Rotor coordinates (dq)

$$\boldsymbol{i}_{\mathrm{s}} = \boldsymbol{i}_{\mathrm{s}}^{\mathrm{s}}\,\mathrm{e}^{-\mathrm{j}\vartheta_{\mathrm{m}}}$$

Outline

Synchronous Machine Model in Rotor Coordinates

- Substitute $\boldsymbol{\psi}_\mathrm{s}^\mathrm{s} = \boldsymbol{\psi}_\mathrm{s}\mathrm{e}^{\mathrm{j}\vartheta_\mathrm{m}}$, $\boldsymbol{u}_\mathrm{s}^\mathrm{s} = \boldsymbol{u}_\mathrm{s}\mathrm{e}^{\mathrm{j}\vartheta_\mathrm{m}}$, and $\boldsymbol{i}_\mathrm{s}^\mathrm{s} = \boldsymbol{i}_\mathrm{s}\mathrm{e}^{\mathrm{j}\vartheta_\mathrm{m}}$

$$\boldsymbol{u}_\mathrm{s}\mathrm{e}^{\mathrm{j}\vartheta_\mathrm{m}} = R_\mathrm{s}\boldsymbol{i}_\mathrm{s}\mathrm{e}^{\mathrm{j}\vartheta_\mathrm{m}} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{\psi}_\mathrm{s}\mathrm{e}^{\mathrm{j}\vartheta_\mathrm{m}}\right) \quad \Rightarrow \quad \boxed{\boldsymbol{u}_\mathrm{s} = R_\mathrm{s}\boldsymbol{i}_\mathrm{s} + \frac{\mathrm{d}\boldsymbol{\psi}_\mathrm{s}}{\mathrm{d}t} + \mathrm{j}\omega_\mathrm{m}\boldsymbol{\psi}_\mathrm{s}}$$

$$\boldsymbol{\psi}_\mathrm{s}\mathrm{e}^{\mathrm{j}\vartheta_\mathrm{m}} = L_\mathrm{s}\boldsymbol{i}_\mathrm{s}\mathrm{e}^{\mathrm{j}\vartheta_\mathrm{m}} + \psi_\mathrm{f}\mathrm{e}^{\mathrm{j}\vartheta_\mathrm{m}} \quad \Rightarrow \quad \boxed{\boldsymbol{\psi}_\mathrm{s} = L_\mathrm{s}\boldsymbol{i}_\mathrm{s} + \psi_\mathrm{f}}$$

- Torque is proportional to i_q

$$\boxed{\tau_\mathrm{M} = \frac{3n_\mathrm{p}}{2}\,\mathrm{Im}\left\{\boldsymbol{i}_\mathrm{s}\boldsymbol{\psi}_\mathrm{s}^*\right\} = \frac{3n_\mathrm{p}}{2}\psi_\mathrm{f}i_\mathrm{q}}$$

while i_d does not contribute to the torque

Power Balance

- Stator voltage can be rewritten as

$$\boldsymbol{u}_\mathrm{s} = R_\mathrm{s}\boldsymbol{i}_\mathrm{s} + L_\mathrm{s}\frac{\mathrm{d}\boldsymbol{i}_\mathrm{s}}{\mathrm{d}t} + \mathrm{j}\omega_\mathrm{m}L_\mathrm{s}\boldsymbol{i}_\mathrm{s} + \boldsymbol{e}_\mathrm{s}$$

  where $\boldsymbol{e}_\mathrm{s} = \mathrm{j}\omega_\mathrm{m}\psi_\mathrm{f}$ is the back-emf
- Power balance is obtained from the stator voltage equation

$$p_\mathrm{s} = \frac{3}{2}\,\mathrm{Re}\left\{\boldsymbol{u}_\mathrm{s}\boldsymbol{i}_\mathrm{s}^*\right\} = \underbrace{\frac{3}{2}R_\mathrm{s}|\boldsymbol{i}_\mathrm{s}|^2}_{\text{Losses}} + \underbrace{\frac{3}{2}\frac{L_\mathrm{s}}{2}\frac{\mathrm{d}|\boldsymbol{i}_\mathrm{s}|^2}{\mathrm{d}t}}_{\text{Rate of change of energy in } L_\mathrm{s}} + \underbrace{\tau_\mathrm{M}\frac{\omega_\mathrm{m}}{n_\mathrm{p}}}_{\text{Mechanical power}}$$

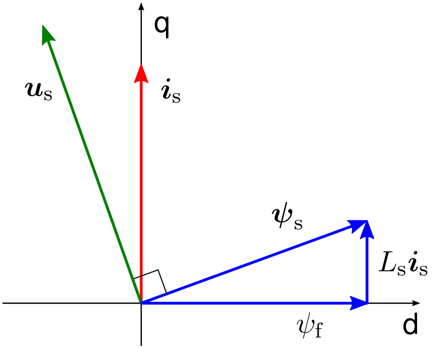
- Middle term is zero in steady state

Vector Diagram

- In steady state, $\mathrm{d}/\mathrm{d}t = 0$ holds in rotor coordinates
- Stator voltage

$$\begin{aligned}\boldsymbol{u}_\mathrm{s} &= R_\mathrm{s}\boldsymbol{i}_\mathrm{s} + \mathrm{j}\omega_\mathrm{m}\boldsymbol{\psi}_\mathrm{s} \\ &= R_\mathrm{s}\boldsymbol{i}_\mathrm{s} + \mathrm{j}\omega_\mathrm{m}(L_\mathrm{s}\boldsymbol{i}_\mathrm{s} + \psi_\mathrm{f})\end{aligned}$$

- Steady-state operating points can be illustrated by means of vector diagrams



Assumption: $R_\mathrm{s} \approx 0$