SHAPES IN ACTION

12.10.2018

Projective Geometry cont.

& Rotation Around a Plane

Taneli Luotoniemi

Boy's Surface (Mathematical Research Institute of Oberwolfach, 1991)

Pole-polar duality

Apollonius of Perga (circa 260-190 B.C.)

Exercise:

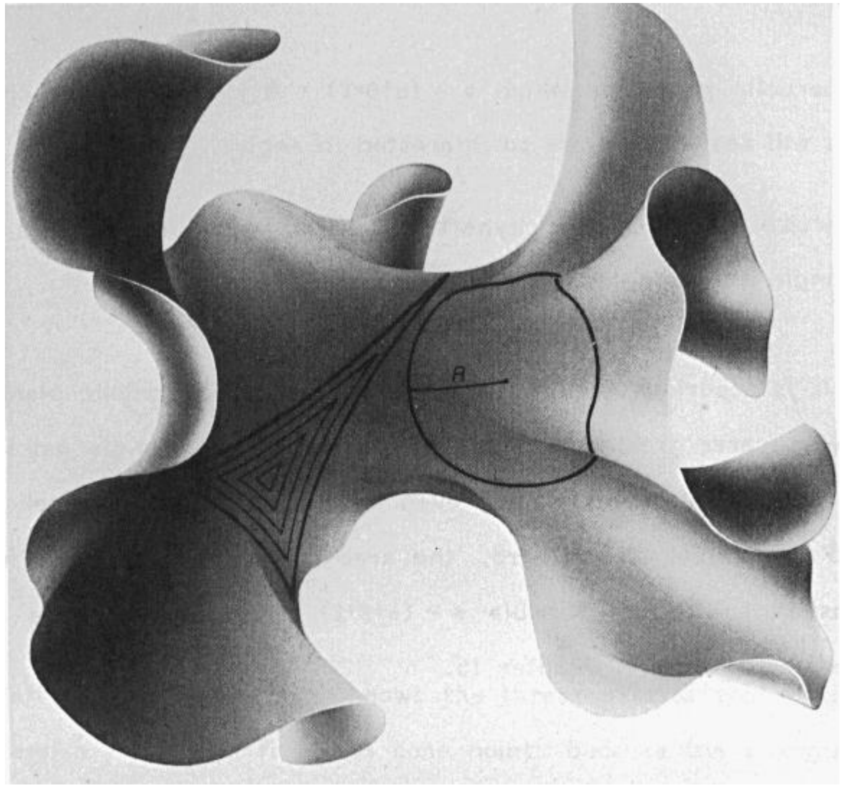
Does the construction work for other conics?

Hyperbolic geometry

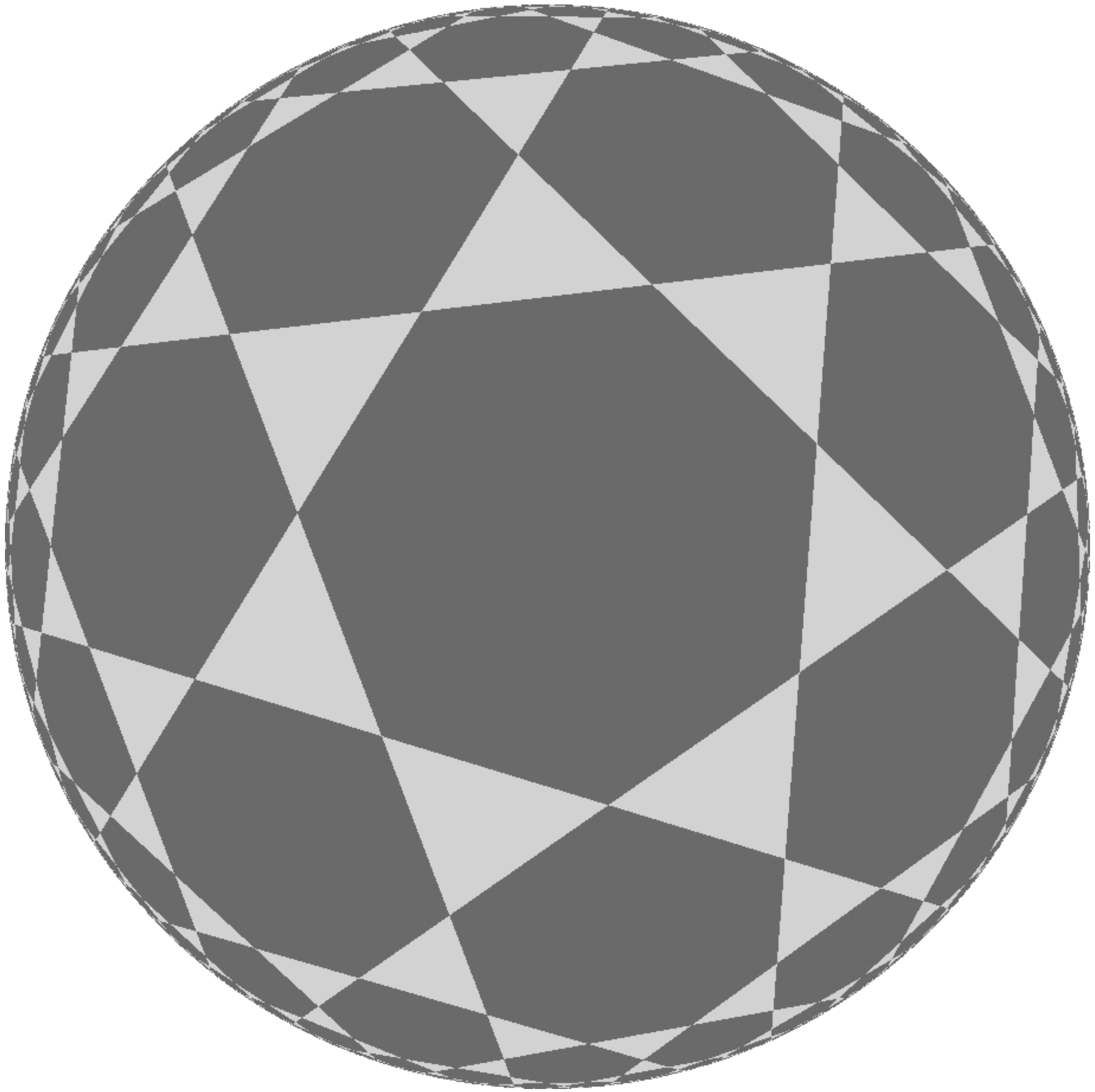
More parallels to a line through a point

Sum of triangle angles < 180 degrees

Circumference of a circle > 2 x pi x radius



Beltrami-Klein disk model



Hyperbolic geometry

parallels

ultraparallels

perpendicularity

reflection with respect to a point

THE DESARGUES CONFIGURATION AS A GNOMONIC PROJECTION

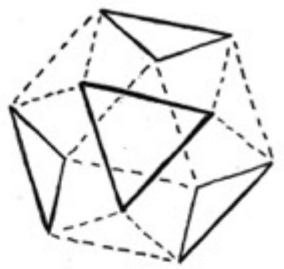
TETRAHEDRON
(3-SIMPLEX)
4 vertices
6 edges
4 faces (triangles)

TETRAHEDRON
(3-SIMPLEX)

4 vertices
6 edges
4 faces (triangles)

CUBOCTAHEDRON
(EXPANDED
TETRAHEDRON)

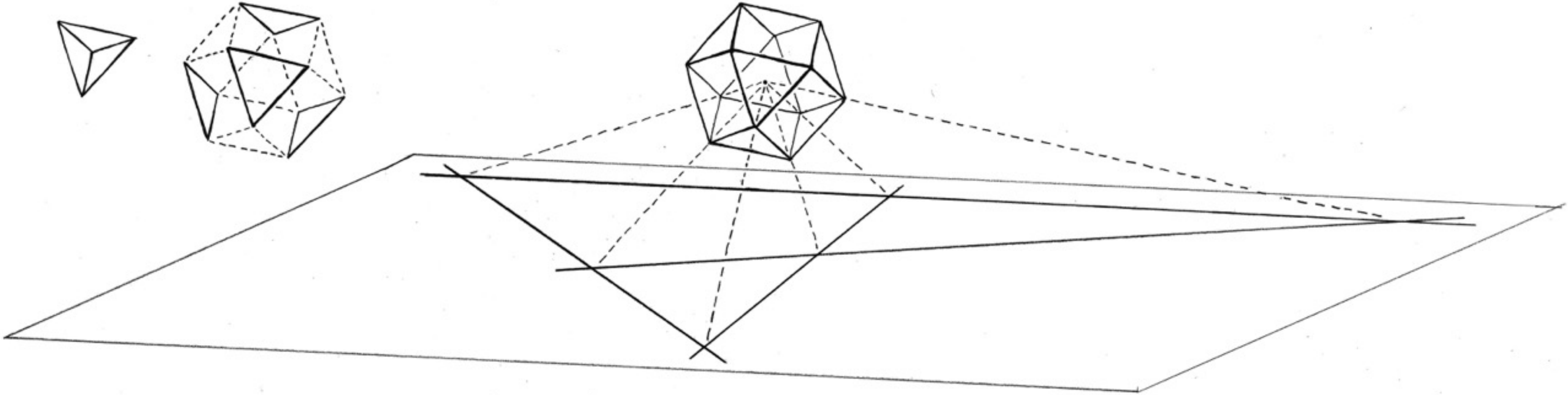
12 vertices
24 edges
14 faces (8 triangles & 6 squares)



TETRAHEDRON
(3-SIMPLEX)
4 vertices
6 edges
4 faces (triangles)

CUBOCTAHEDRON
(EXPANDED
TETRAHEDRON)
12 vertices
24 edges
14 faces (8 triangles & 6 squares)

GNOMONIC
PROJECTION



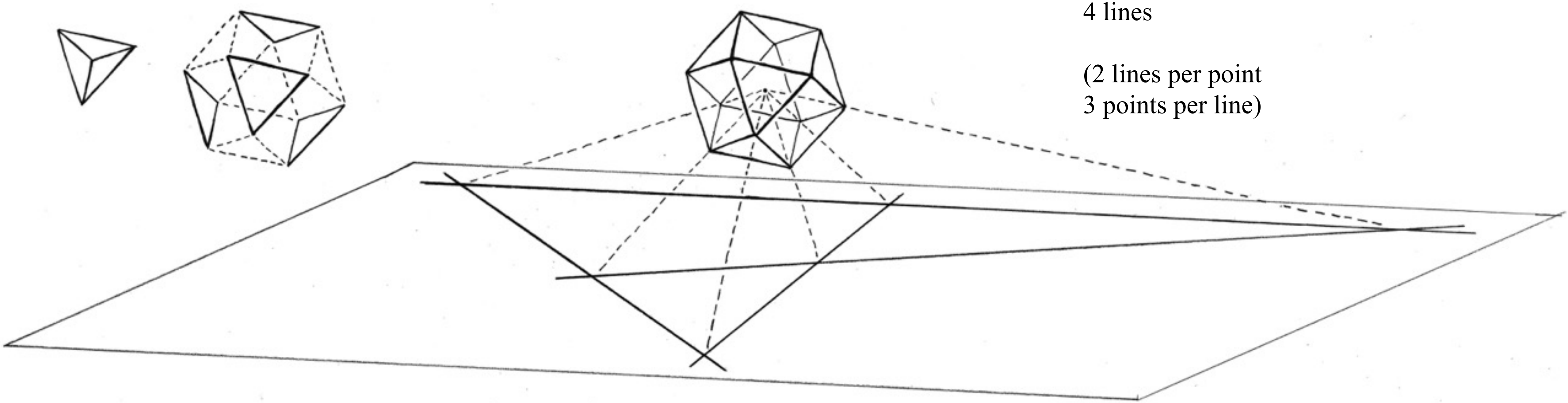
TETRAHEDRON
(3-SIMPLEX)
4 vertices
6 edges
4 faces (triangles)

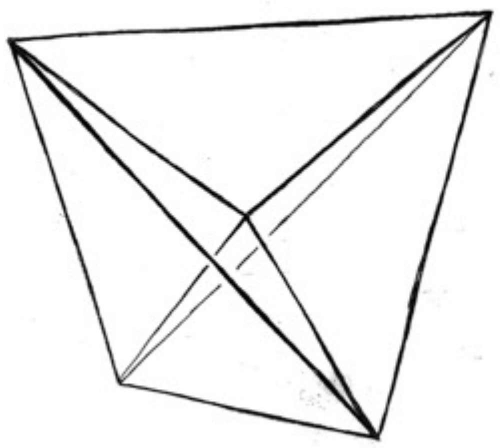
CUBOCTAHEDRON
(EXPANDED
TETRAHEDRON)
12 vertices
24 edges
14 faces (8 triangles & 6 squares)

GNOMONIC
PROJECTION

THE COMPLETE
QUADRILATERAL
6 points
4 lines

(2 lines per point
3 points per line)





PENTACHORON

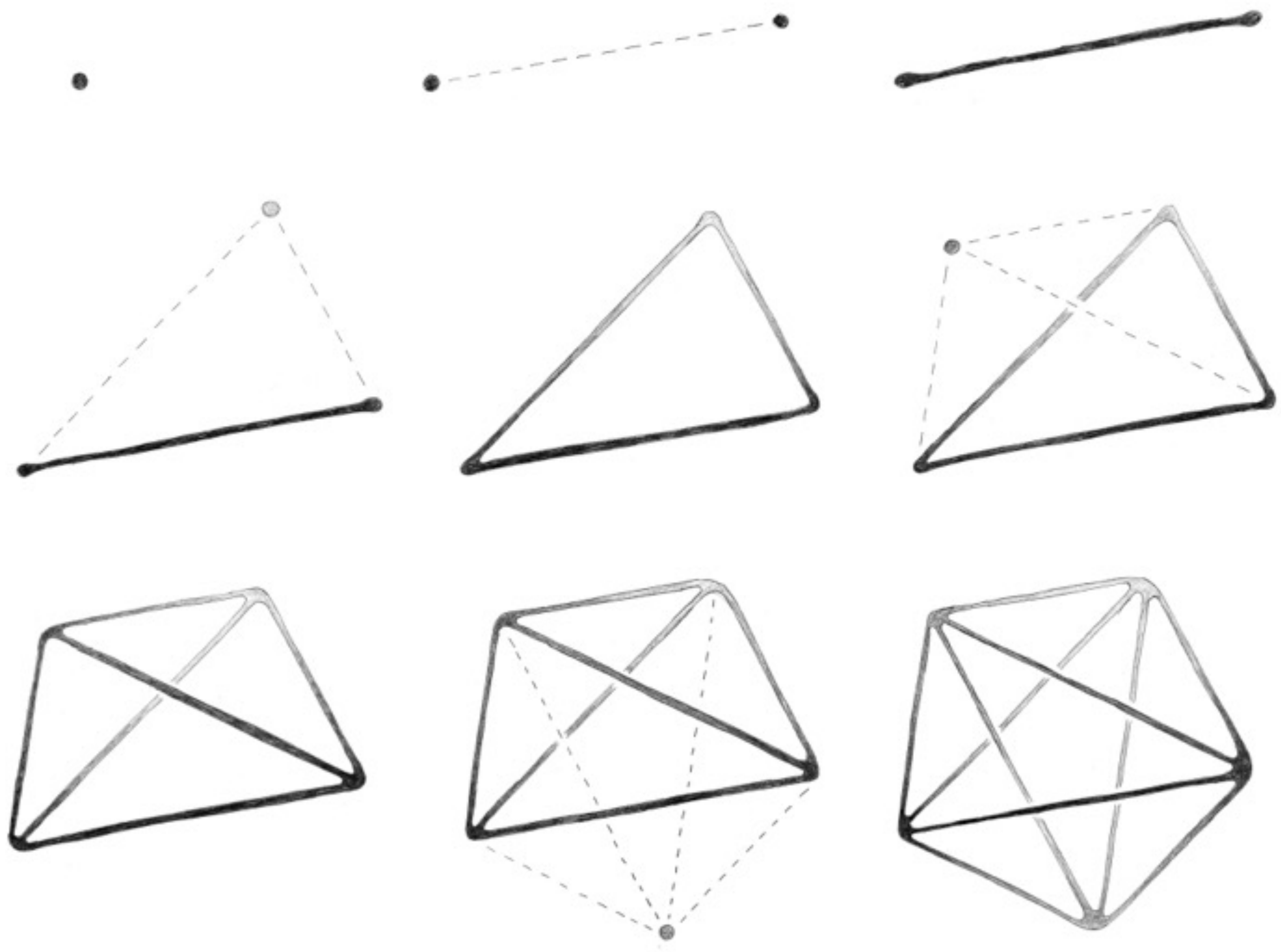
5 vertices

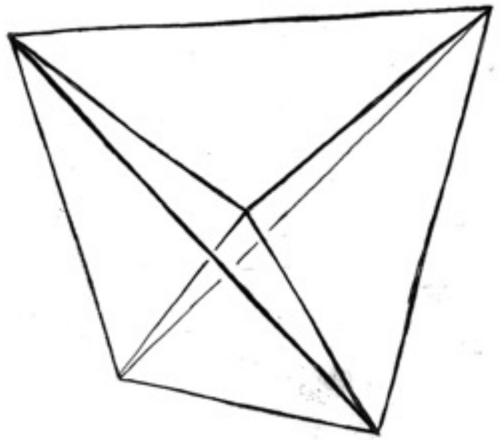
10 edges

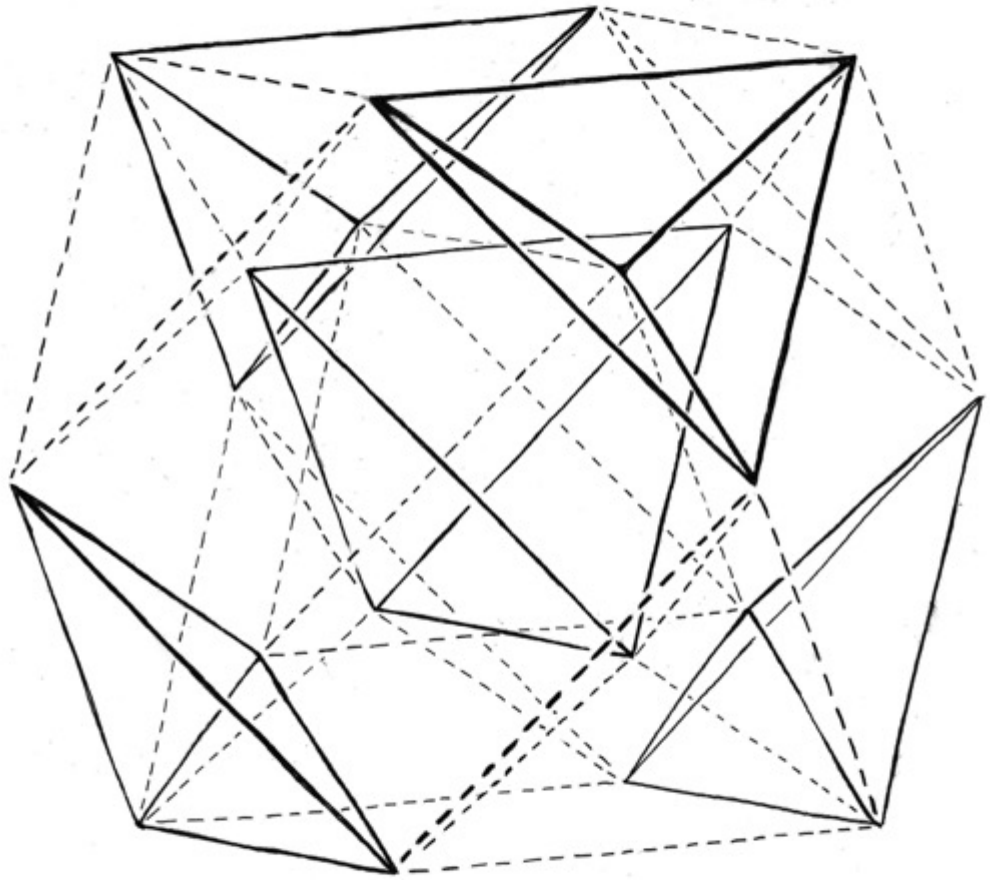
10 faces (triangles)

5 cells (tetrahedra)

Constructing the pentachoron







PENTACHORON → EXPANDED PENTACHORON

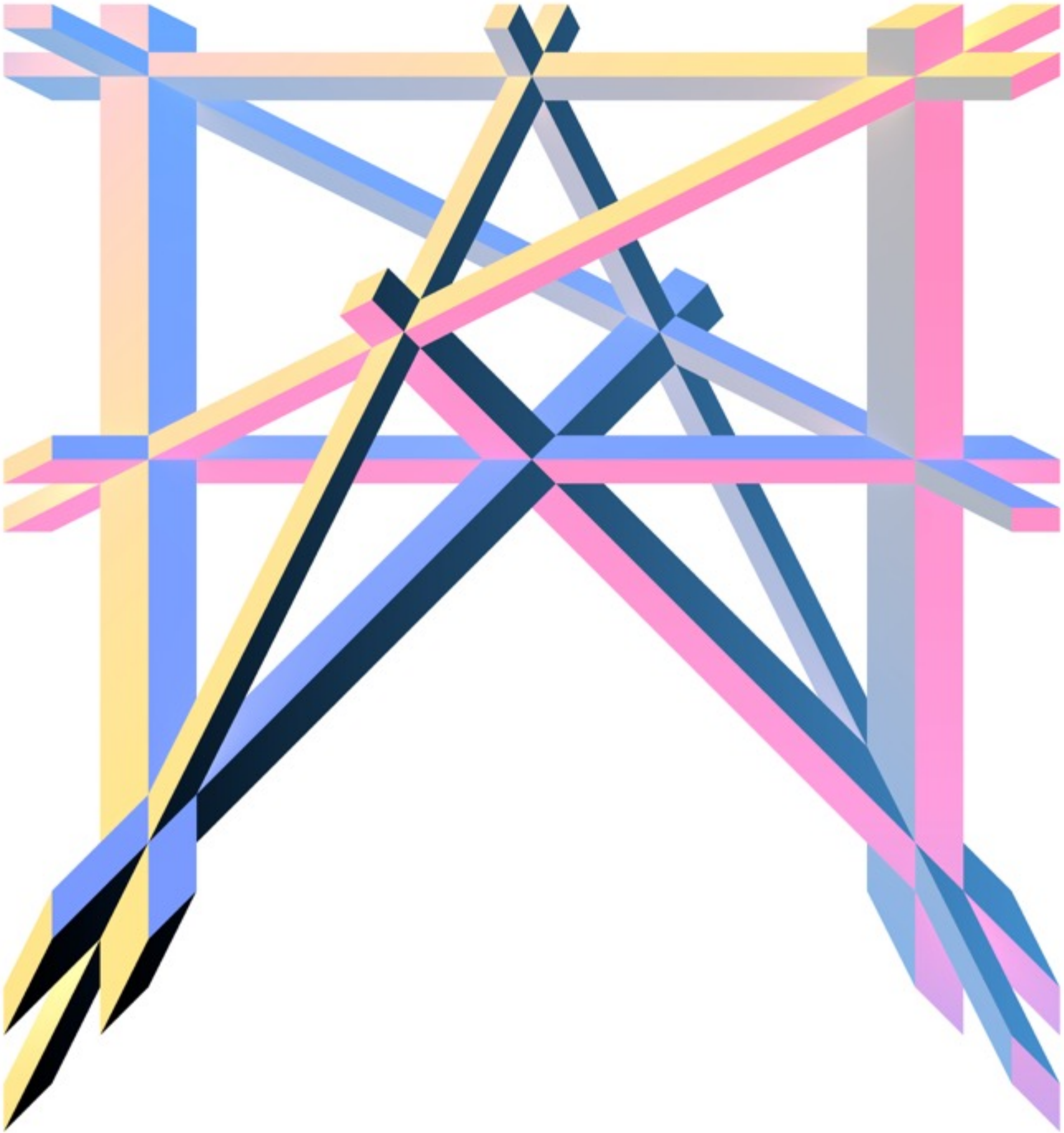
PENTACHORON
5 vertices
10 edges
10 faces (triangles)
5 cells (tetrahedra)

EXPANDED PENTACHORON
20 vertices
60 edges
70 faces (40 triangles + 30 squares)
30 cells (10 tetrahedra + 20 triangular prisms)

GNOMONIC PROJECTION
OF THE EXPANDED PENTACHORON
=
THE DESARGUES
CONFIGURATION

10 points
10 lines
5 planes

(3 lines and 3 planes per point
3 points and 2 planes per line
6 points and 4 lines per plane)



5-SIMPLEX

6 vertices
15 edges
20 faces (triangles)
15 cells (tetrahedra)
6 hypercells (pentachora)

5-SIMPLEX

6 vertices
15 edges
20 faces (triangles)
15 cells (tetrahedra)
6 hypercells (pentachora)

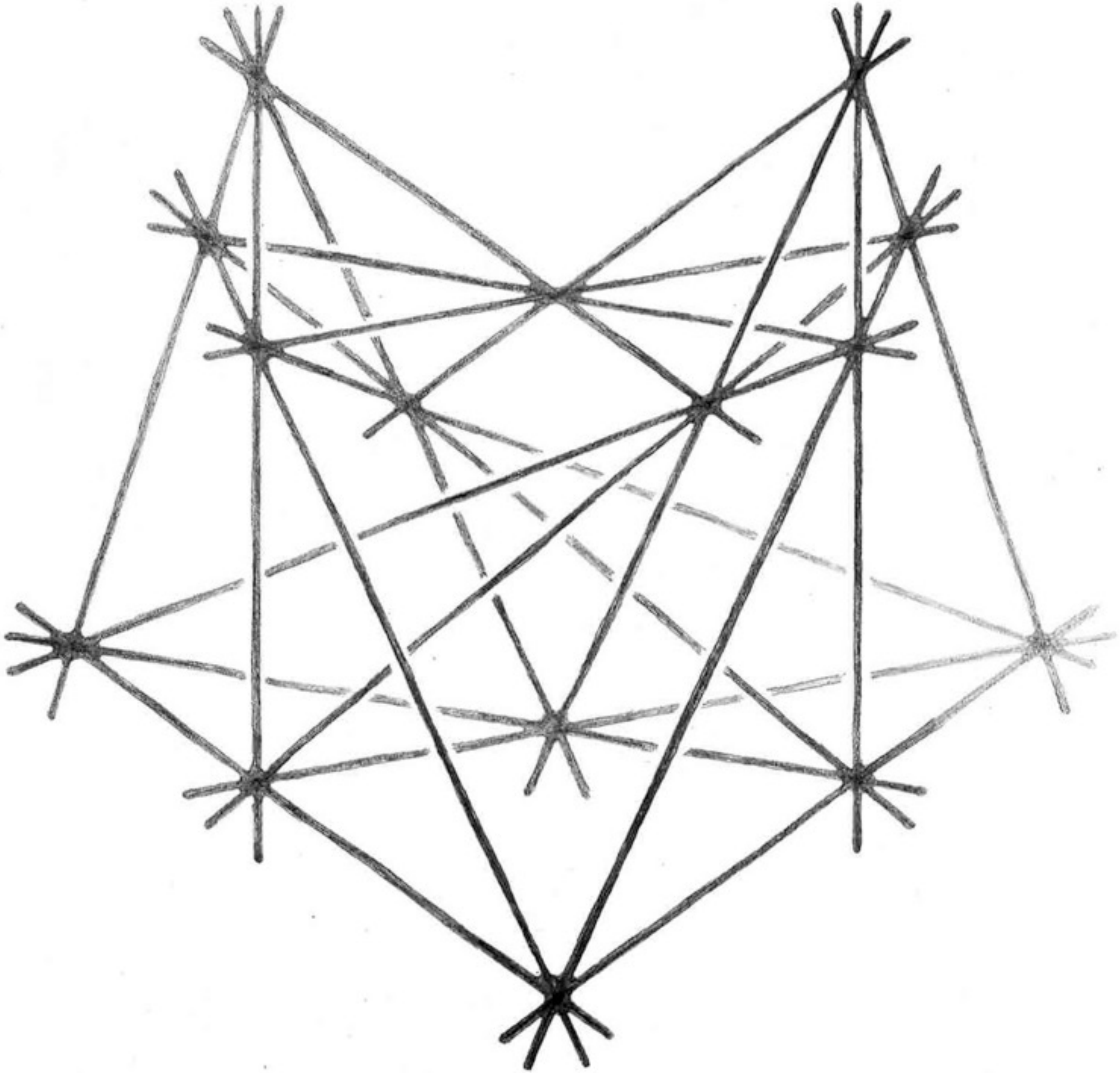
EXPANDED 5-SIMPLEX

30 vertices
120 edges
210 faces (120 triangles + 90 squares)
180 cells (60 tetrahedra +120 triangular prisms)
62 hypercells (12 pentachora + 30 tetrahedral prisms + 20 three-by-three duoprisms)

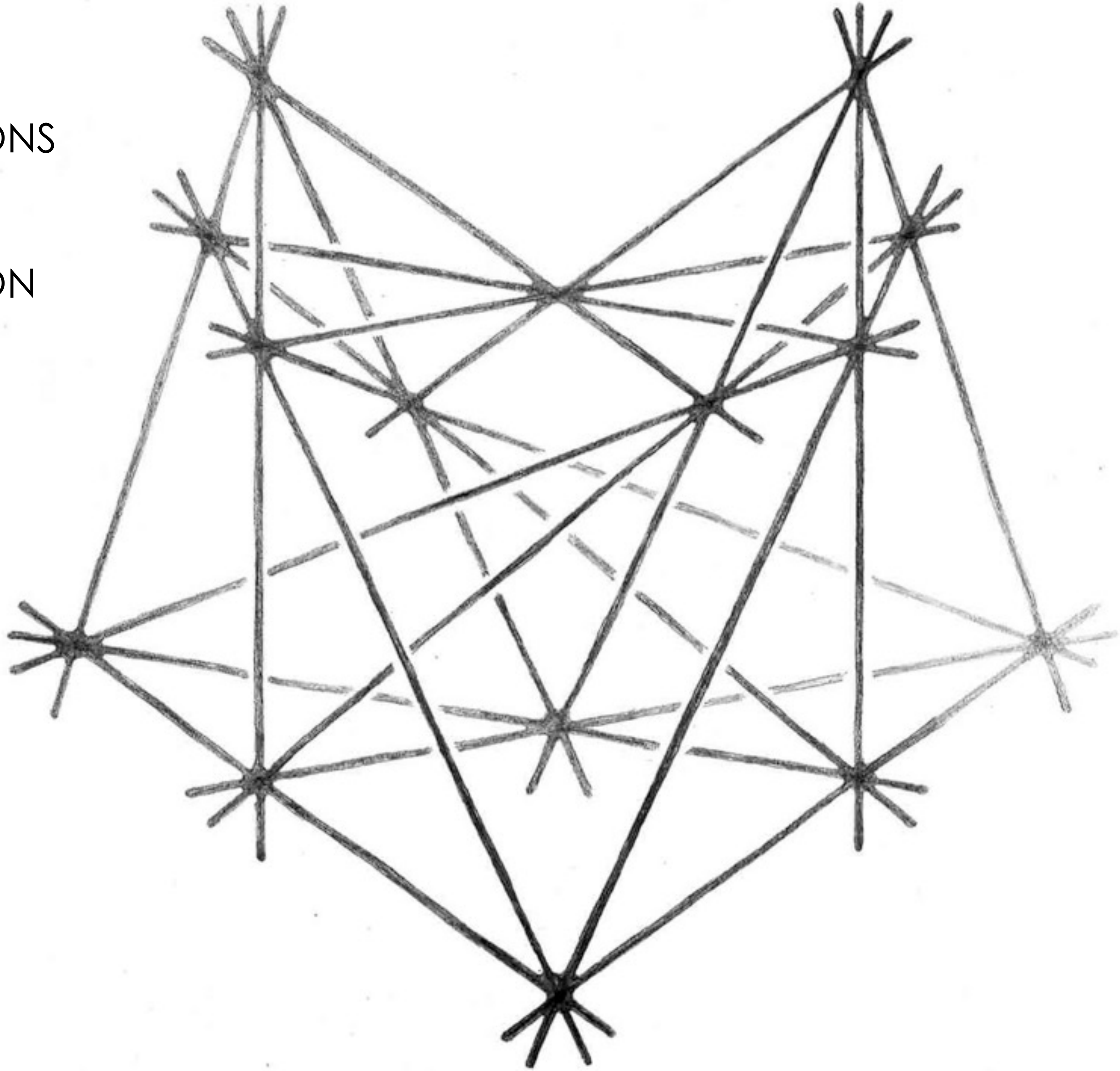
GNOMONIC PROJECTION OF THE EXPANDED 5-SIMPLEX
=
'THE LARGE DESARGUES CONFIGURATION'

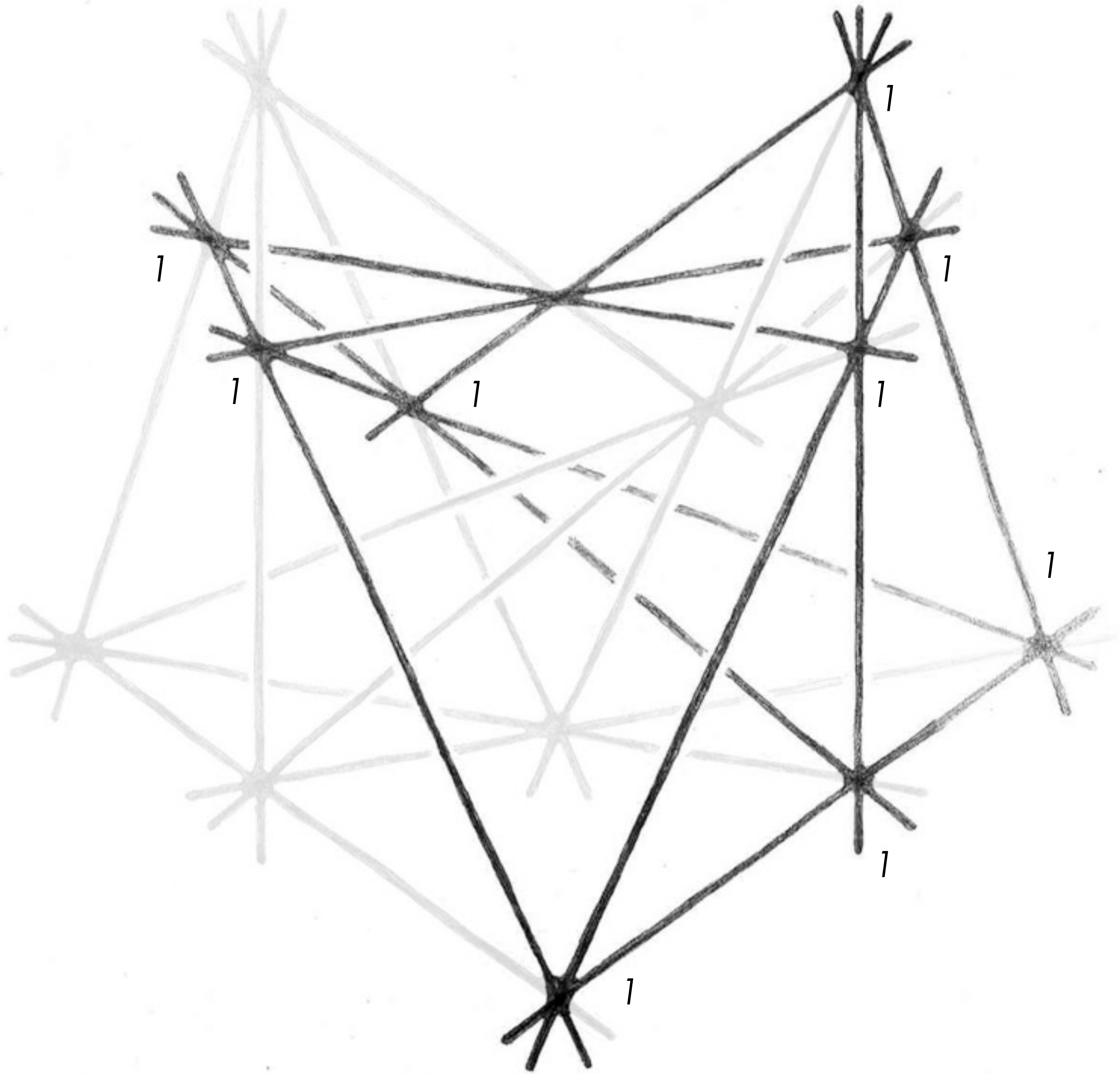
15 points
20 lines
15 planes
6 hyperplanes

(4 lines, 6 planes, and 4 hyperplanes per point
3 points, 3 planes, and 3 hyperplanes per line
6 points, 4 lines, and 2 hyperplanes per plane
10 points, 10 lines, and 5 planes per hyperplane)



SIX DESARGUES CONFIGURATIONS IN THE LARGE DESARGUES CONFIGURATION

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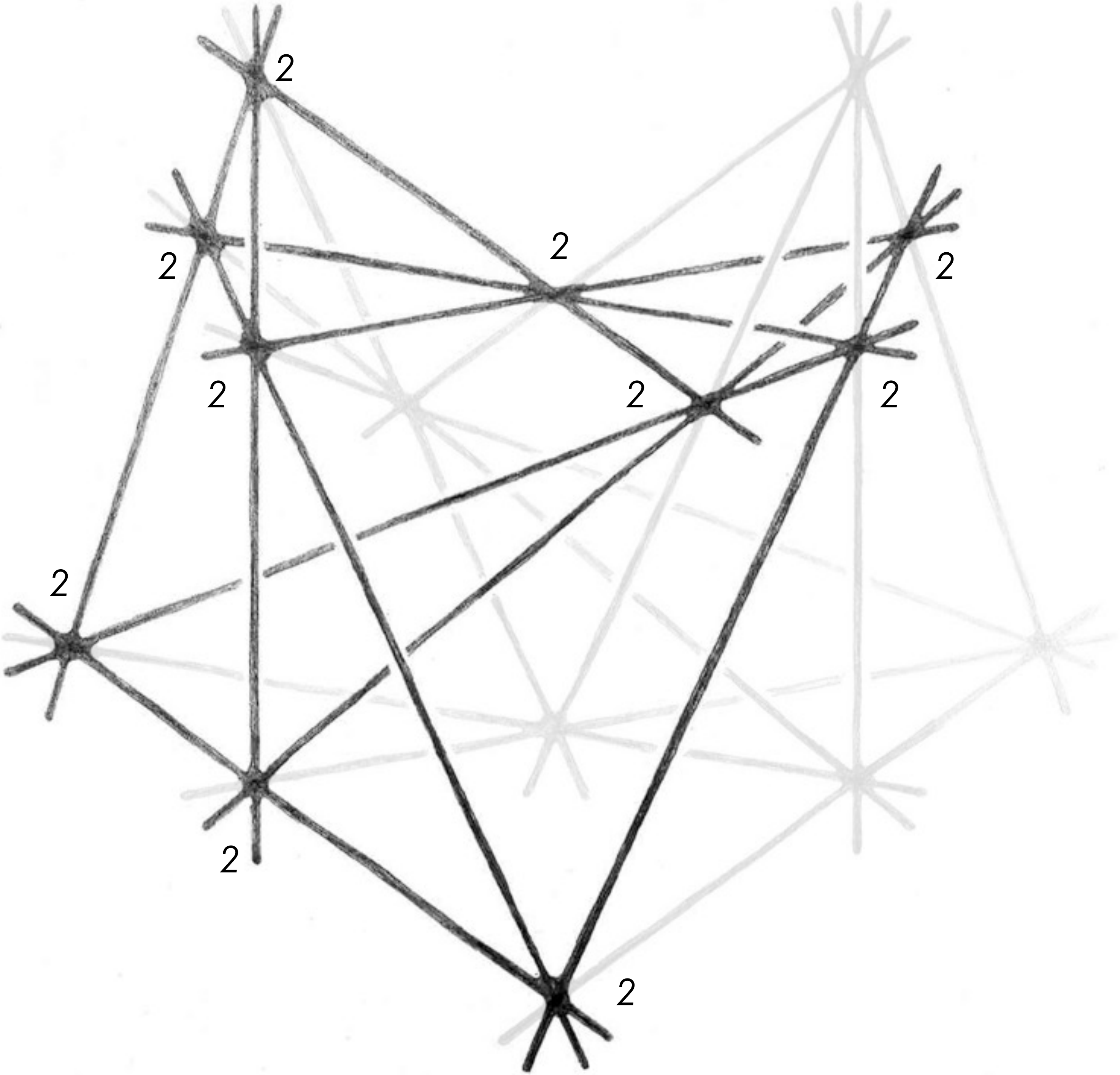
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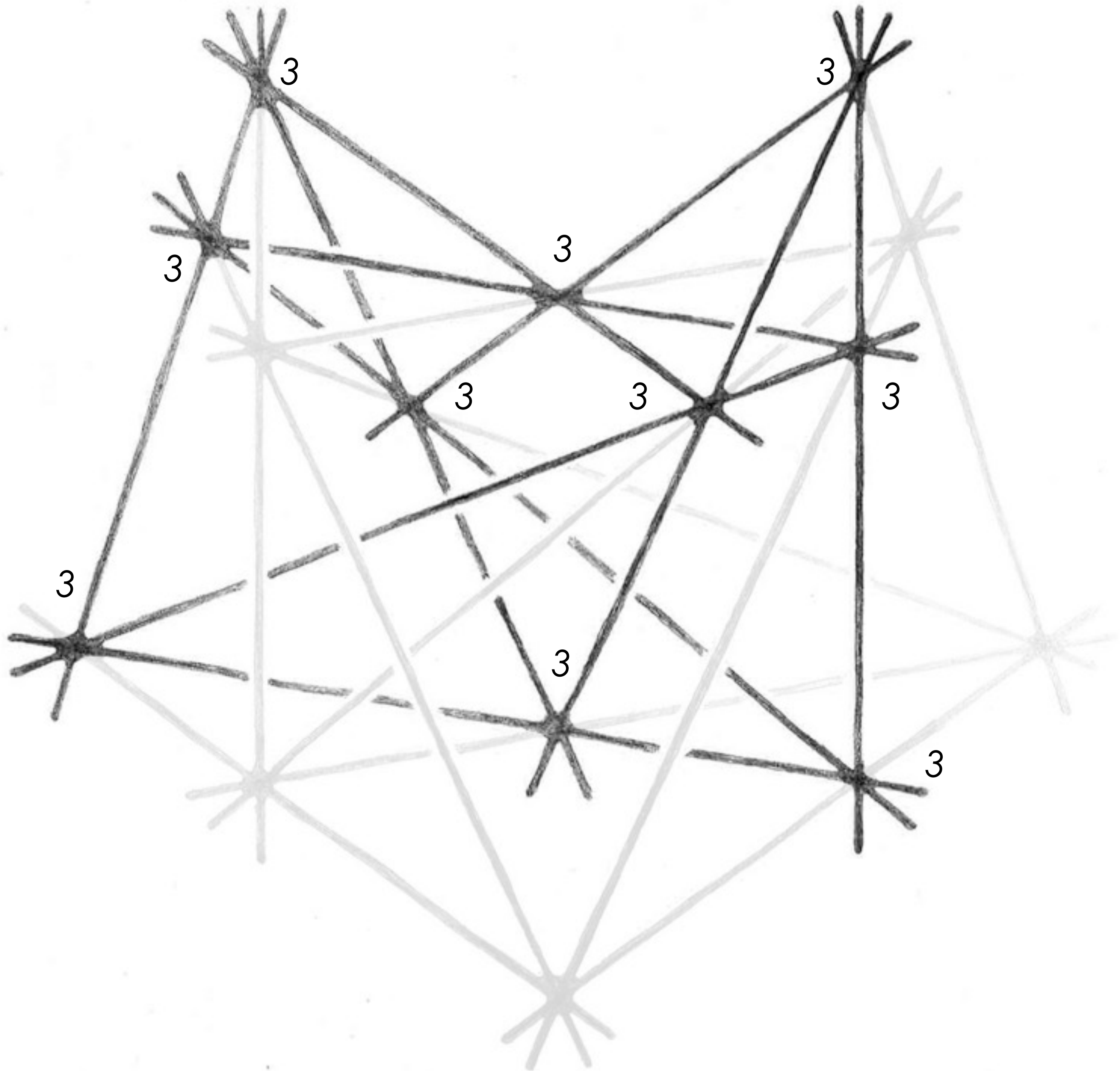
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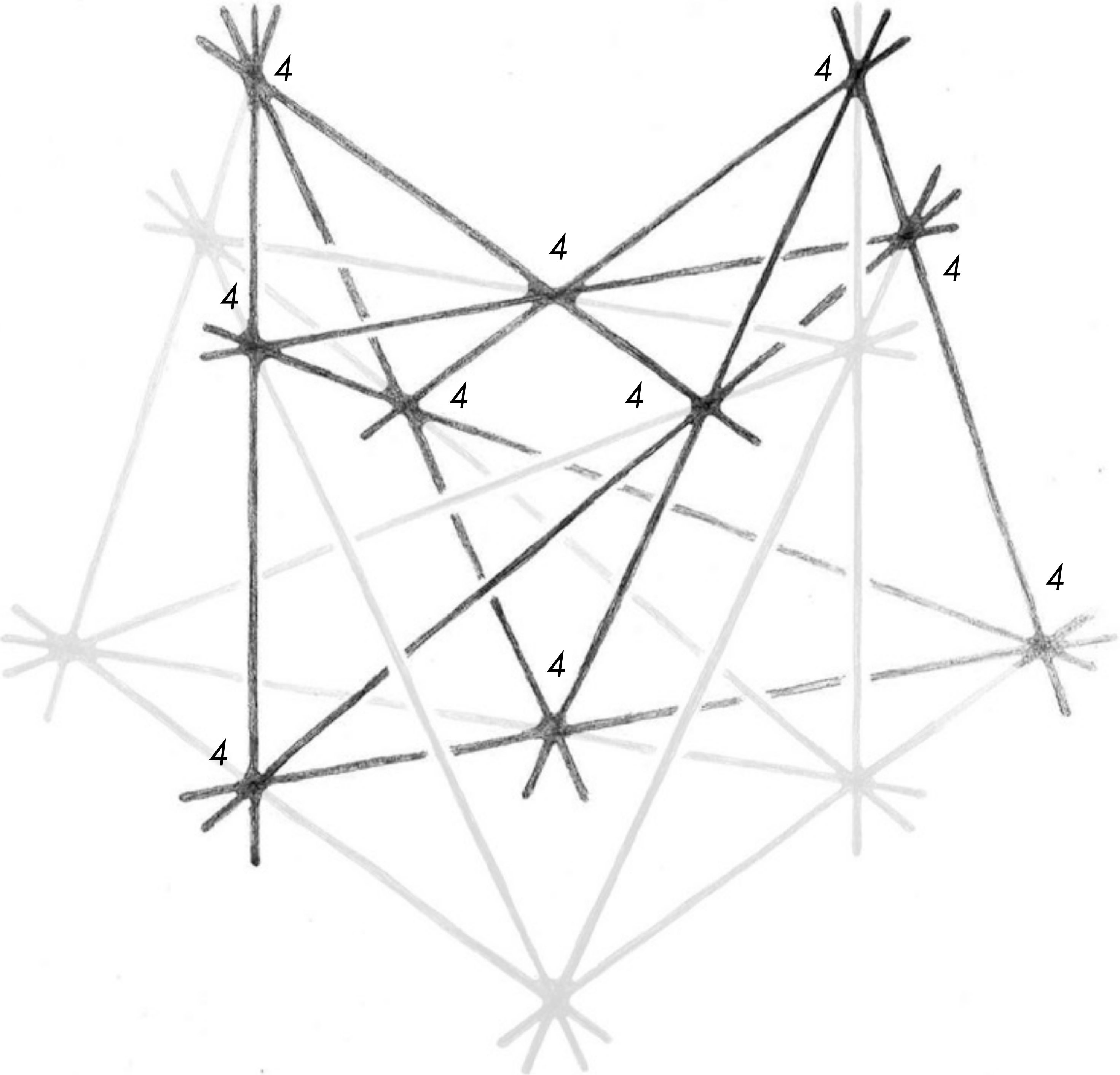
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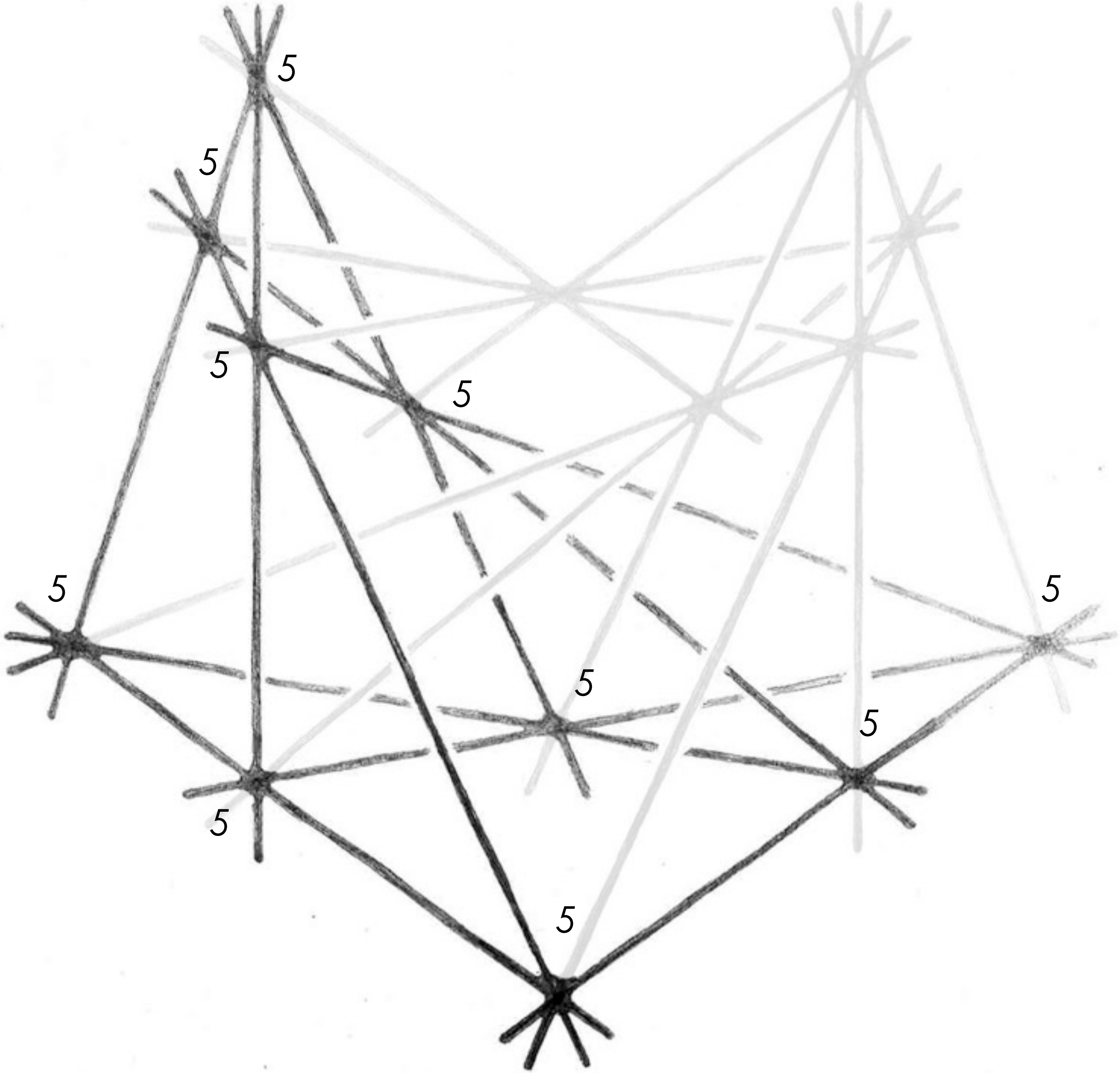
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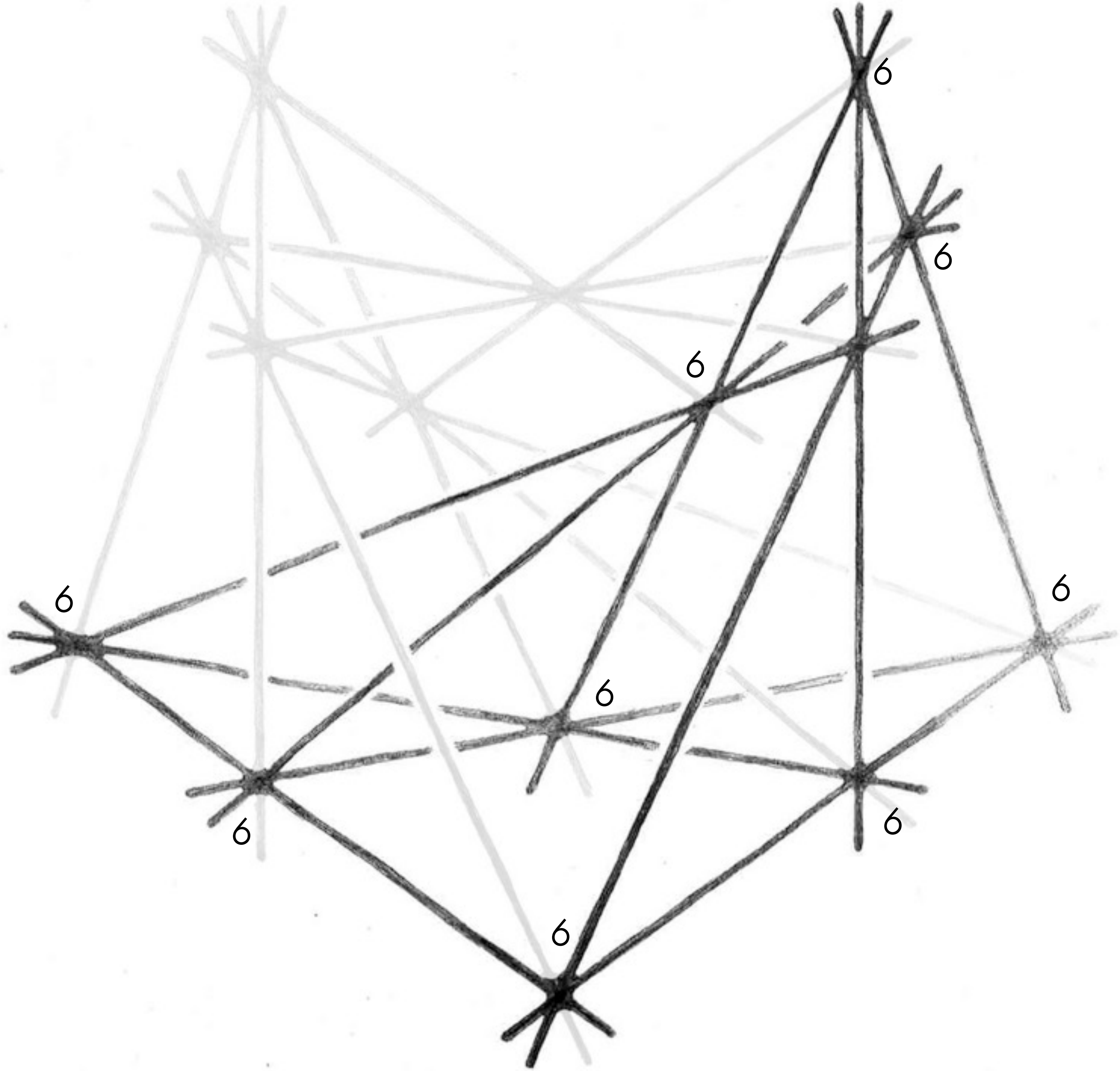
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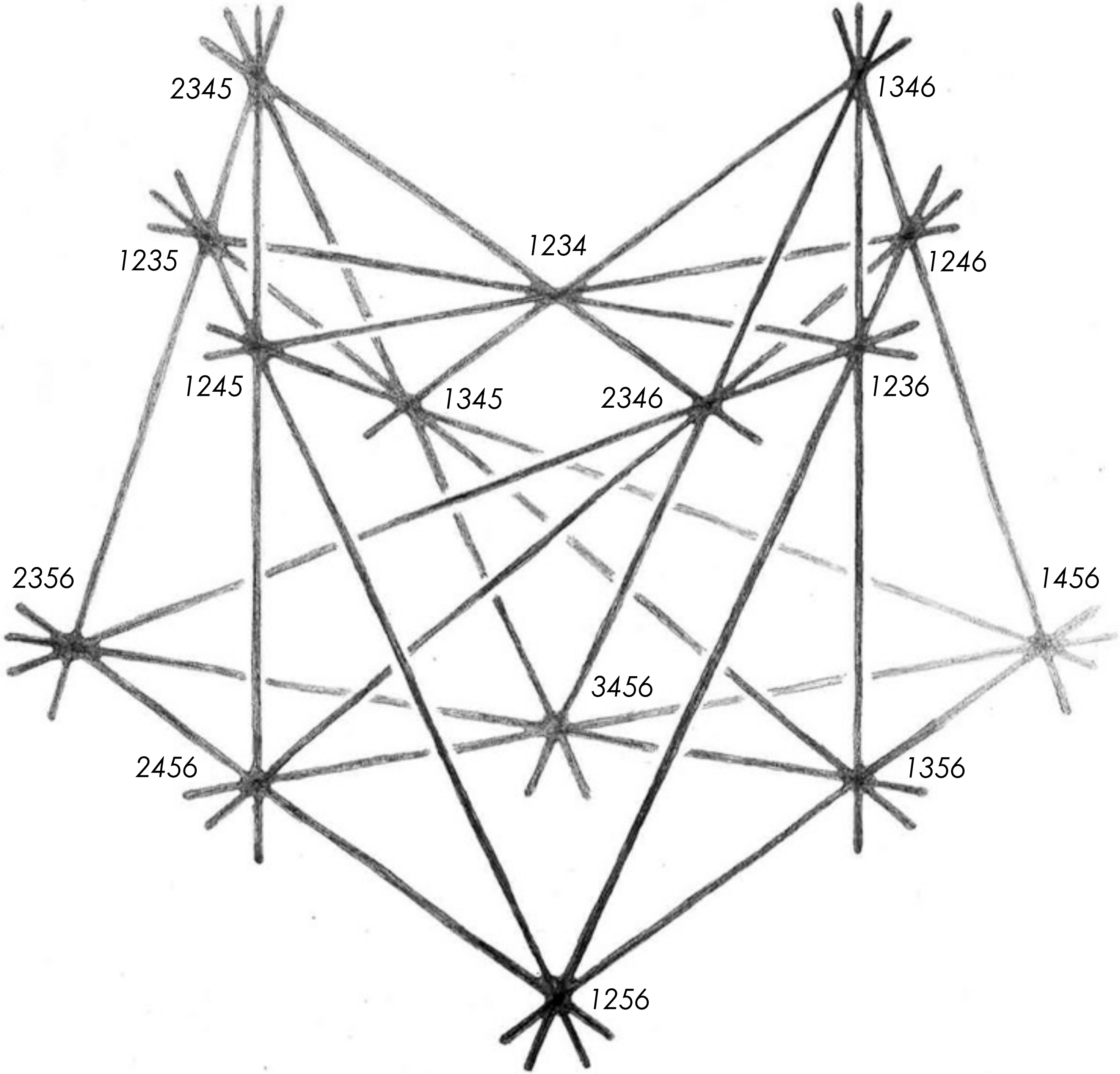
2345
1346
1234
1235
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2356
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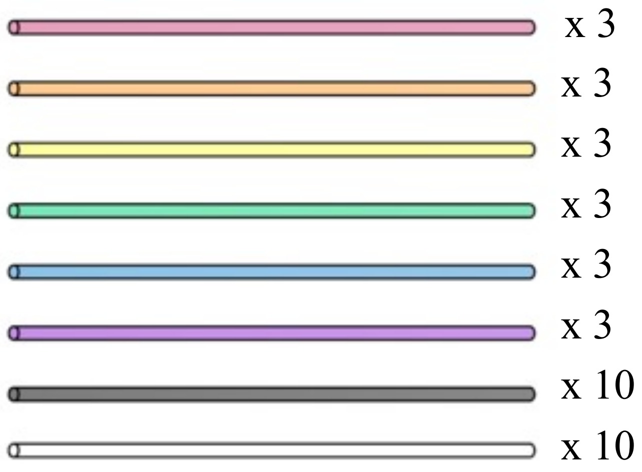
1356
1256

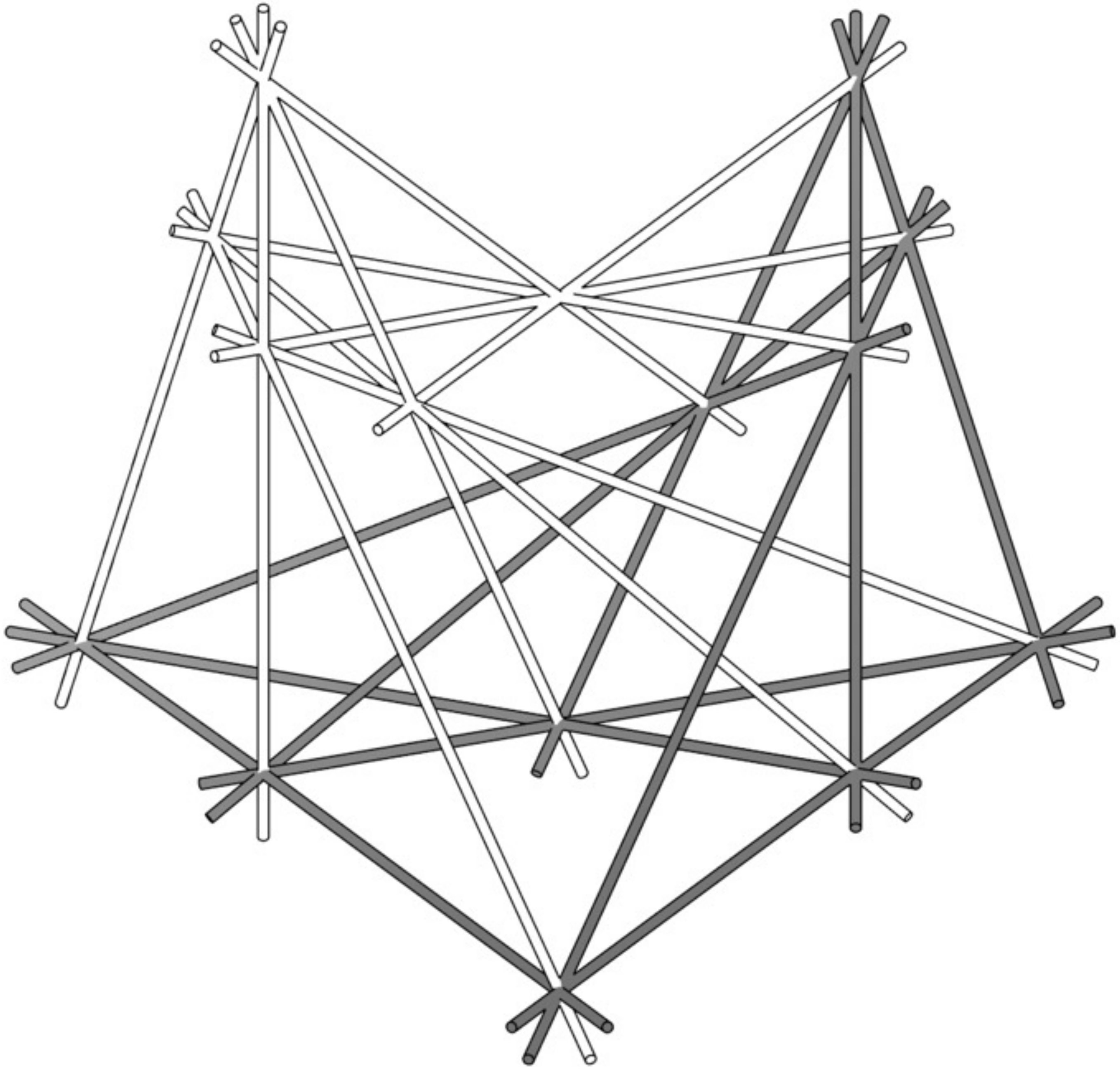
PHYSICAL MODEL

20 acrylic tubes & 38 painted wooden rods



CONSTRUCTION OF THE DESARGUES CONFIGURATION

Take 5 random points floating in 4-dimensional space, and connect them with 10 lines and 10 planes (white). If the lines and planes are cut with a hyperplane that neither contains nor is parallel to any of them, the intersections of the lines and planes will be 10 points and 10 lines forming an instance of Desargues configuration (black).

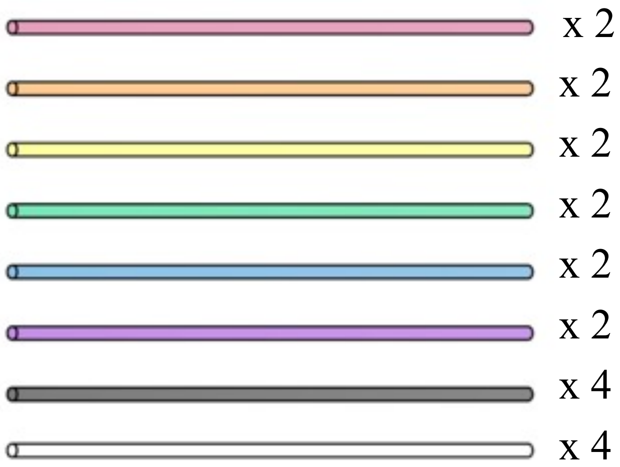


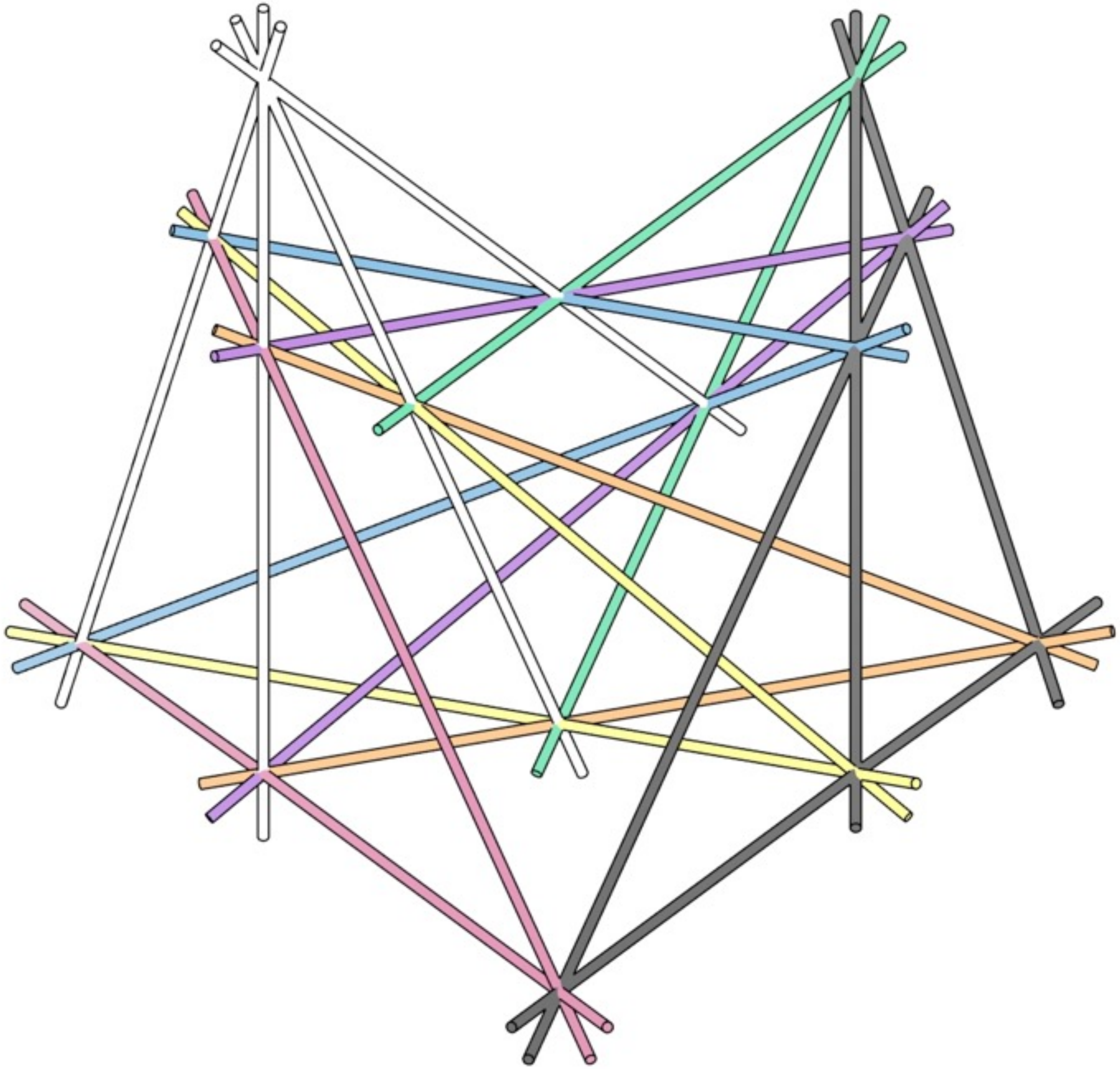
x 10

x 10

PERSPECTIVITY BETWEEN TWO TETRAHEDRA

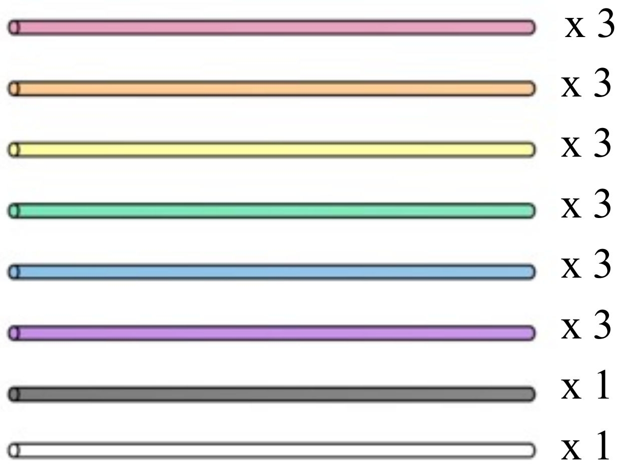
If two tetrahedra are in perspective from a point (white), then the intersections of the extensions of the corresponding edges are coplanar (black).

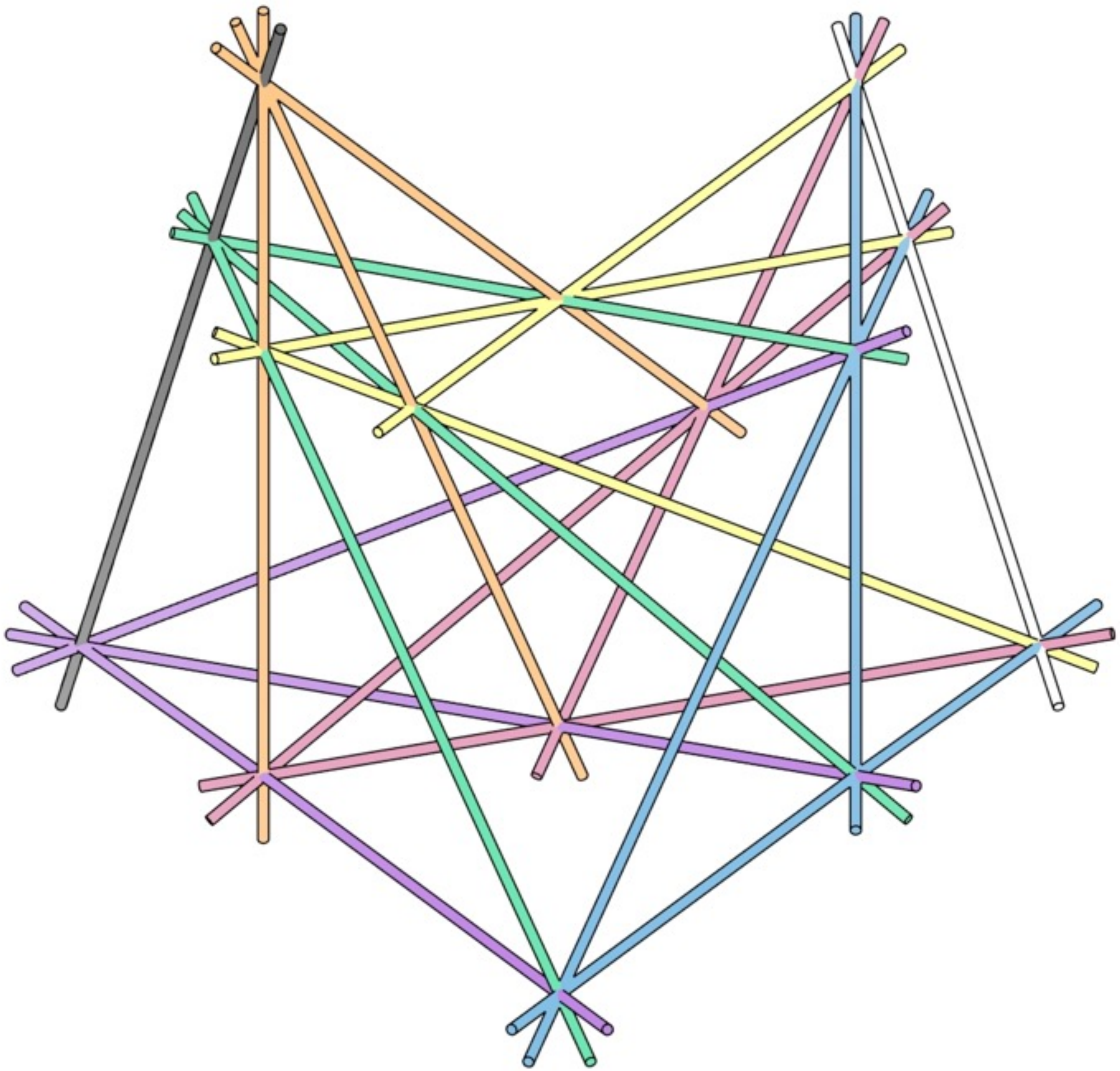




PERSPECTIVITY BETWEEN THREE TRIANGLES

If the three centers of perspectivity (orange, green, purple) determined by three triangles (blue, pink, yellow) are collinear (black), then the intersections of the extensions of corresponding edges are collinear (white) as well.

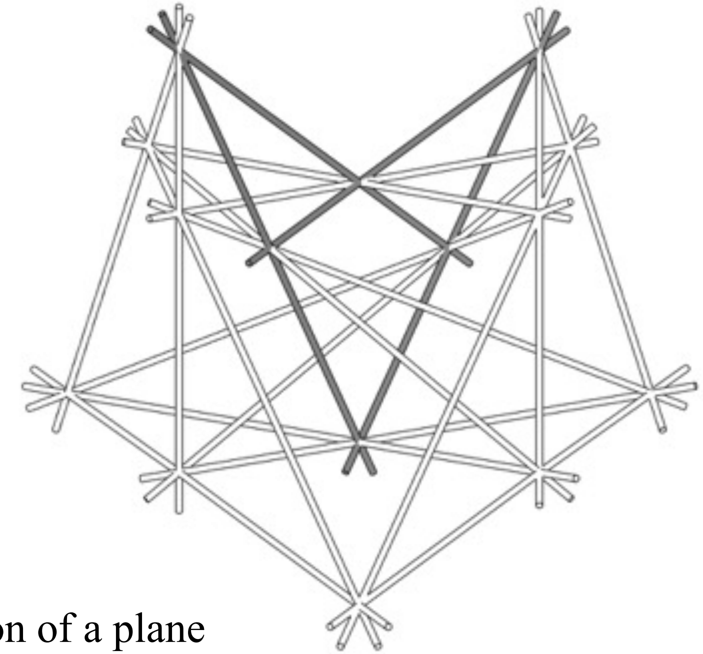




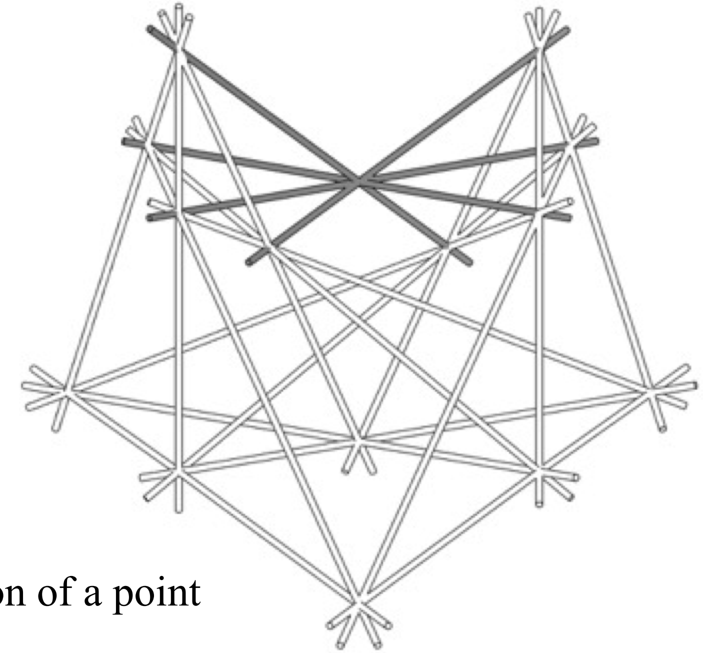
TIC-TAC-TOE

Game for two players (black & white), who take turns inserting wooden rods into the tubes of the configuration.
The first player to occupy an entire plane or an entire point wins.

Exercise: Play few rounds of 'projective' tic-tac-toe with your classmate



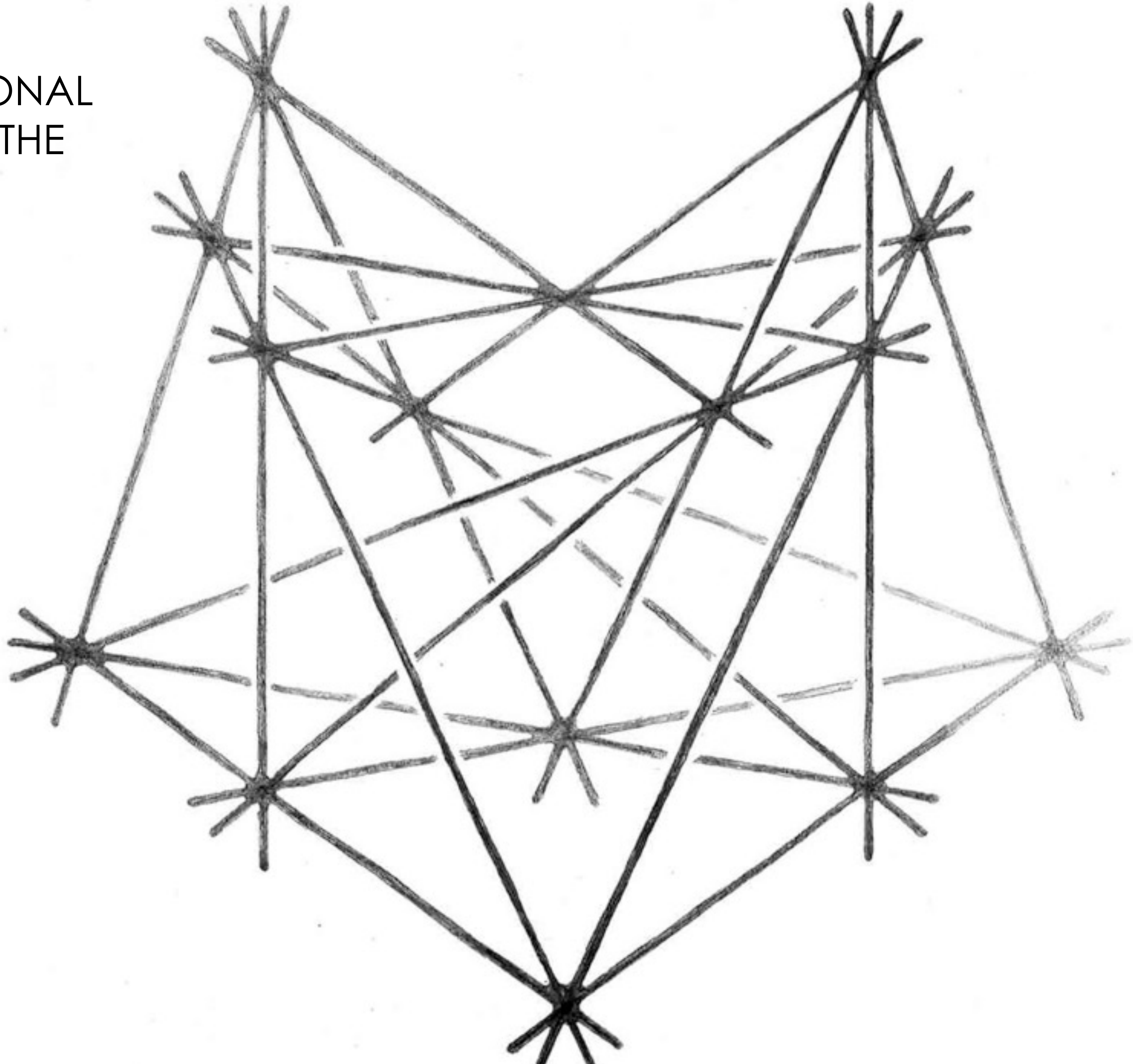
Win by an occupation of a plane



Win by an occupation of a point

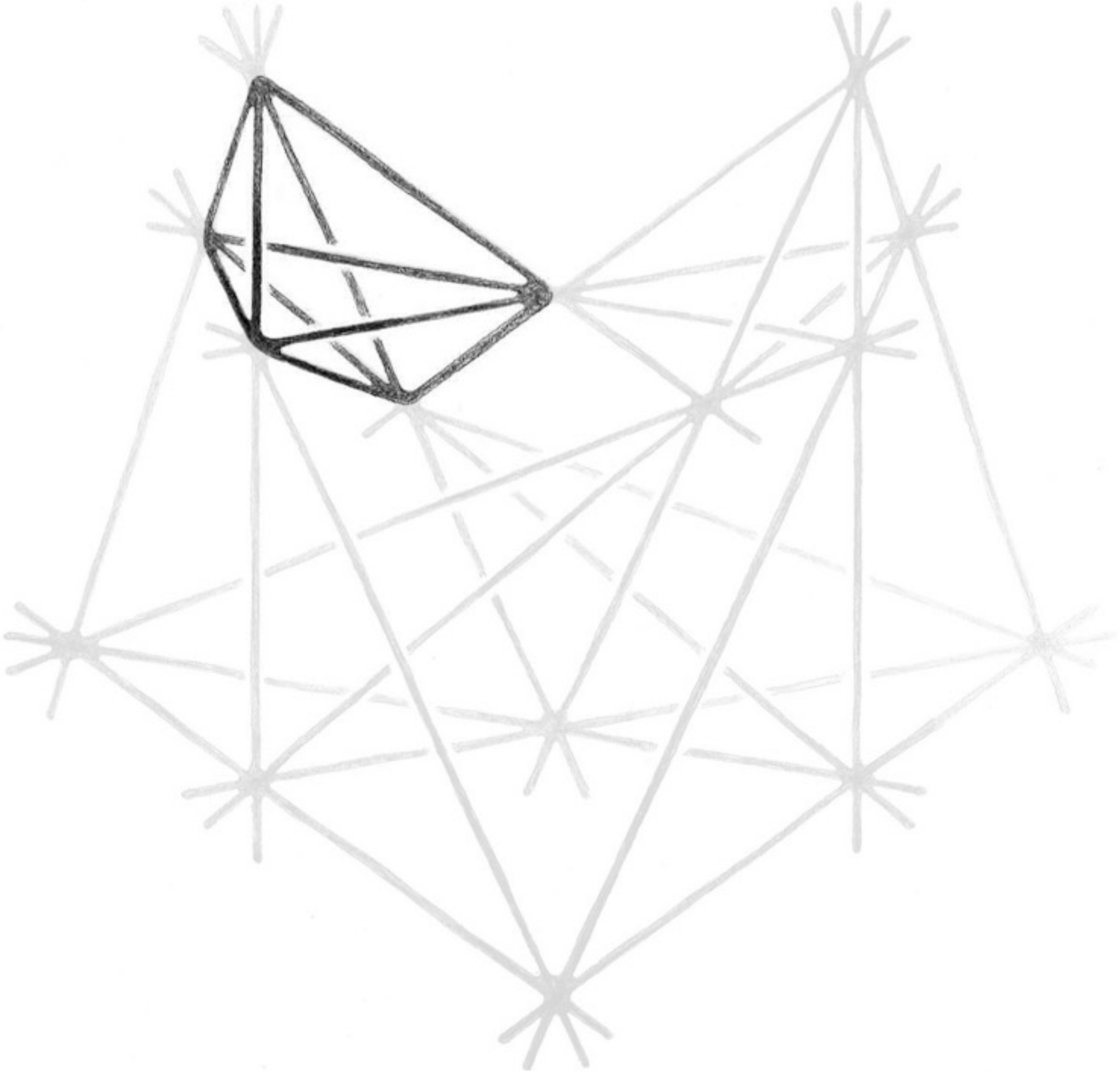
THE “PRISMARY”: A MODEL OF THE 3-3 DUOPRISM

FOUR-DIMENSIONAL
HYPERCELLS IN THE
EXPANDED
5-SIMPLEX



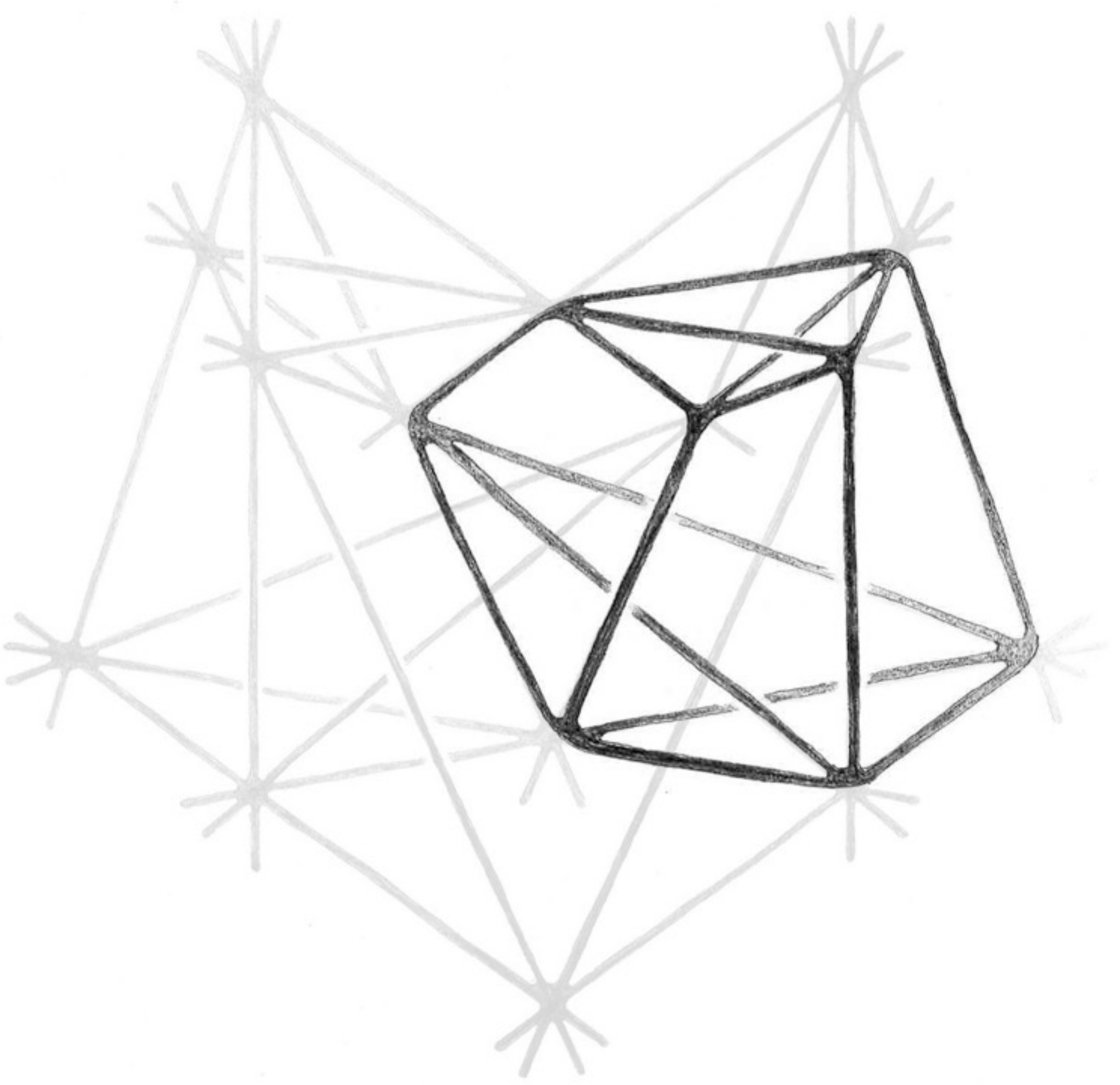
PENTACHORON

5 vertices
10 edges
10 faces (triangles)
5 cells (tetrahedra)



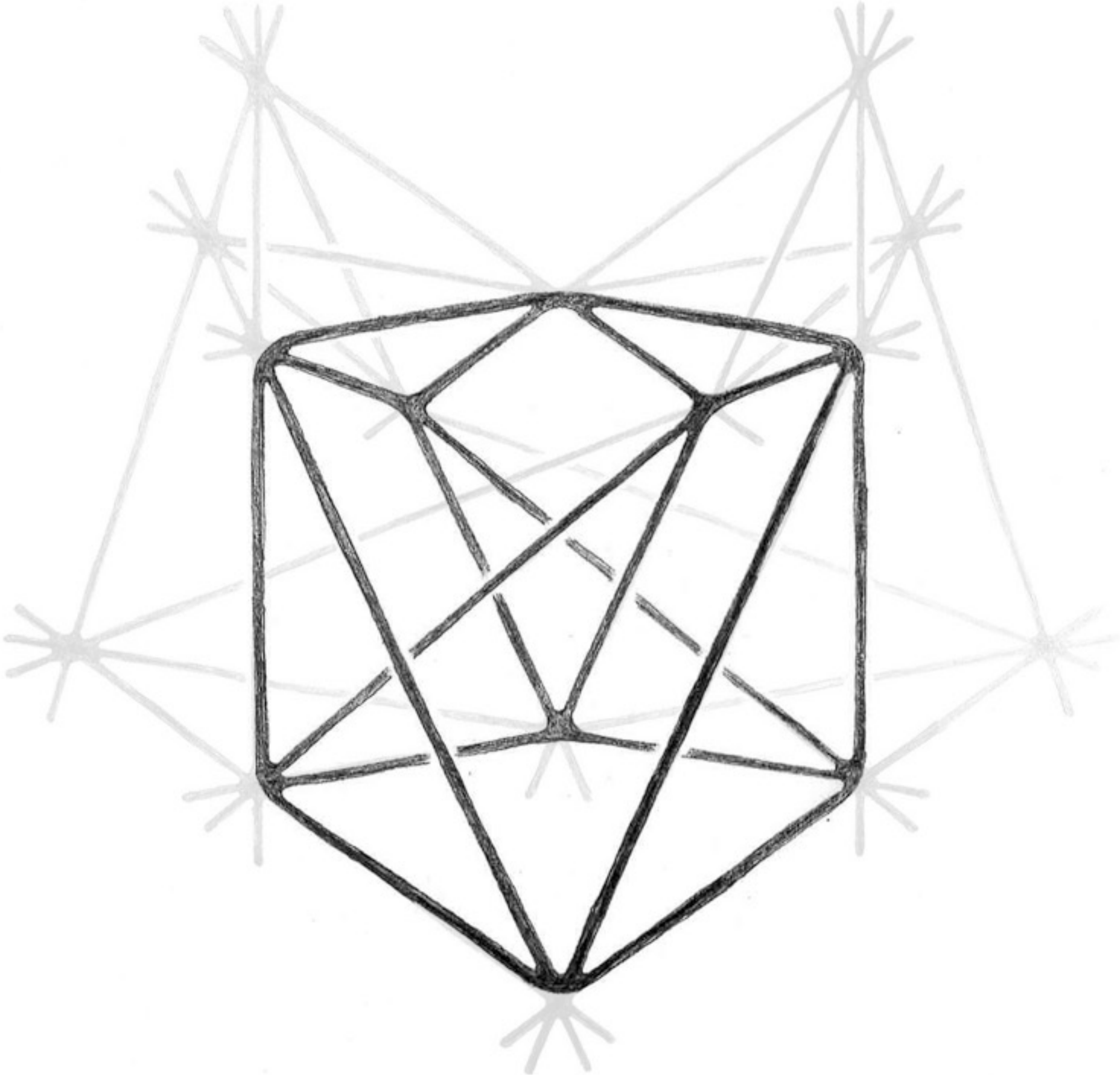
TETRAHEDRAL PRISM

8 vertices
16 edges
14 faces (8 triangles + 6 squares)
6 cells (2 tetrahedra + 4 triangular prisms)

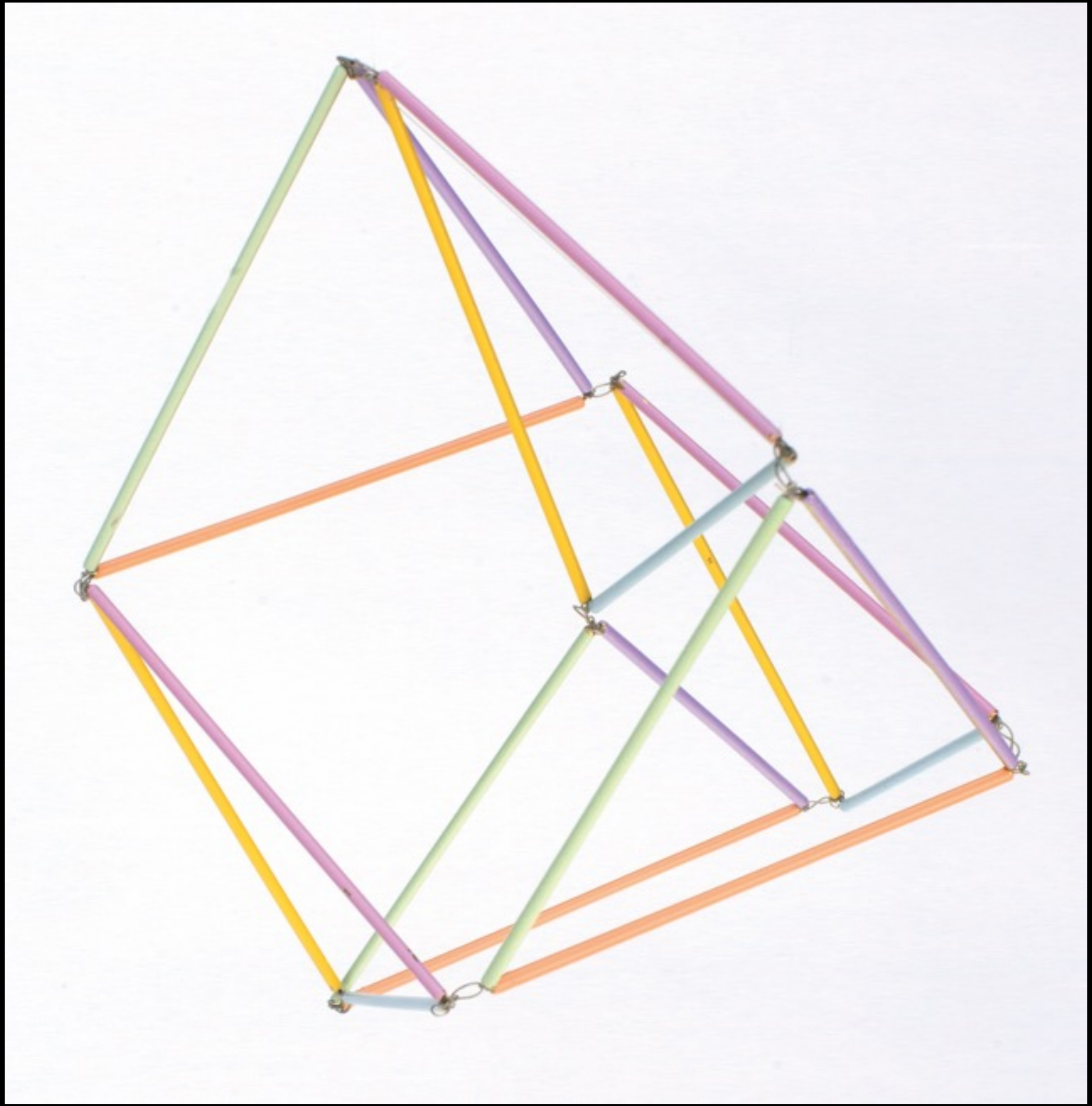


3-3 DUOPRISM

9 vertices
18 edges
15 faces (9 triangles + 6 squares)
6 cells (triangular prisms)



The ‘Prismary’



The parallel edges of the 3-3 duoprism converge towards the vanishing points on the horizon lines, of which there are two – one for both of the cycles of prisms

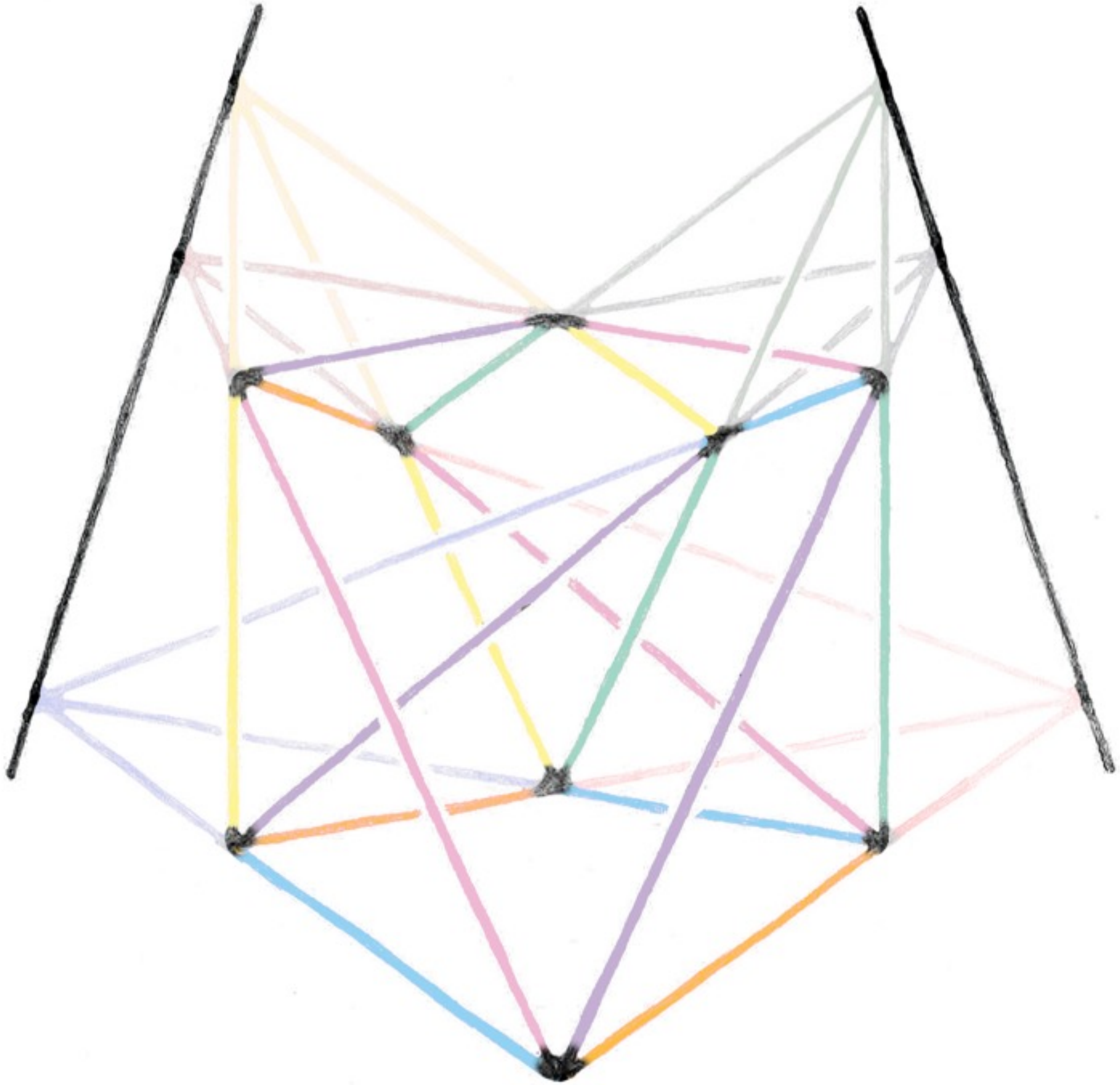
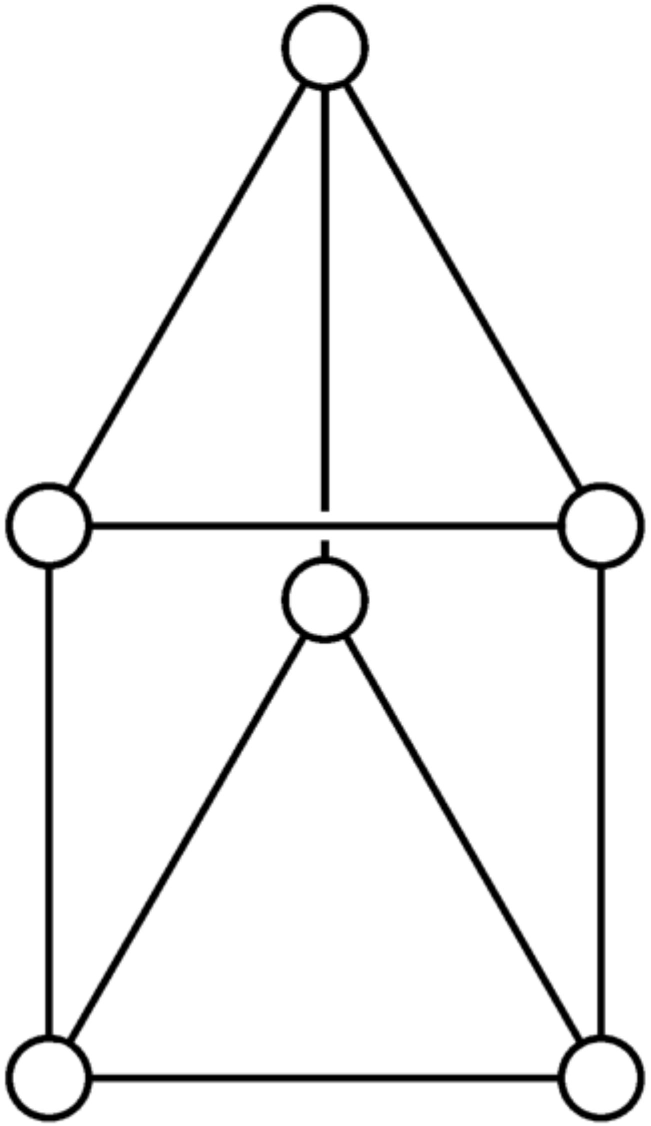


Illustration of the triangular prism with all nine edges having the same length:

the contour = triangle & square meeting along an edge, as if unfolded down to the same plane

the interior = one triangle (undistorted) & two distorted squares, all meet at a vertex in the middle.

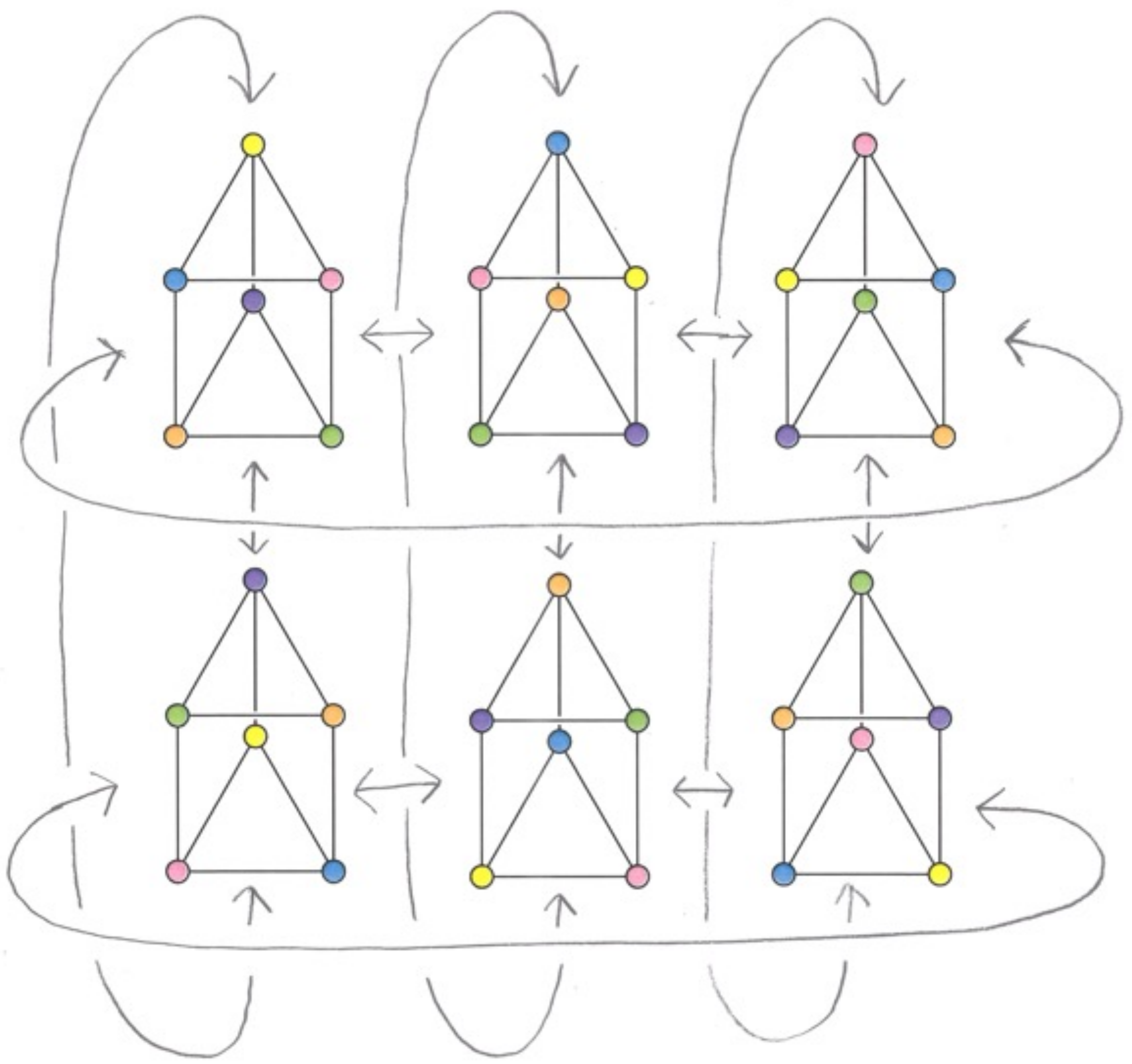
The undistorted triangle and square are in ‘front’, and the other polygons are in the ‘back’.



One third rotation along the plane parallel to the pair of triangles makes one square from the back to appear in the front, and one square in the front disappears to the back.

If the prism does a half-turn around the plane parallel to the front square, one of the triangular faces in the back appears in the front, and one triangle in the front disappears to the back.

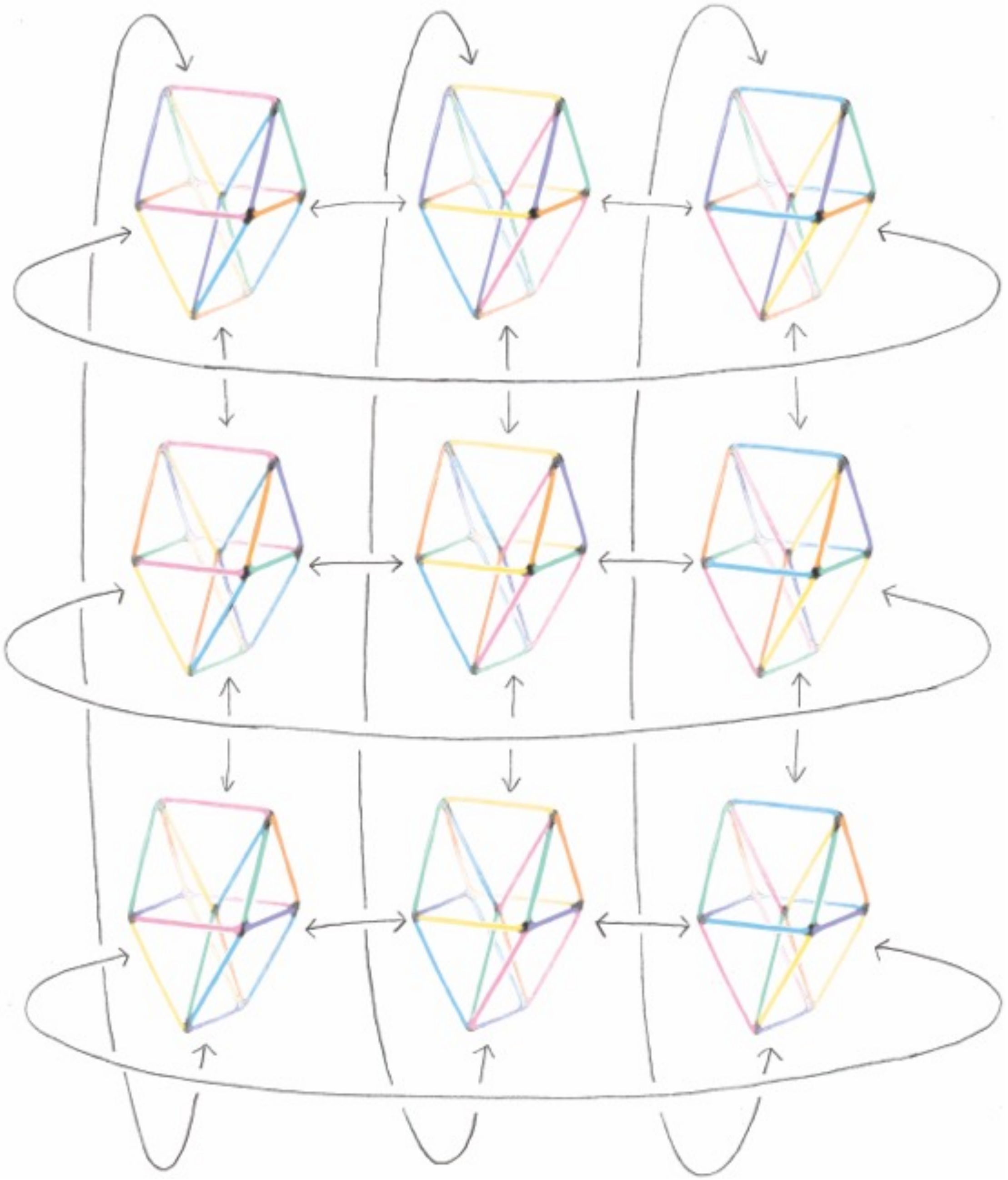
the graph-like structure of this map itself appears like a triangular prism, as the six possible states correspond to the six vertices of the solid.



A map the nine different stages the ‘Prismary’

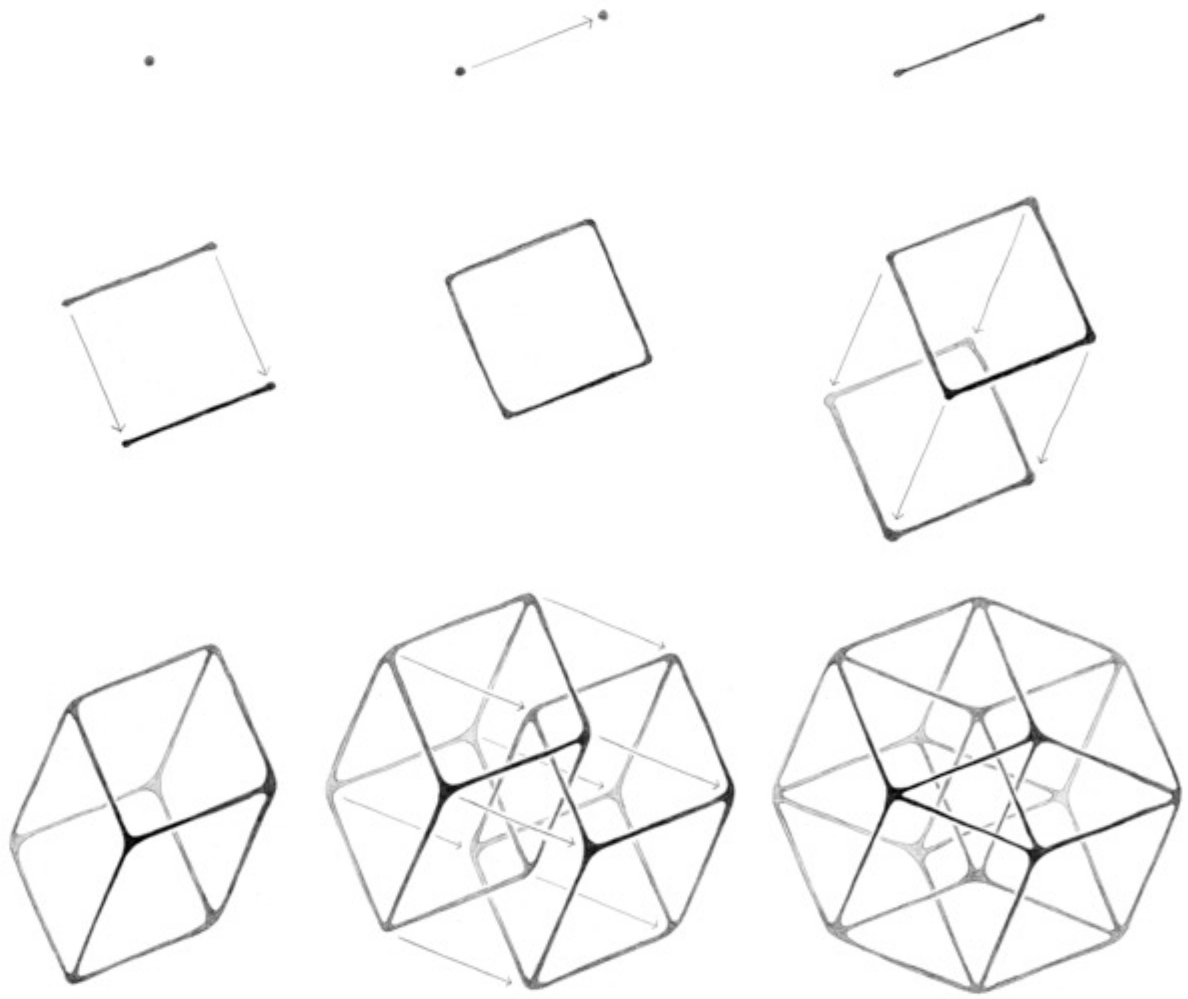
they are connected by one third rotations along either of the cycles.

the nine possible states correspond to the nine vertices
-> the graph like structure of this map itself appears like a 3-3 duoprism

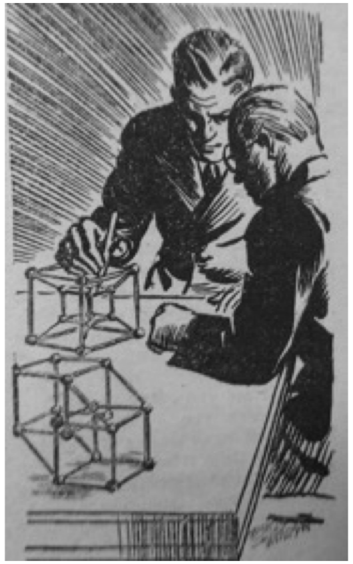
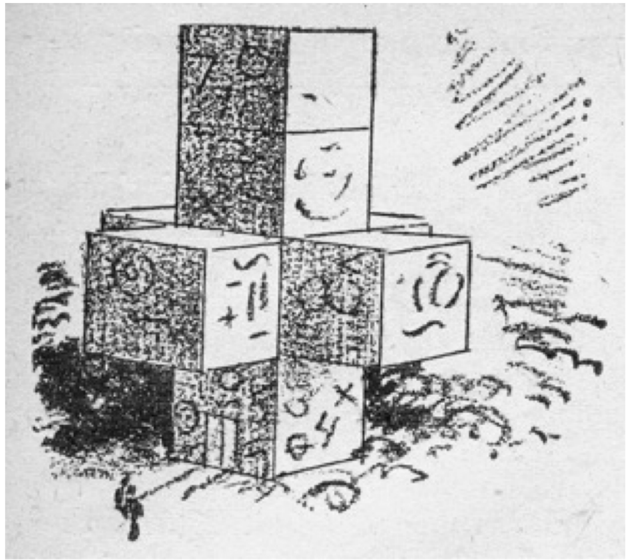
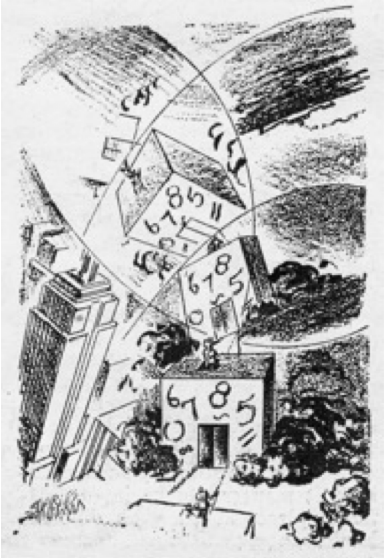


CROOKED HOUSES: VISUALIZING THE POLYCHORA WITH HYPERBOLIC PATCHWORKS

Constructing the hypercube

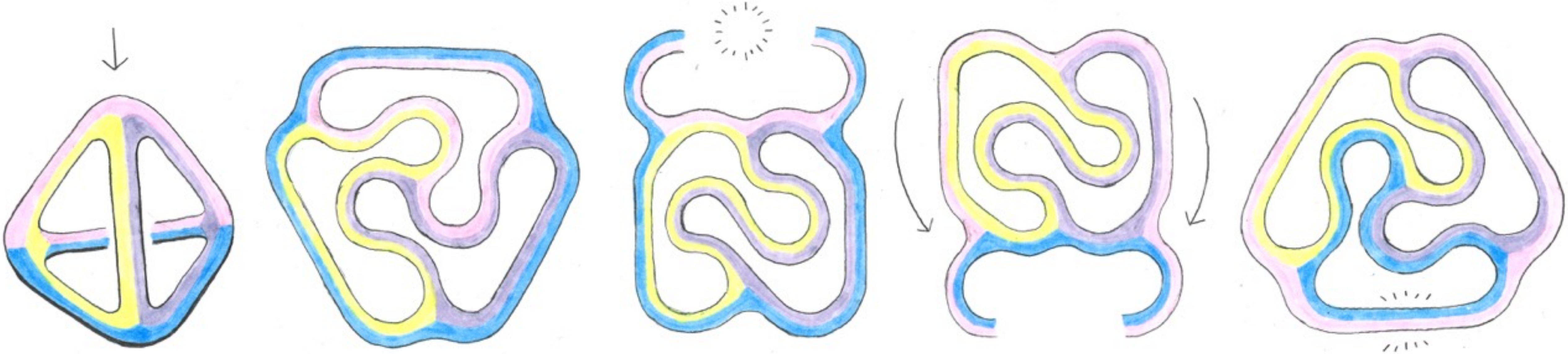


Robert A. Heinlein:
"…And He Built a Crooked House"
in *Astounding Science Fiction*, February 1941

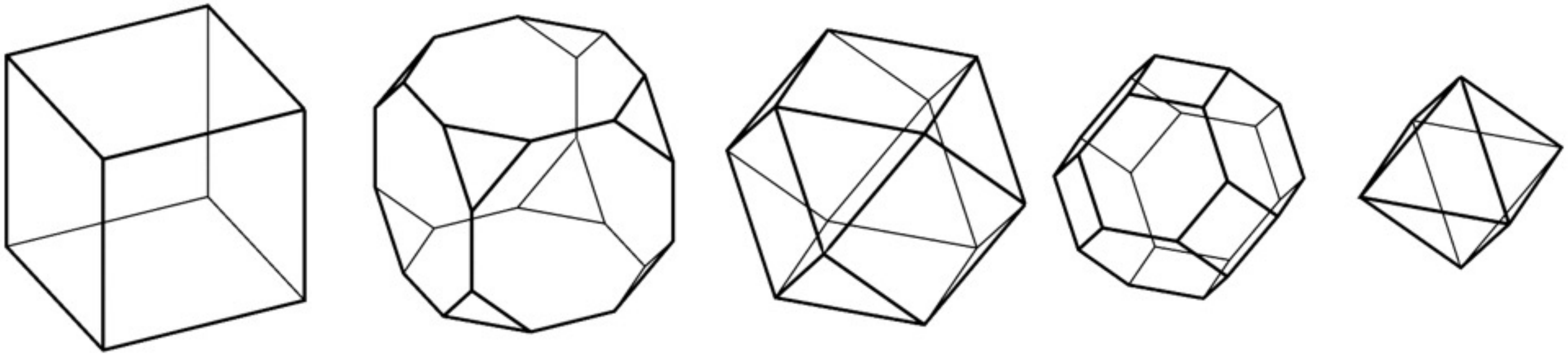





(Illustrations by Charles Schneeman)

Flattening a soft tetrahedron

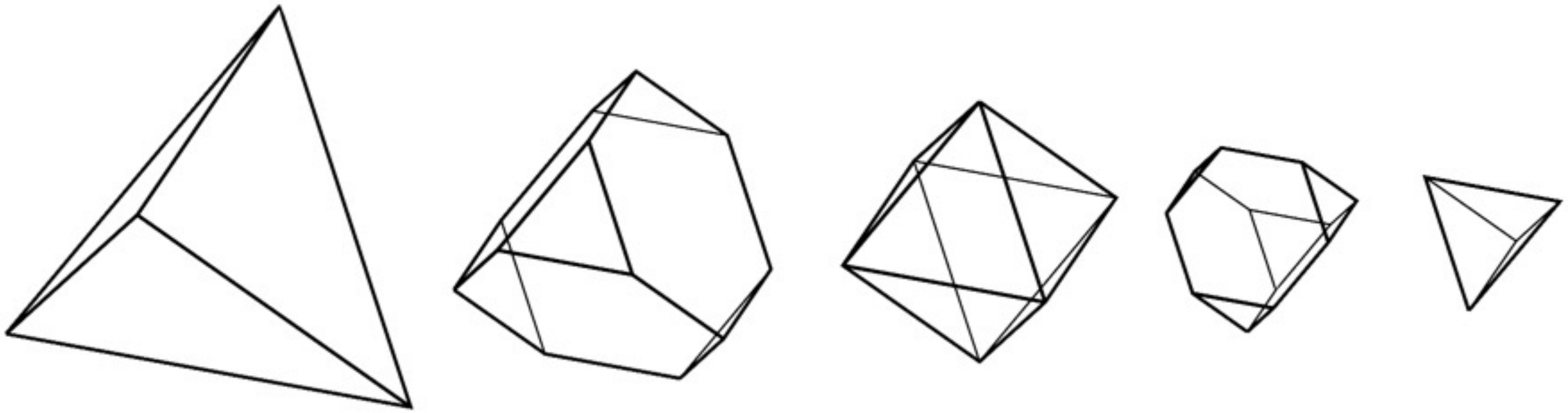


Truncation of the cube



Cube | Truncated Cube | Cuboctahedron (Rectified Cube) | Truncated Octahedron (Bitruncated Cube) | Octahedron (Birectified Cube)

Truncation of the tetrahedron



Tetrahedron

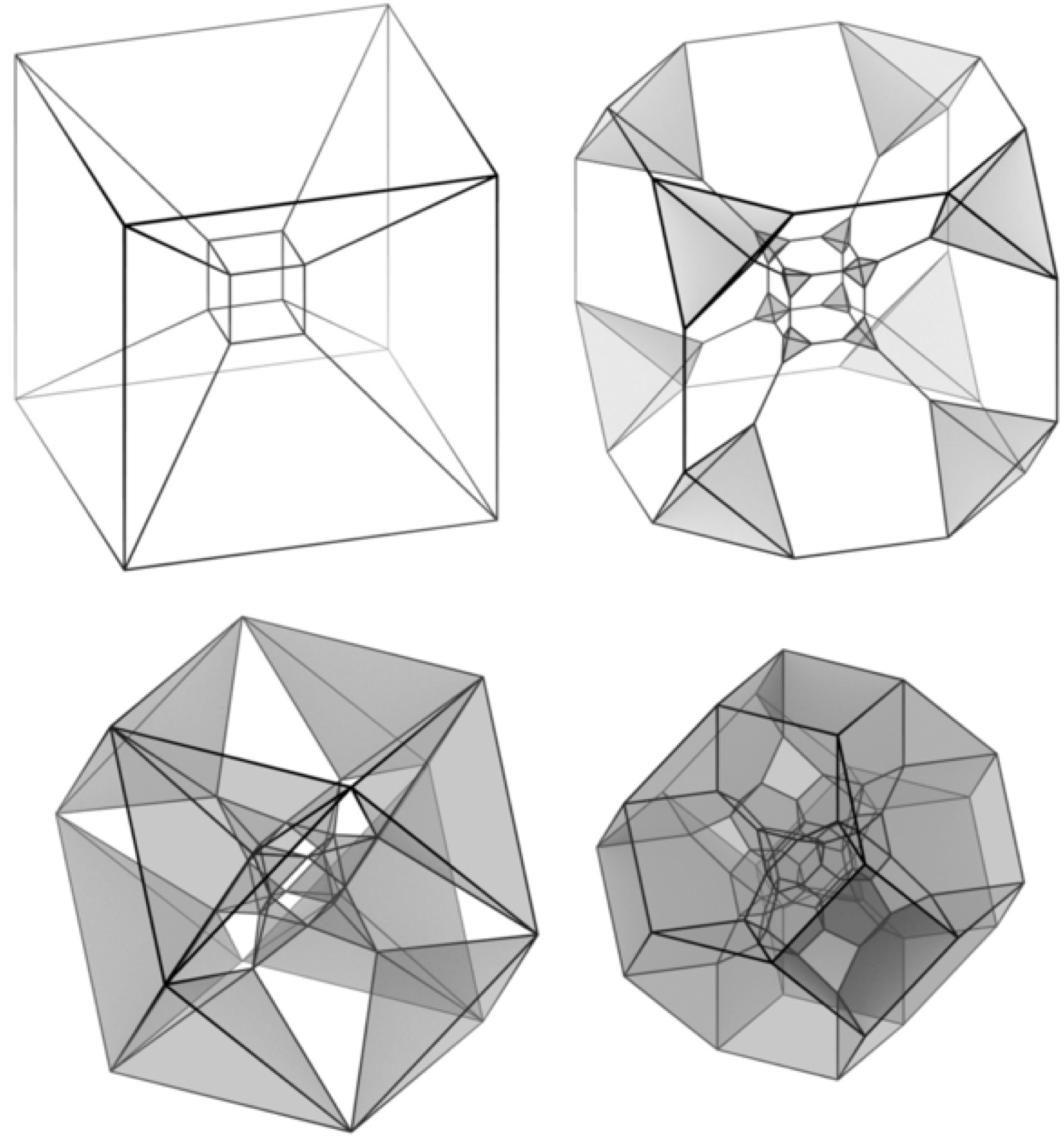
Truncated Tetrahedron

Octahedron (Rectified Tetrahedron)

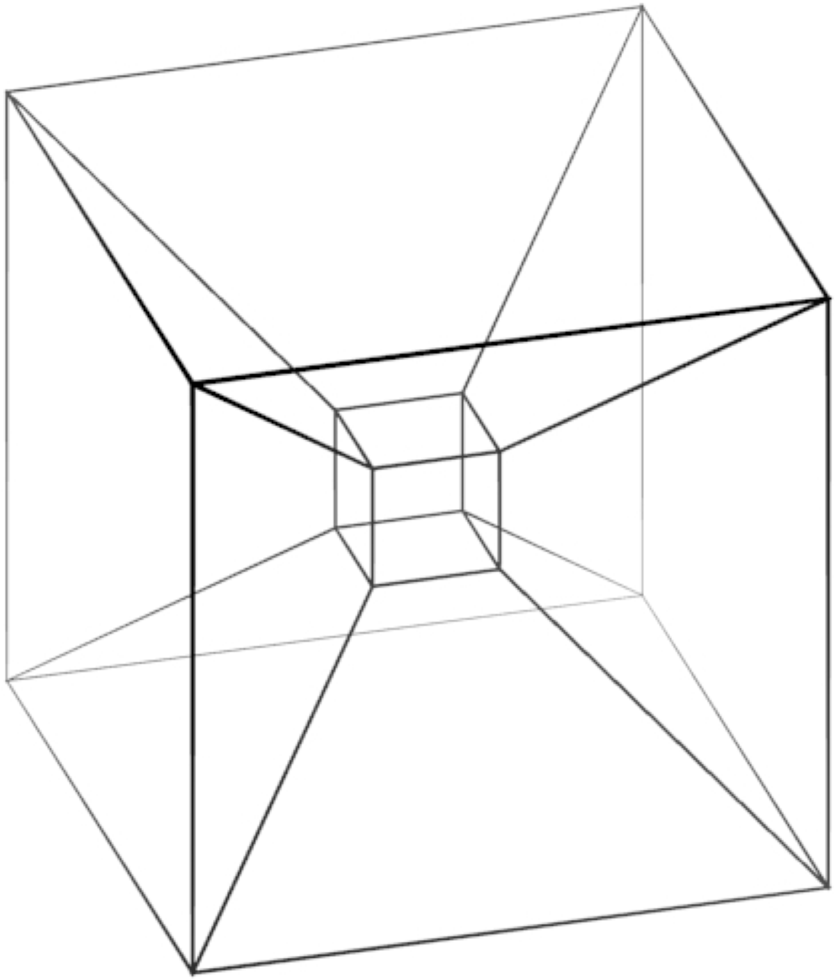
Truncated Tetrahedron (Bitruncated Tetrahedron)

Tetrahedron (Birectified Tetrahedron)

Truncation of the hypercube

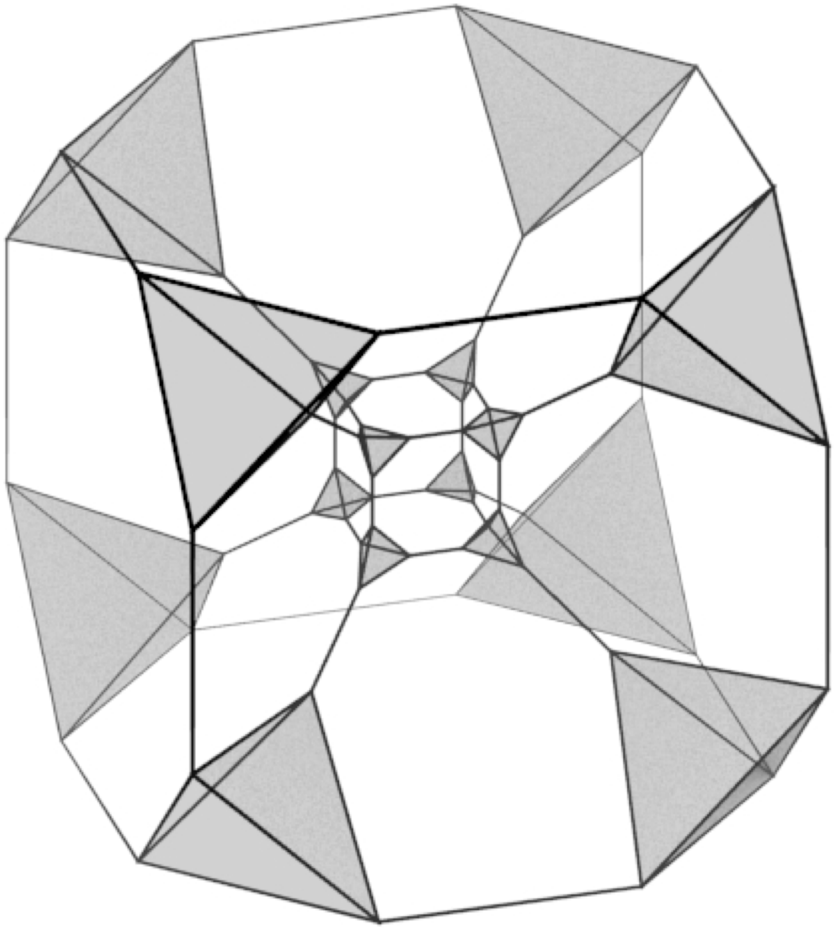


Hypercube



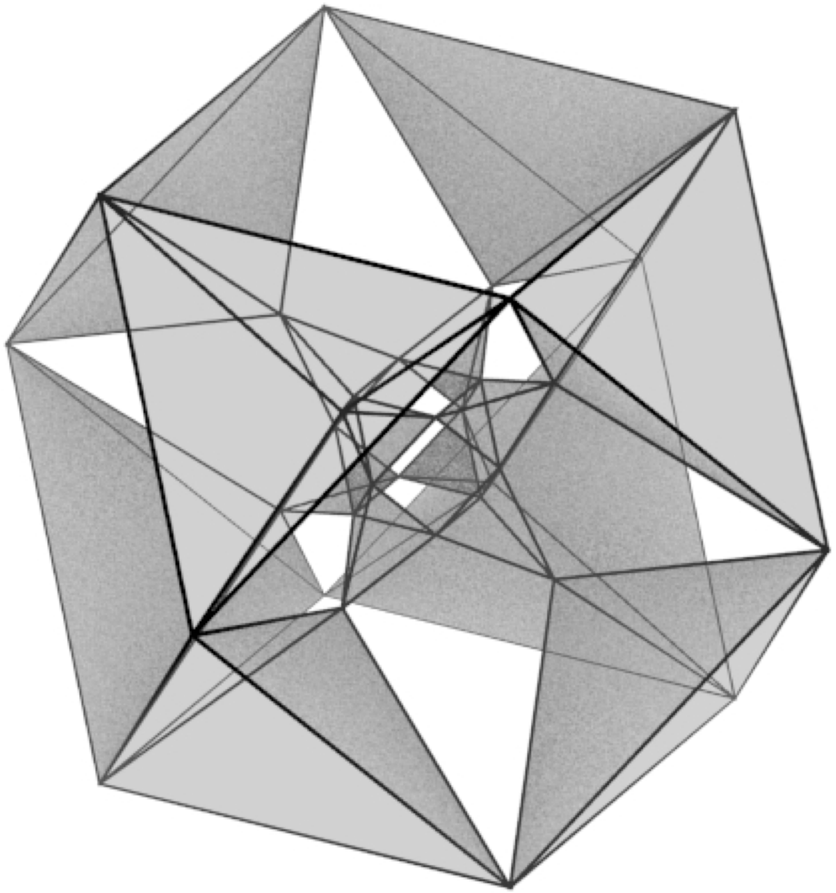
16 vertices
32 edges
24 faces (squares)
8 cells (cubes)

Truncated Hypercube



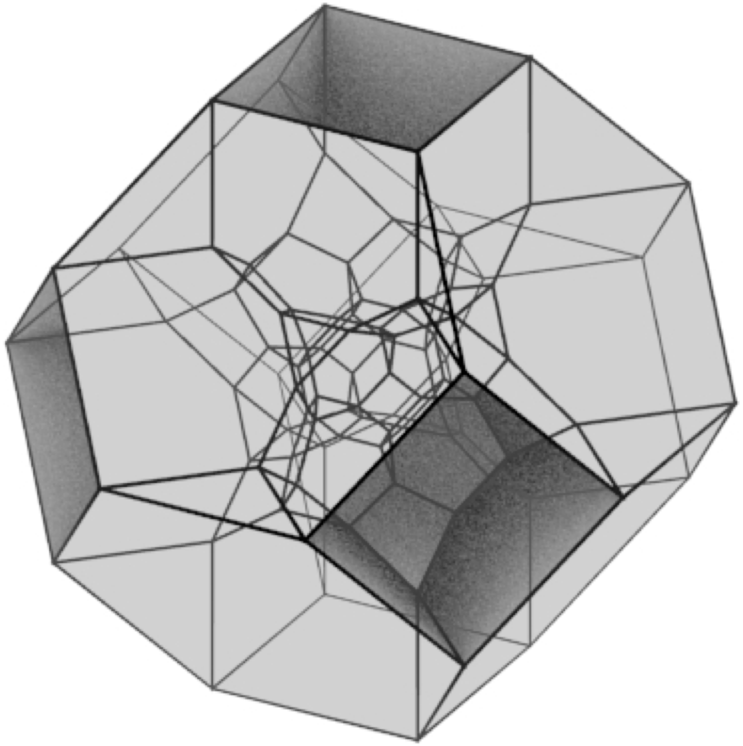
64 vertices
128 edges
88 faces (64 triangles & 24 octagons)
24 cells (8 truncated cubes & 16 tetrahedra)

Rectified Hypercube



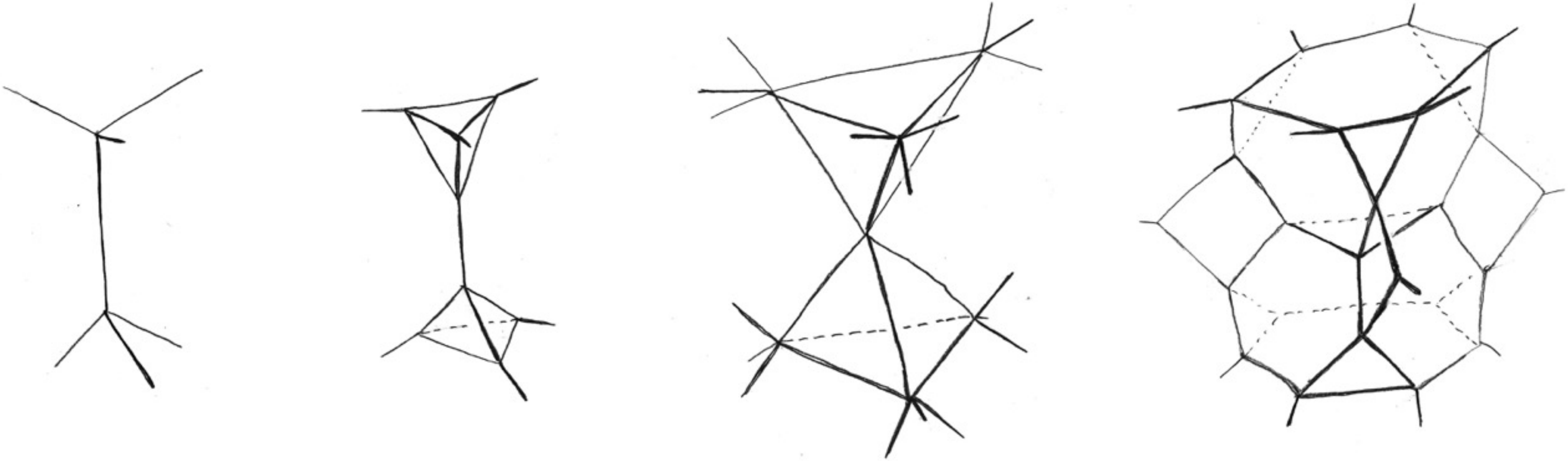
32 vertices
96 edges
88 faces (64 triangles & 24 squares)
24 cells (8 cuboctahedra & 16 tetrahedra)

Bitruncated Hypercube



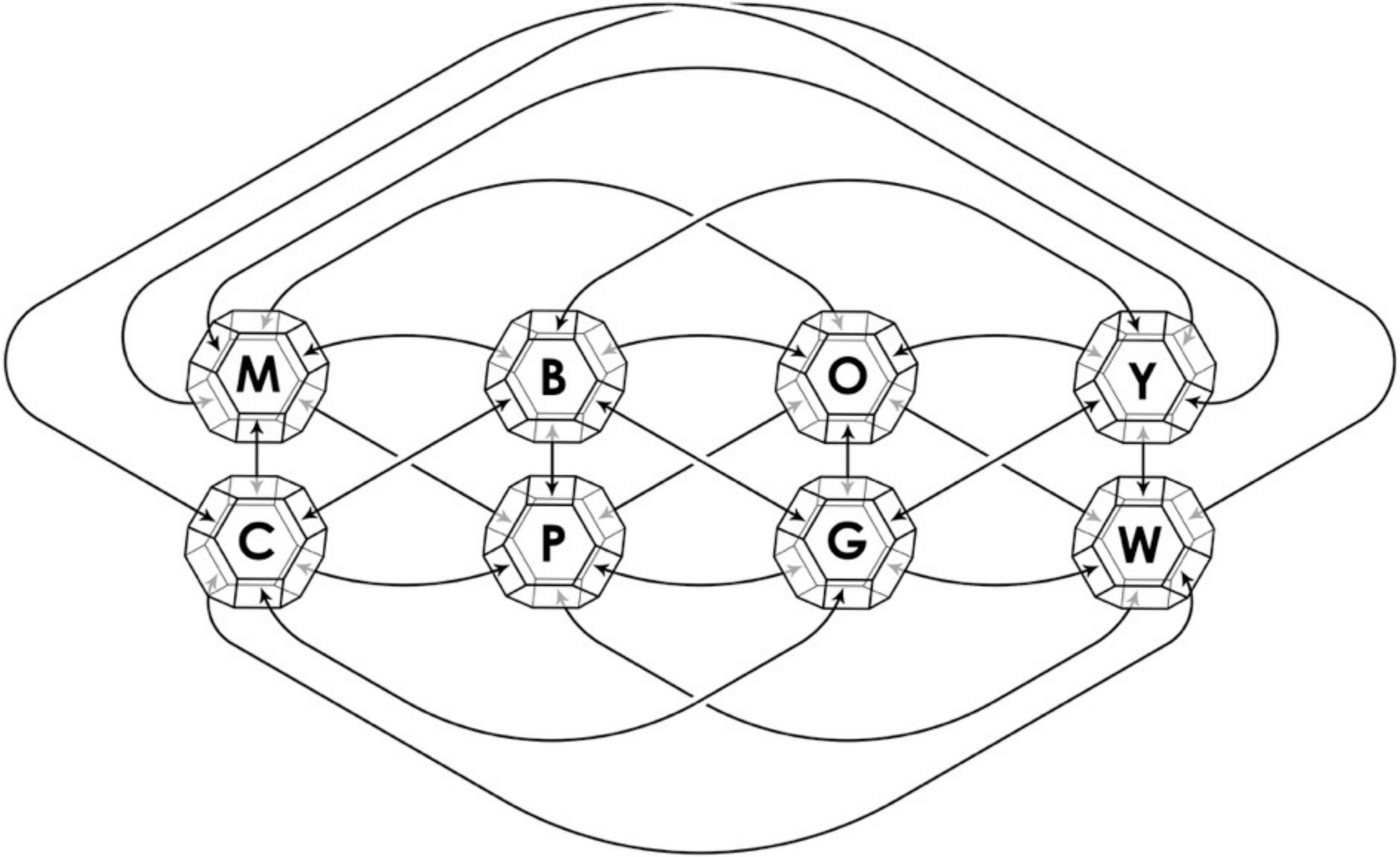
96 vertices
192 edges
120 faces (32 triangles & 24 squares & 64 hexagons)
24 cells (8 truncated octahedra & 16 truncated tetrahedra)

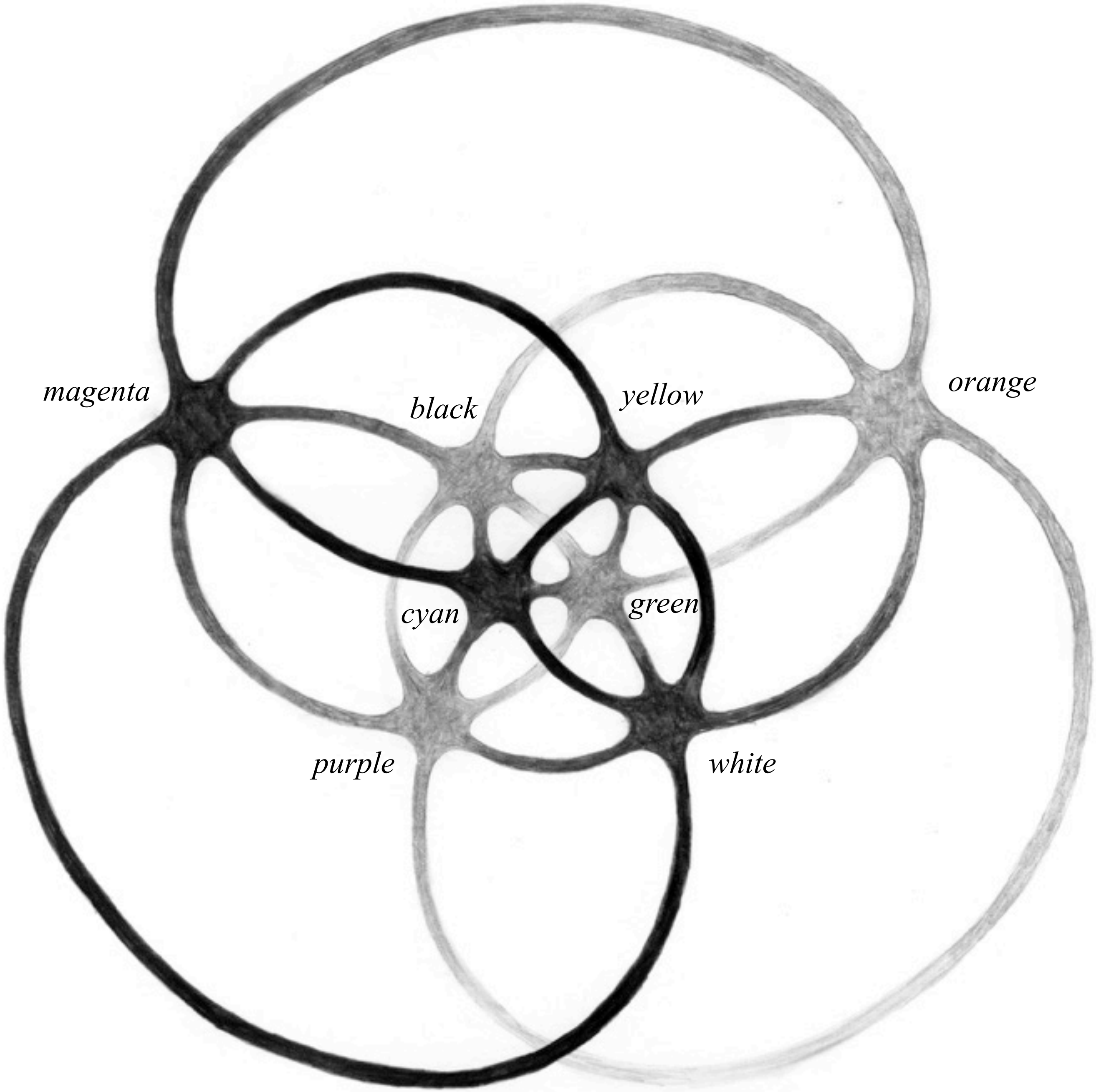
Truncation of a tesseract edge



64 hexagons, 8 colors, 8 of each

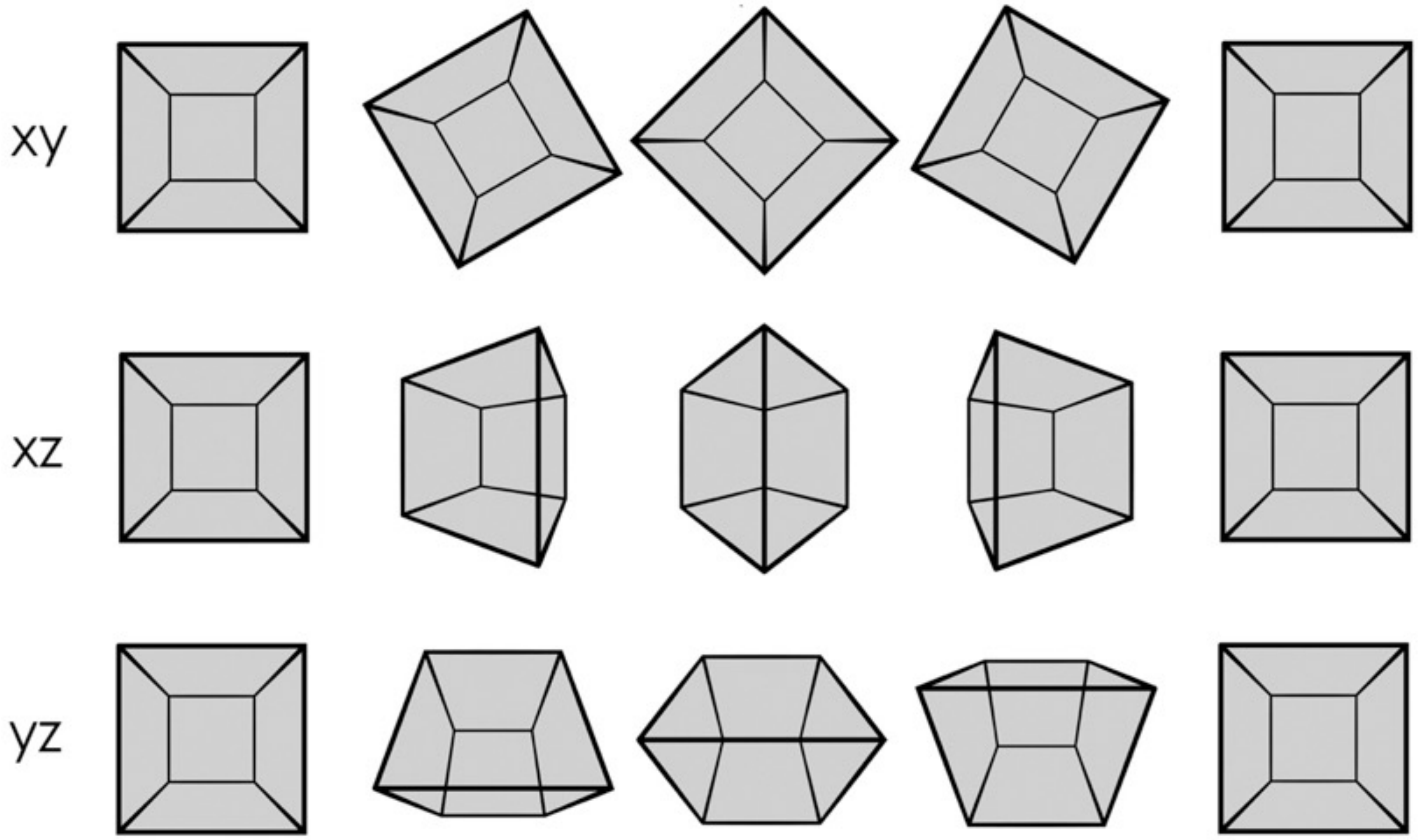
Connections



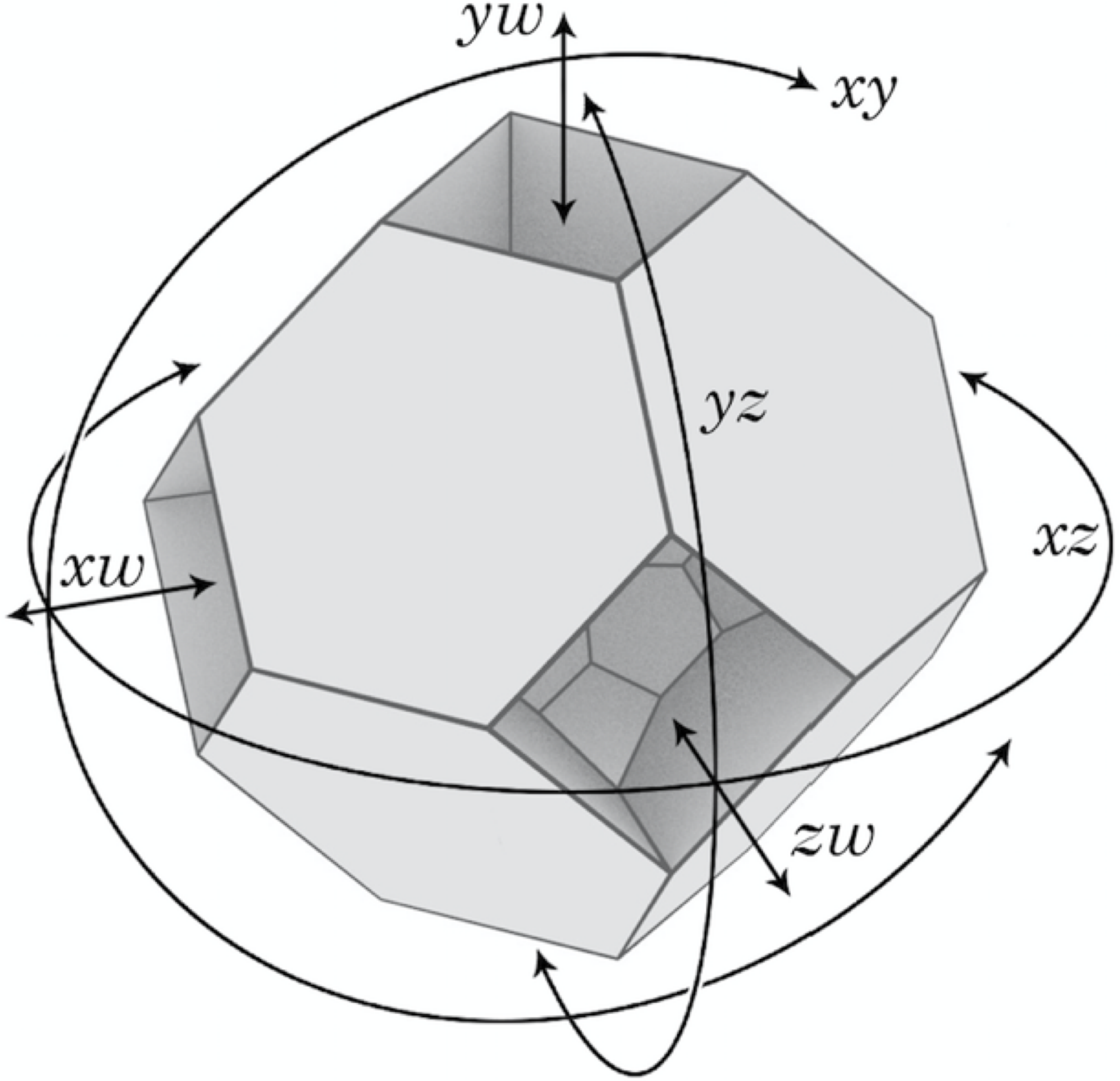

magenta
black
yellow
orange
cyan
green
purple
white

“Crooked House II”:
Hyperbolic Patchwork Model of the Bitruncated Hypercube

The appearance of quarter rotations of a cube along three perpendicular planes



The appearance of rotations of the bitruncated hypercube along six perpendicular planes



Determining the Topology

Euler characteristic (χ) = V – E + F

#faces = 64
#edges = (64 x 6) / 2 = 192
#vertices = (64 x 6) / 4 = 96

V – E + F = 96 – 192 + 64 = -32

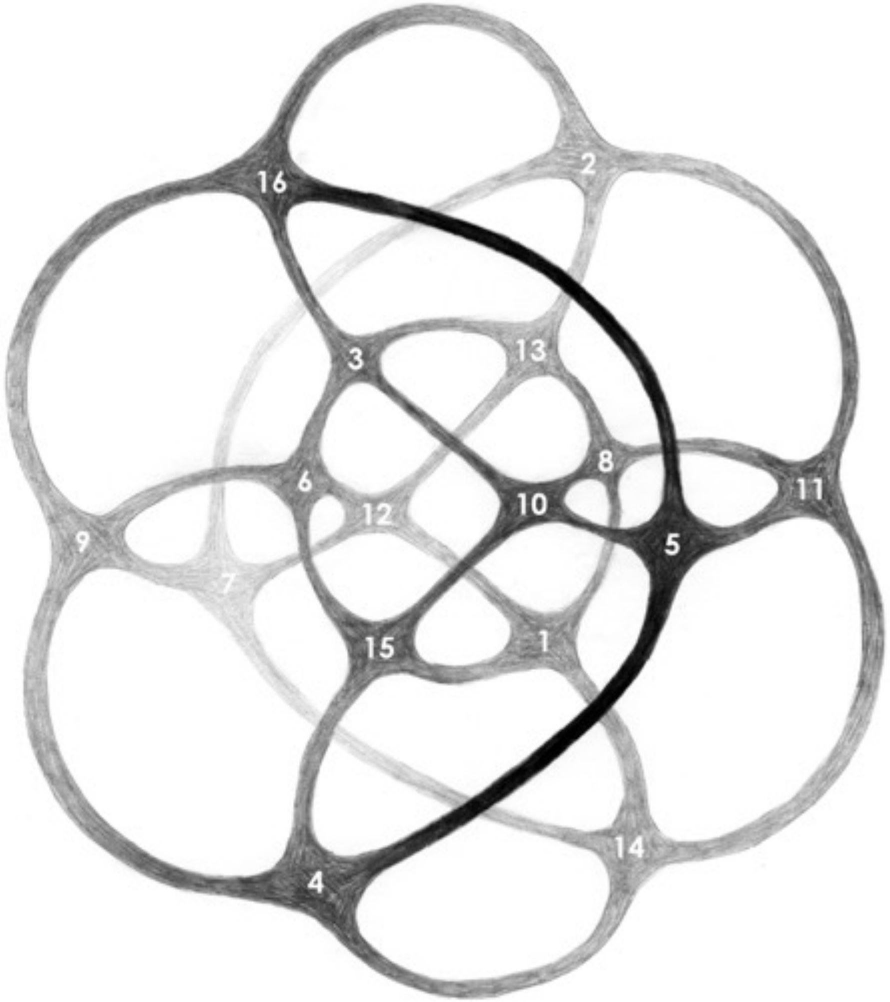
For a closed orientable surface: $\chi = 2 - 2g$

Genus: 17

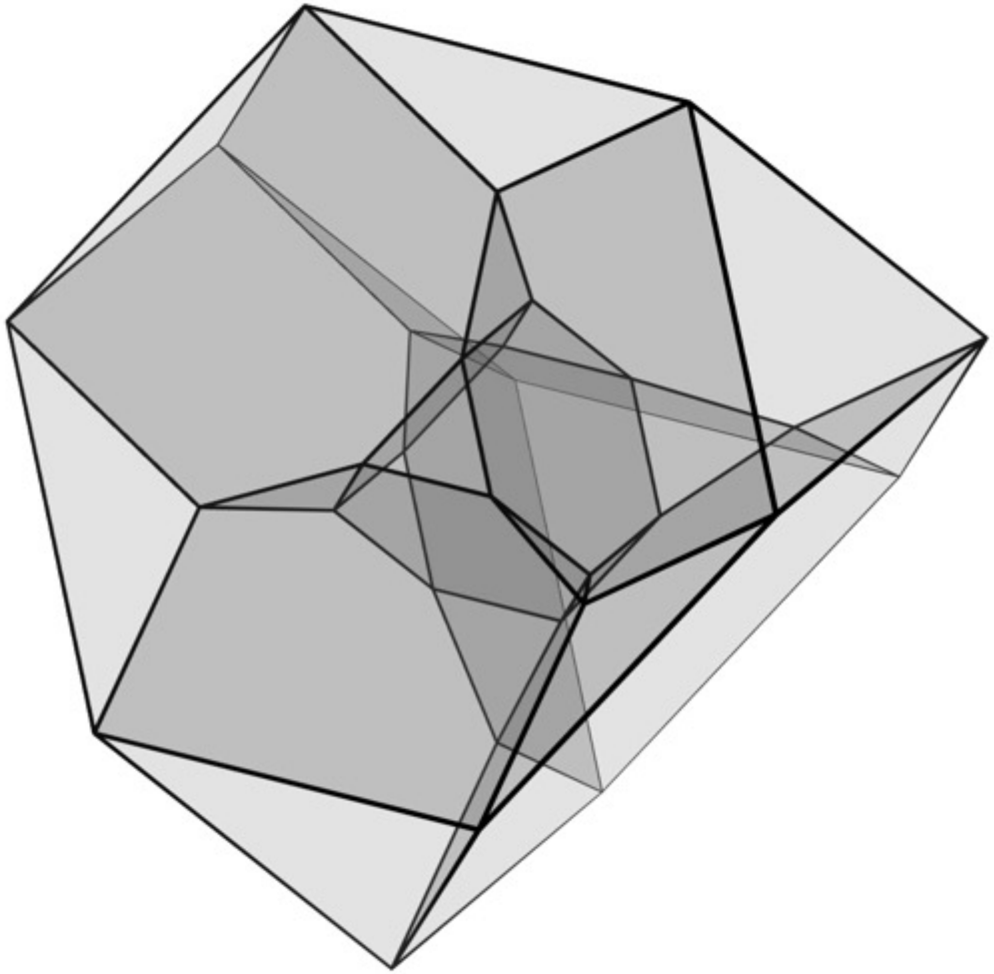
Topology: a torus with 17 handles

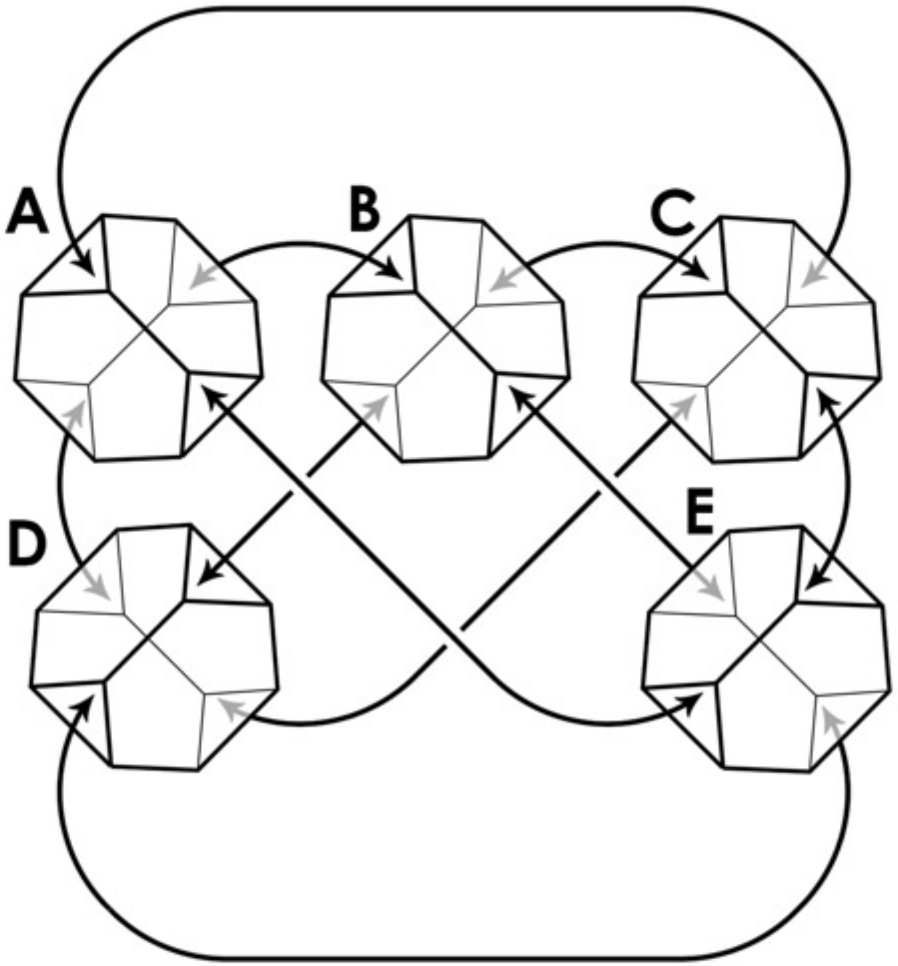
Bitruncations of other regular polychora?

Bitruncated Hexadecachoron
(Bitruncated 16-cell)?



Bitruncated Pentachoron





5 + 5 truncated tetrahedra
Hyperbolic surface of 20 hexagons, meeting 4 at a vertex
Topology: a torus with 6 handles

“Crooked House I”: Hyperbolic Patchwork Model of the Bitruncated Pentachoron

Bitruncated Icositetrachoron
(Bitruncated 24-cell)

24 + 24 truncated cubes
Hyperbolic surface of 144 octagons, meeting 4 at a vertex
Topology: a torus with 73 handles

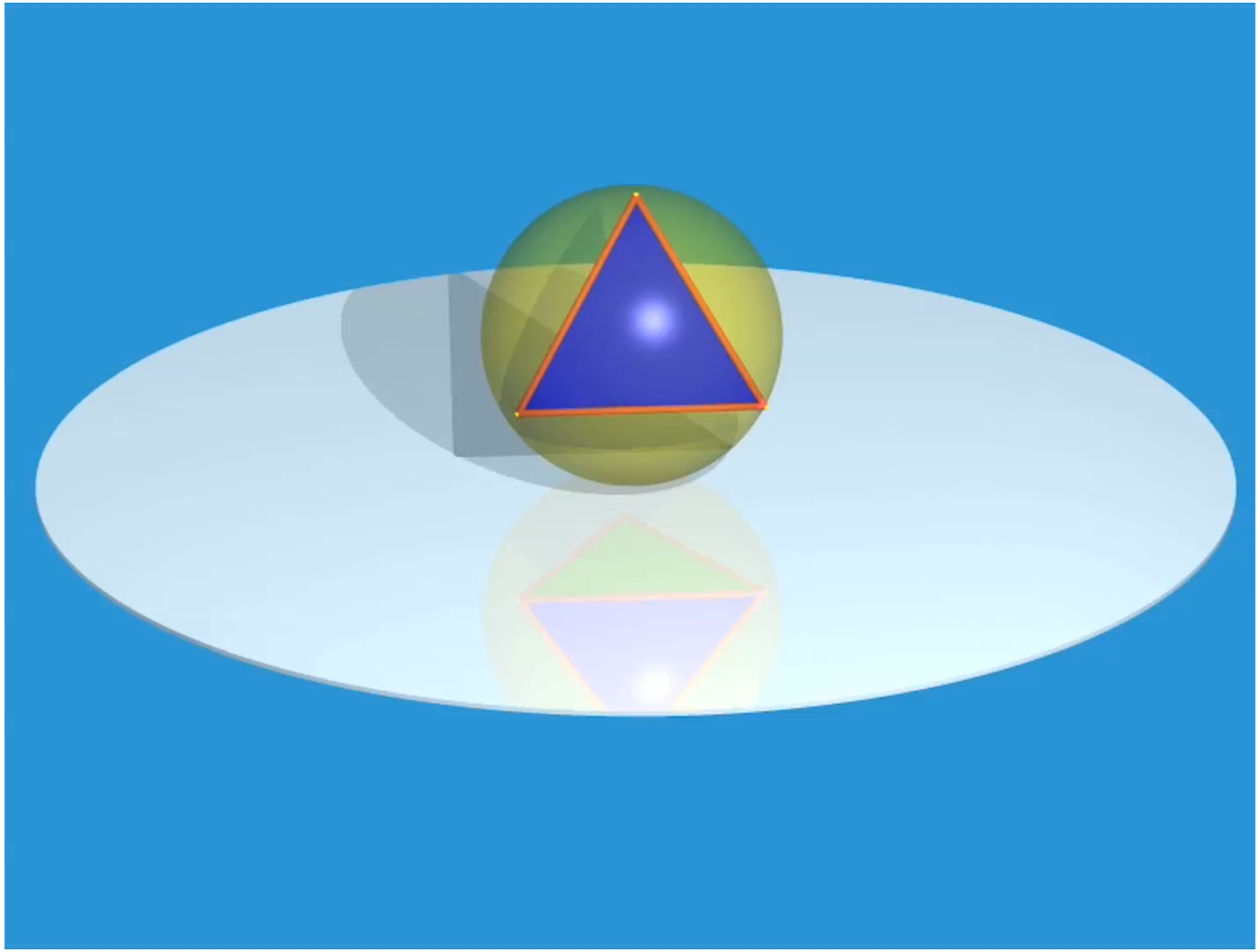
Bitruncated Hecatonicosachoron (Bitruncated 120-cell) & Bitruncated Hexacosichoron (Bitruncated 600-cell)

600 truncated tetrahedra + 120 truncated icosahedra
Hyperbolic surface of 1200 hexagons, meeting 4 at a vertex
Topology: a torus with 299 handles

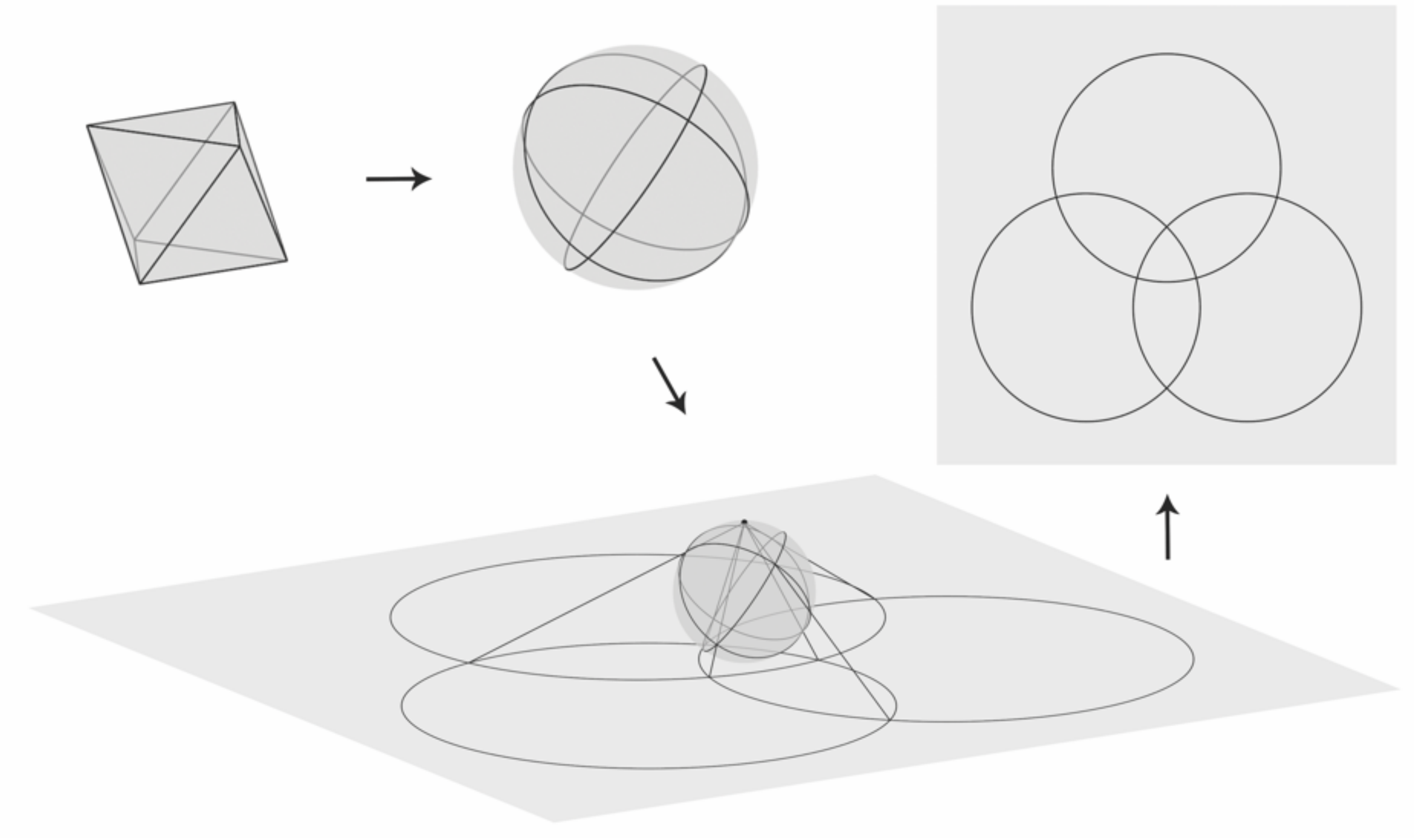
Crocheted version of the ‘Crooked House I’ by Kirsi Peltonen
with a constant negative curvature

THE “KINOCHORON”: MANIPULABLE WIRE MODEL OF THE 16-CELL

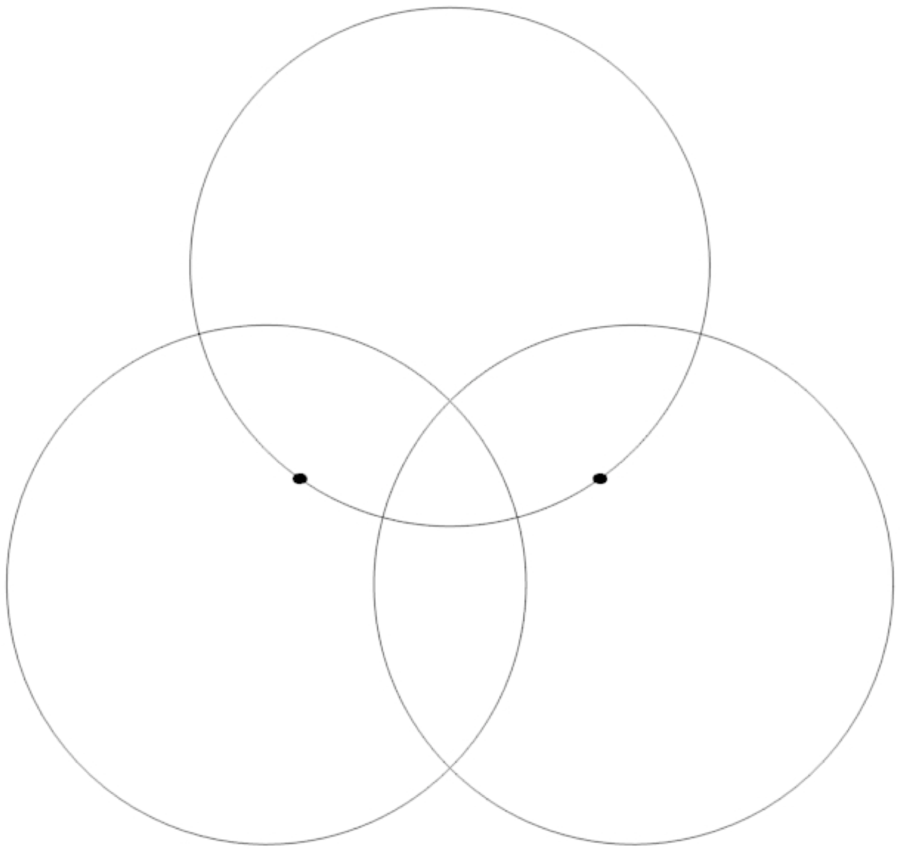
Dimensions by Aurélien Alvarez, Étienne Ghys, and Jos Leys



http://www.dimensions-math.org/

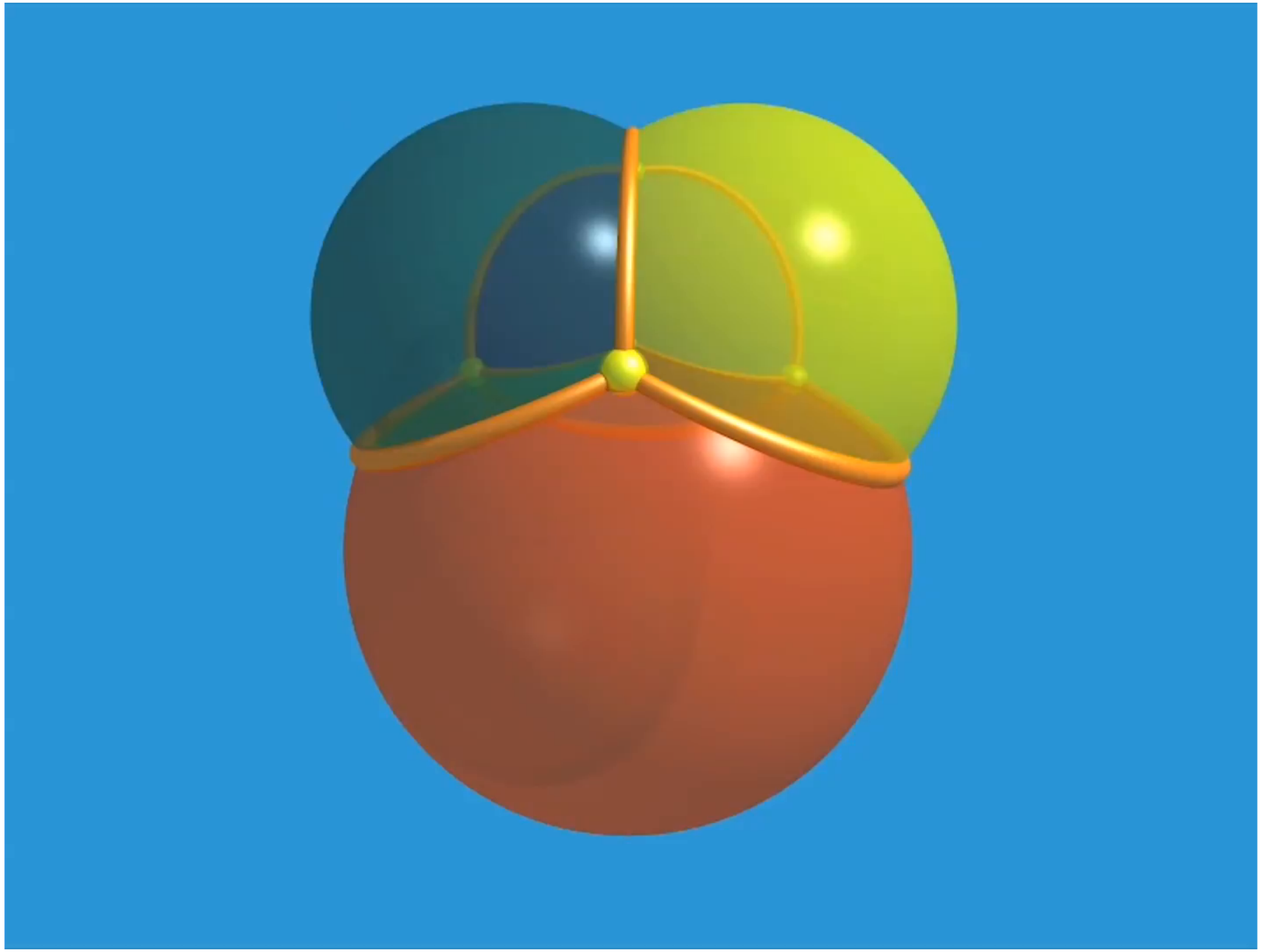


Stereographic projection of the octahedron



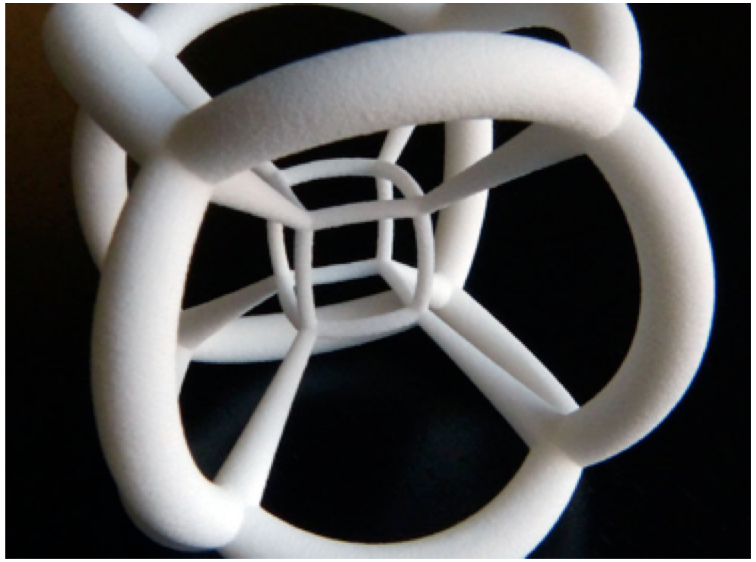
Stereographic projection of the octahedron

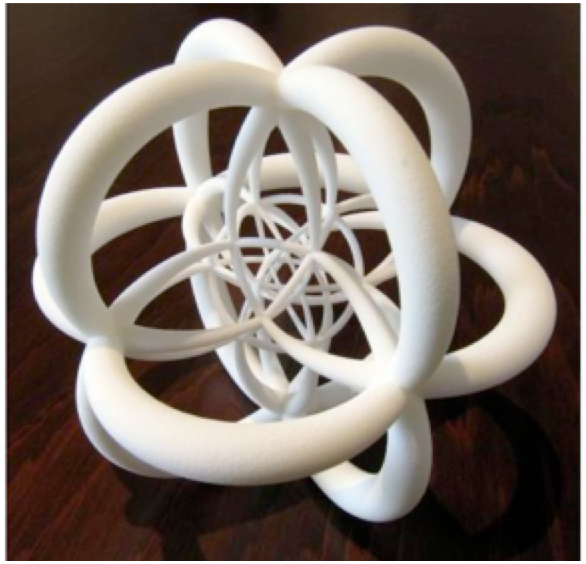
Dimensions by Aurélien Alvarez, Étienne Ghys, and Jos Leys

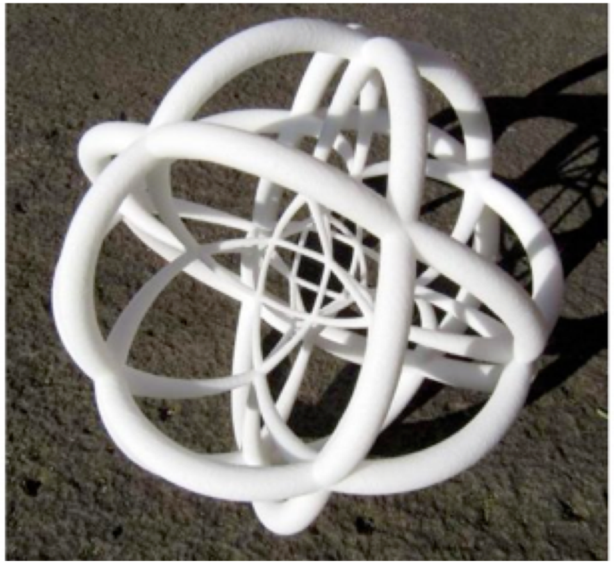


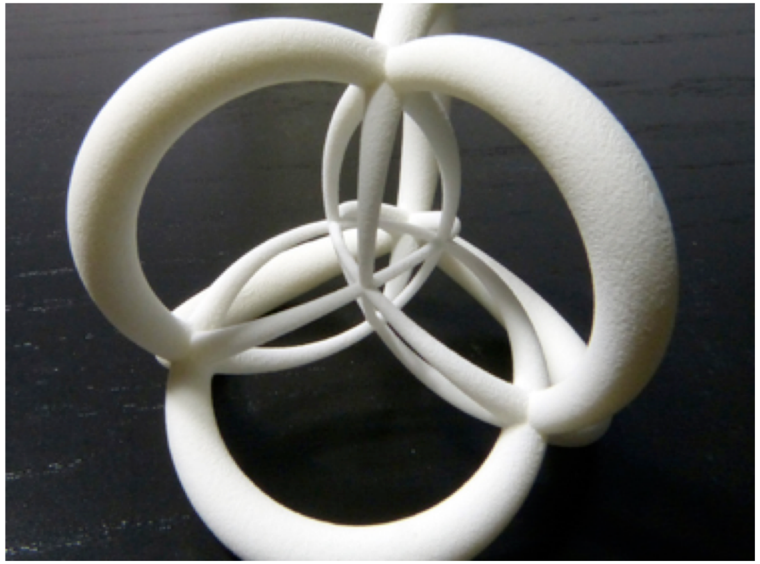
http://www.dimensions-math.org/

3D prints by Henry Segerman



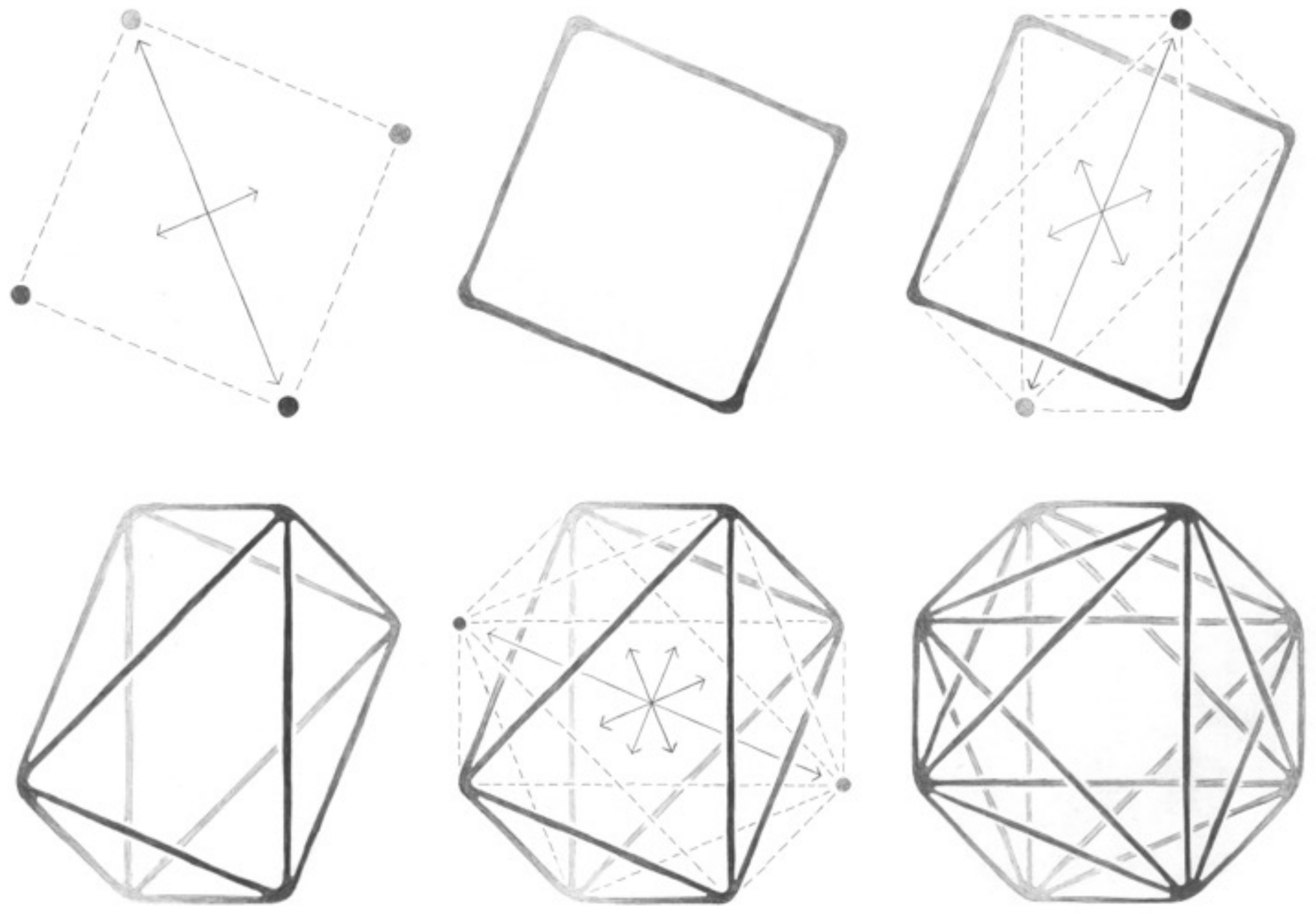




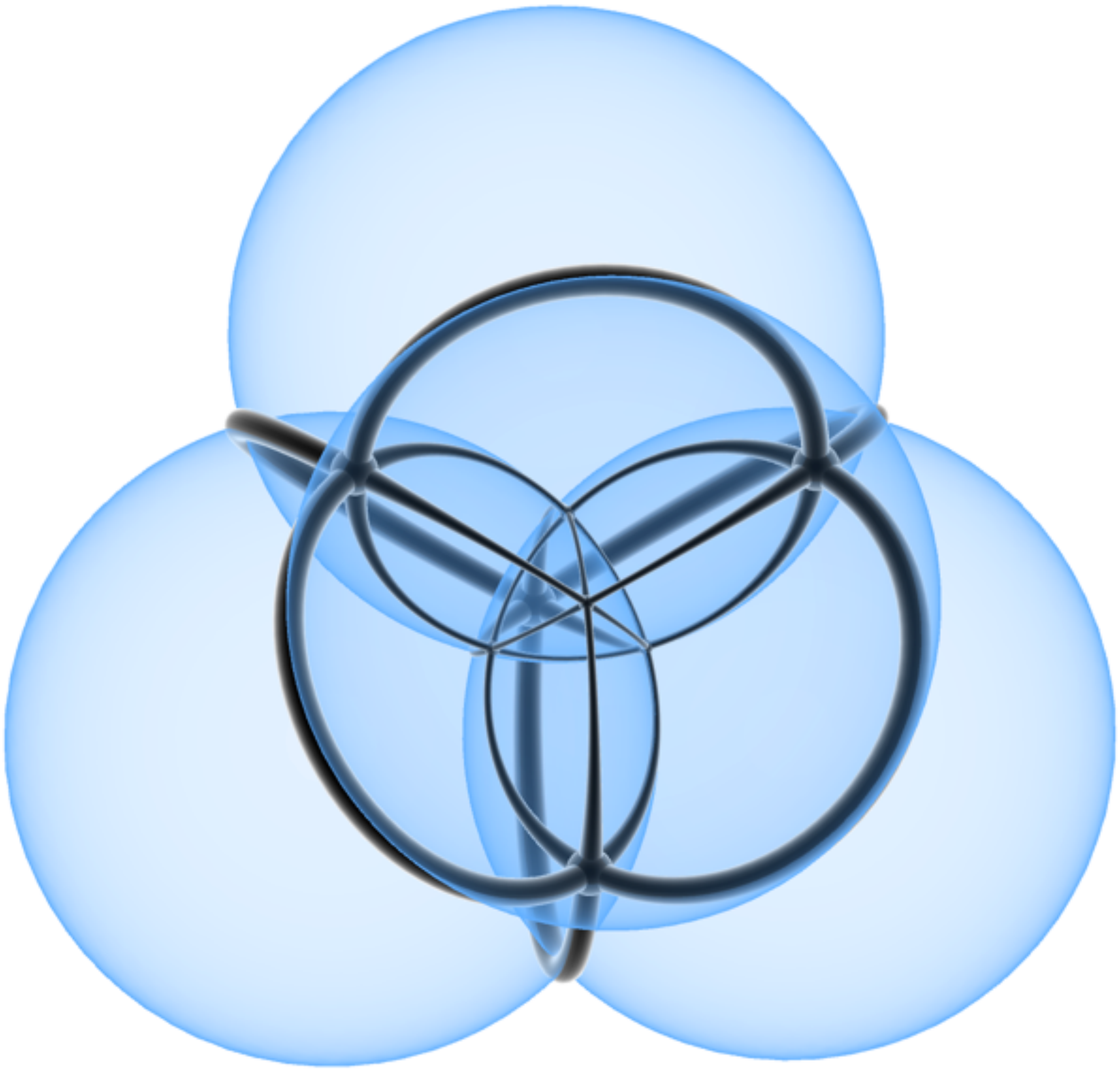


http://www.shapeways.com/designer/henryseg

Constructing the 16-cell

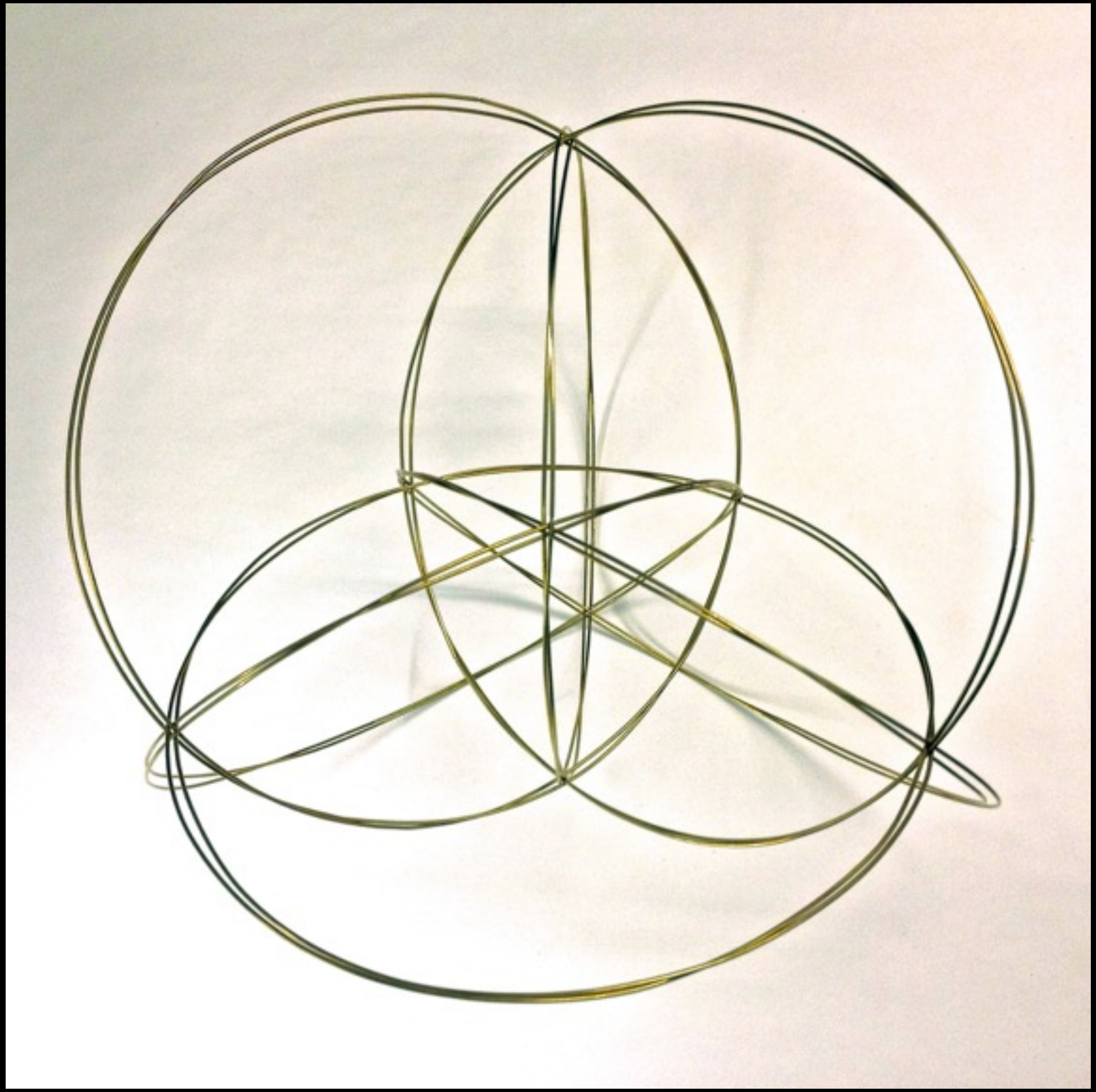


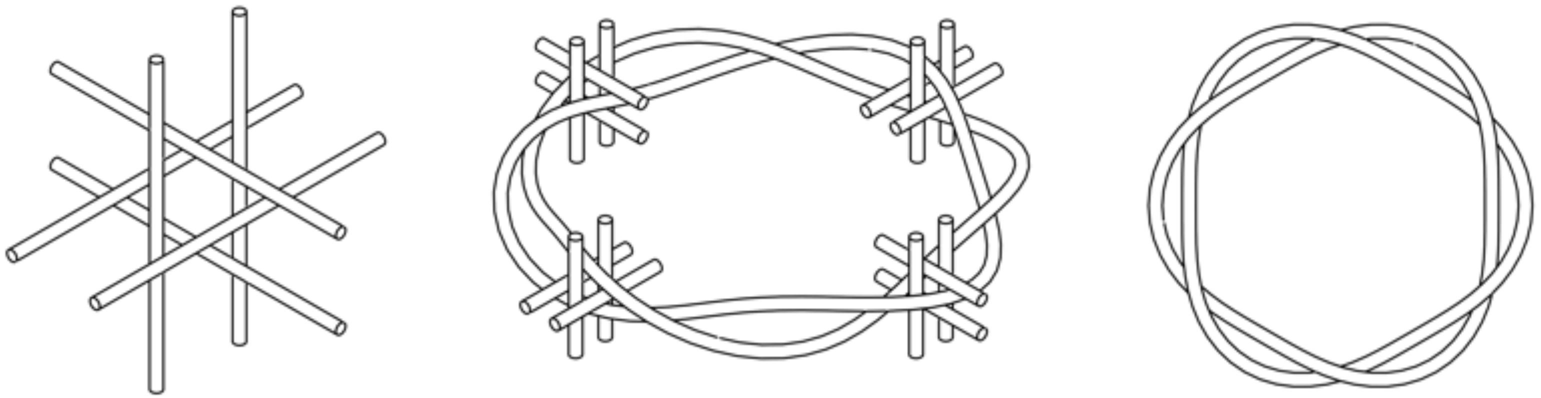
The 16-cell



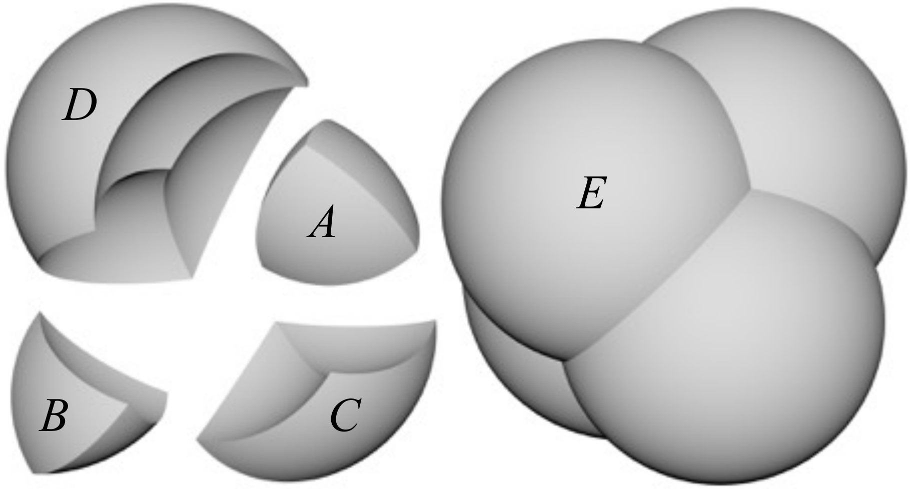
App: Jenn 3d" by Fritz Obermeyer (http://jenn3d.org/)

The Kinochoron





The *Kinochoron*: the tetrahedral cells



Type:	*A*	*B*	*C*	*D*	*E*
Layer:	1	2	3	4	5
Quantity:	1	4	6	4	1
Convex faces:	4	3	2	1	0
Concave faces:	0	1	2	3	4

The *Kinochoron*: the ‘hyper-rotation’