Aalto University

Fractal Geometry

Phenomena that cannot be explained by classical geometry

Shapes in Action Tue 16th Oct

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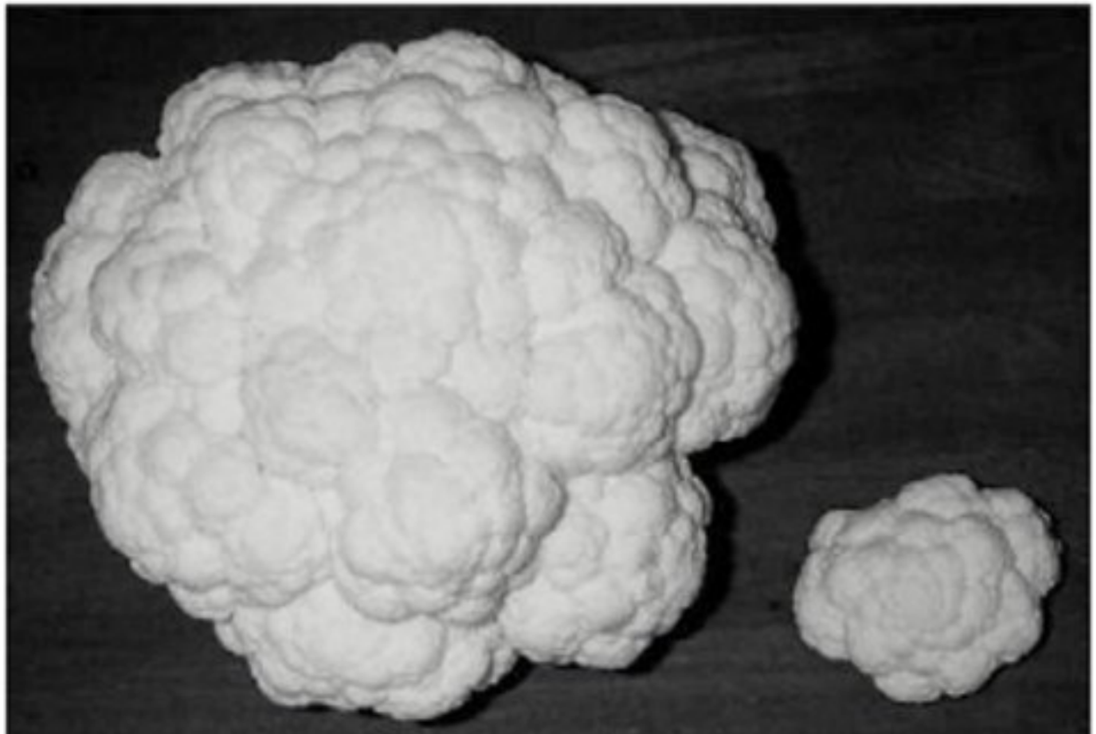
‘Natural’ vs. ‘man-made’ objects

What do these pictures present?

What happened ? Why does the trick work?

Many objects look the same in different scales.

How can one distinguish the correct size?



Who invented ‘fractal geometry’ in the sense of ‘new geometry of nature’?

Many fundamental examples due to classical mathematics !

George Ferdinand Ludwig Philipp Cantor 1845-1918

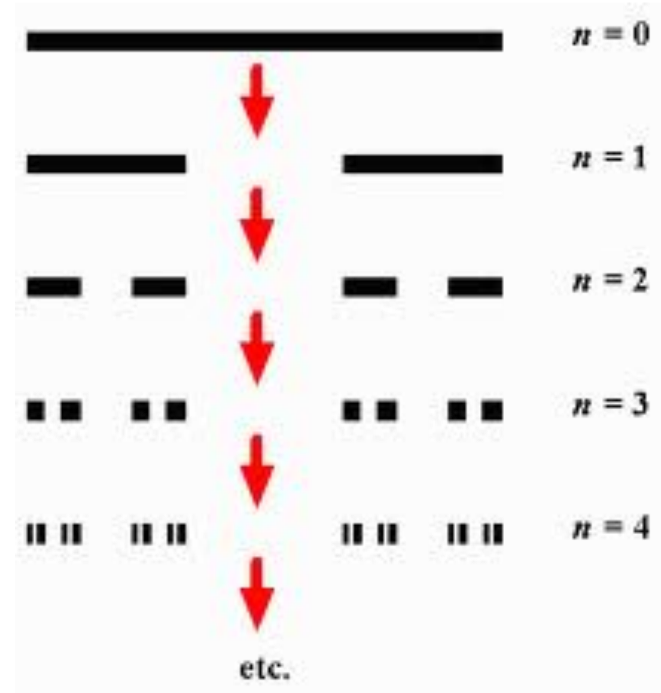
- *the crisis of the dimension*
- *exceptional objects*
- *‘mathematical monsters’*
- *limits of fundamental notions*

(‘curve’, ‘continuous’)

Abnormal Monsters or Typical Nature ?

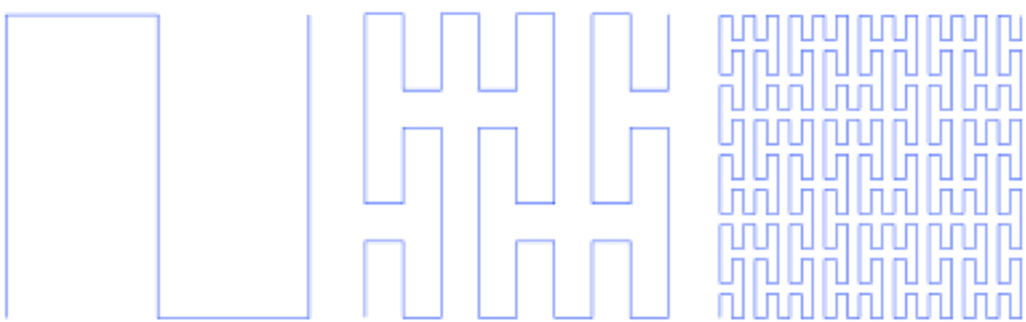
Cantor's middle third set (1883)

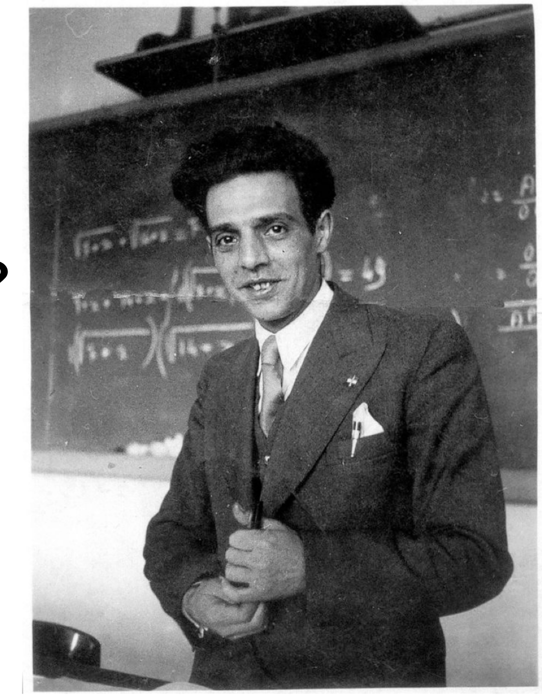
- **are there any points left in the limit?**
  - subintervals left $(2/3)^n \rightarrow 0$ as $n \rightarrow \infty$
  - endpoints never removed !
  - infinite decimal presentation of 0's and 2's in a base 3 (1/3=0.0222…!)
- **is it possible to numerate them ?**
- **size of the limit set vs [0,1]?**
- **dimension of the limit set?**
- **connectedness of the limit set?**
- ***a self-similar set***
- **a prototype of a fractal set**



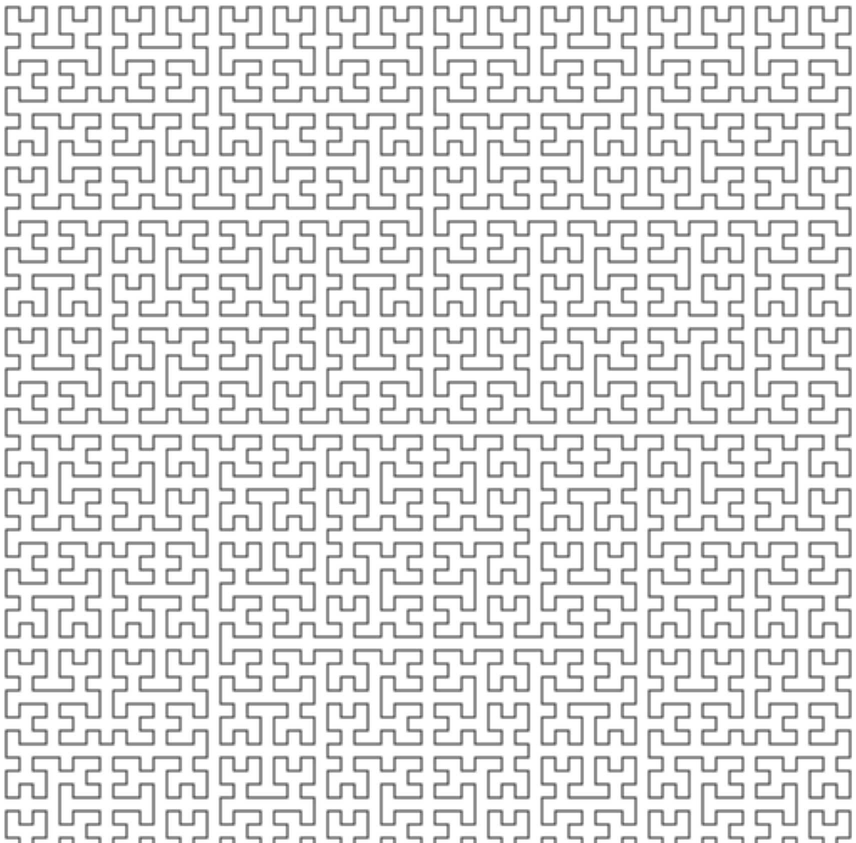
Giuseppe Peano, 1858-1932

What is a curve?
What is the dimension of a curve?
Can a curve fill a square/cube/hypercube/…?





A Peano curve by David Hilbert (1862-1943)



Ueber die stetige Abbildung einer Linie auf ein Flächenstück.*)

Von

David Hilbert in Königsberg i. Pr.

Peano hat kürzlich in den Mathematischen Annalen**) durch eine arithmetische Betrachtung gezeigt, wie die Punkte einer Linie stetig auf die Punkte eines Flächenstückes abgebildet werden können. Die für eine solche Abbildung erforderlichen Functionen lassen sich in übersichtlicherer Weise herstellen, wenn man sich der folgenden geometrischen Anschauung bedient. Die abzubildende Linie — etwa eine Gerade von der Länge 1 — theilen wir zunächst in 4 gleiche Theile 1, 2, 3, 4 und das Flächenstück, welches wir in der Gestalt eines Quadrates von der Seitenlänge 1 annehmen, theilen wir durch zwei zu einander senkrechte Gerade in 4 gleiche Quadrate 1, 2, 3, 4 (Fig. 1). Zweitens theilen wir jede der Theilstrecken 1, 2, 3, 4 wiederum in 4 gleiche Theile, so dass wir auf der Geraden die 16 Theilstrecken 1, 2, 3, . . ., 16 erhalten; gleichzeitig werde jedes der 4 Quadrate 1, 2, 3, 4 in 4 gleiche Quadrate getheilt und den so entstehenden 16 Quadraten

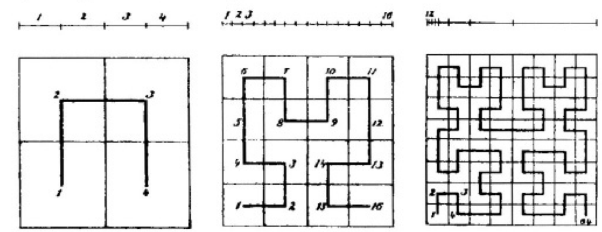


Fig. 1. Fig. 2. Fig. 3.

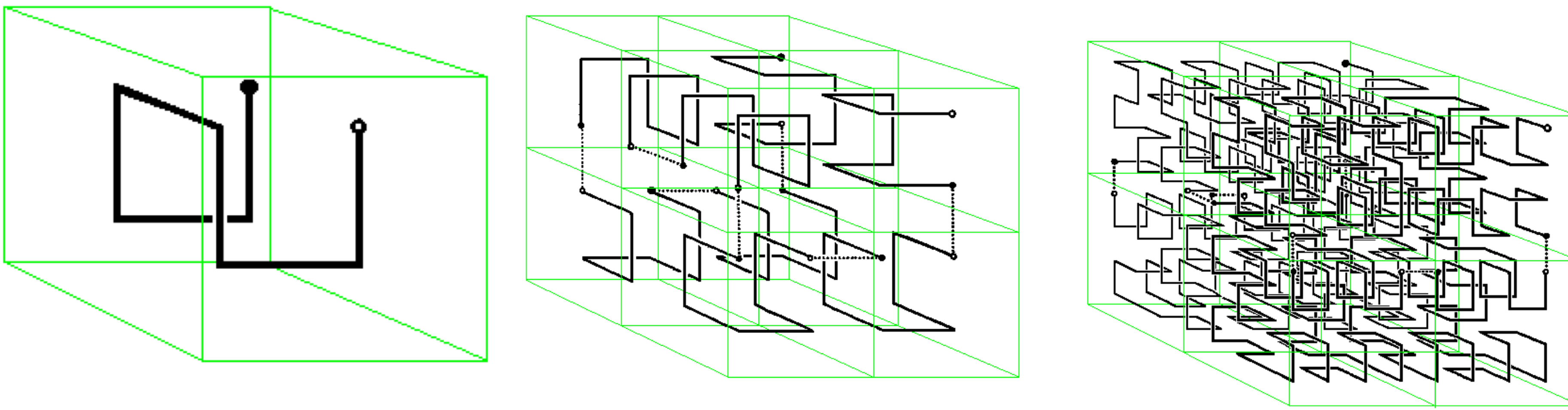
werden dann die Zahlen 1, 2 . . . 16 eingeschrieben, wobei jedoch die Reihenfolge der Quadrate so zu wählen ist, dass jedes folgende Quadrat sich mit einer Seite an das vorhergehende anlehnt (Fig. 2). Denken wir uns dieses Verfahren fortgesetzt — Fig. 3 veranschaulicht den

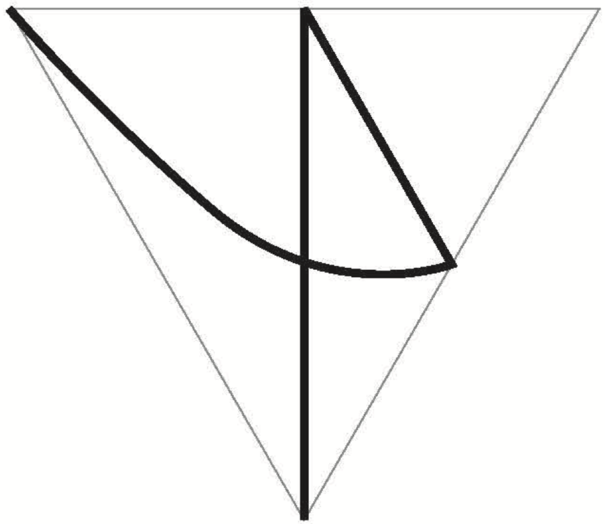
*) Vergl. eine Mittheilung über denselben Gegenstand in den Verhandlungen der Gesellschaft deutscher Naturforscher und Aerzte. Bremen 1890.

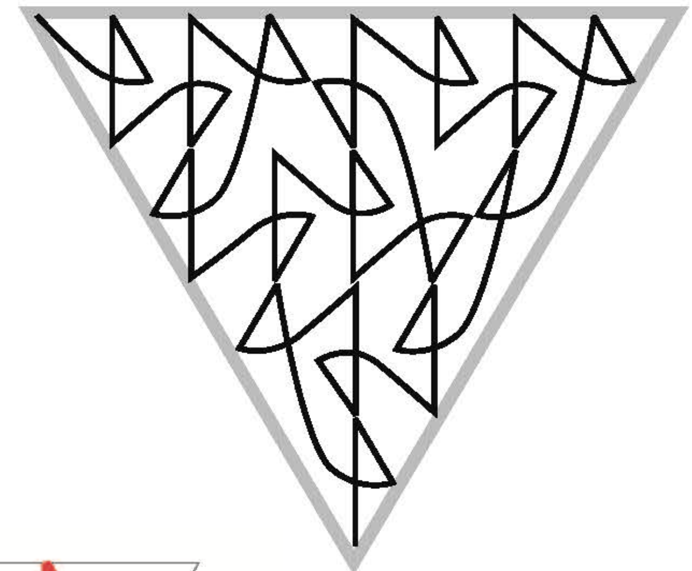
**) Bd. 36, S. 157.

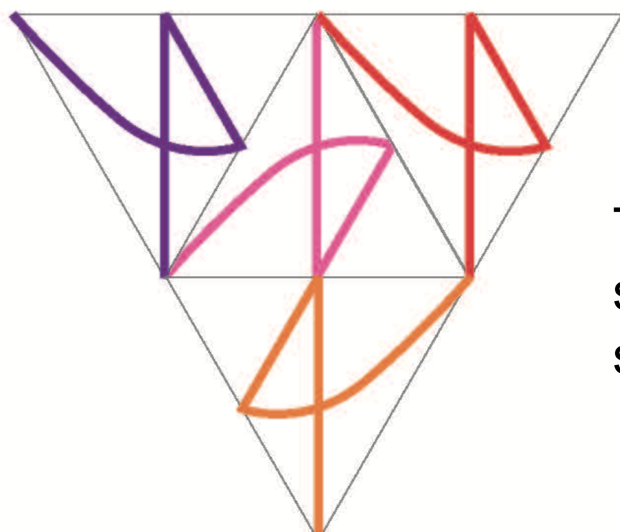
30*

A 3D generalisation of Hilbert's construction by W. Gilbert (Mathematical Intelligencer 6(3) (1984), page 78)



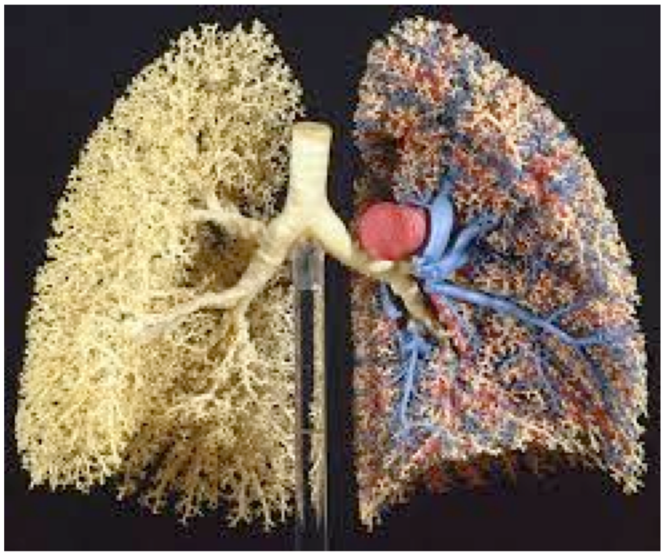


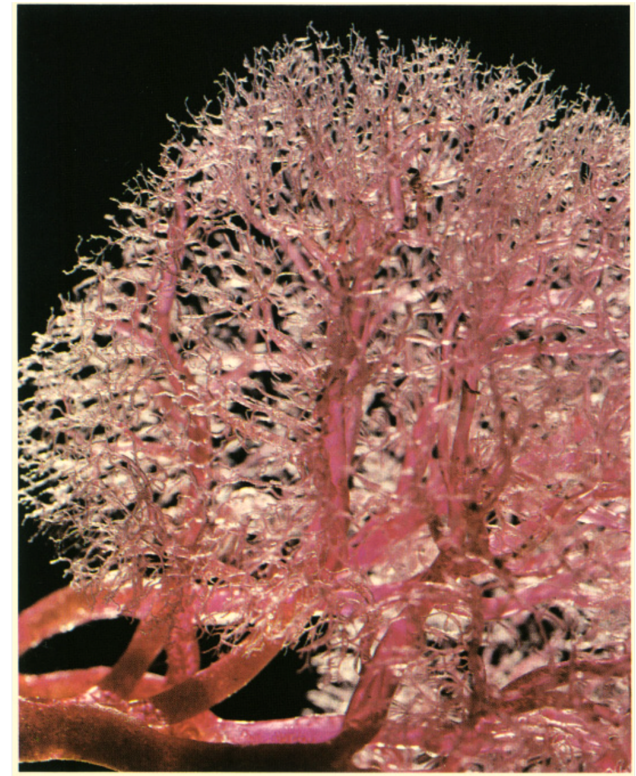




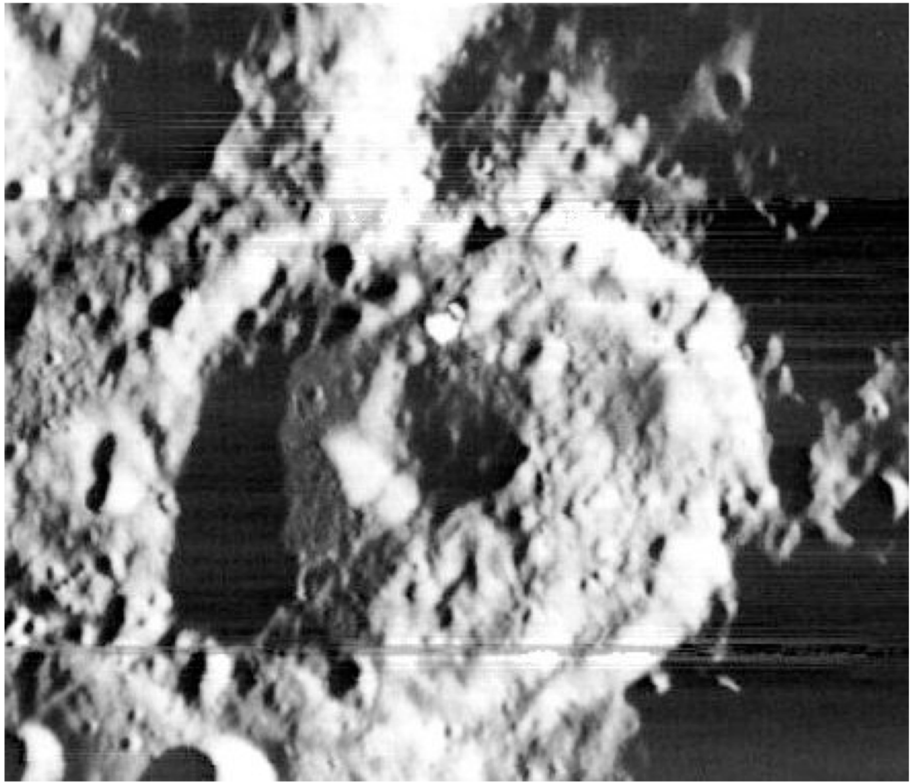
Triangle filling variant (with self intersections on finite scale) by Gross 2007

Space filling structures in Nature

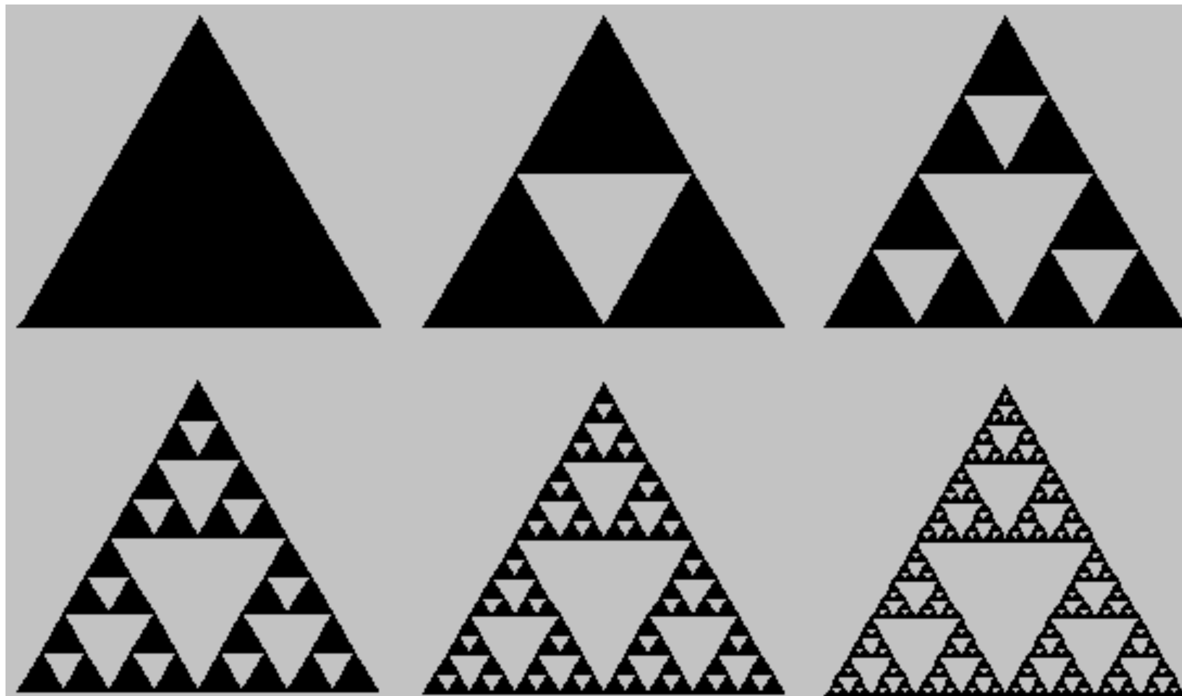




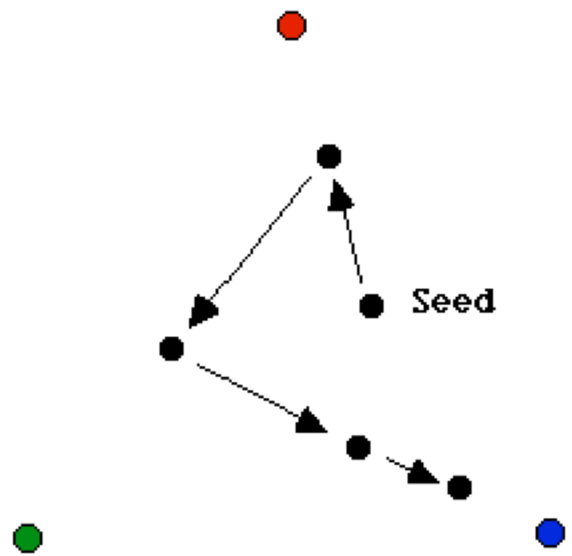
Wacław Franciszek Sierpiński 1882-1969

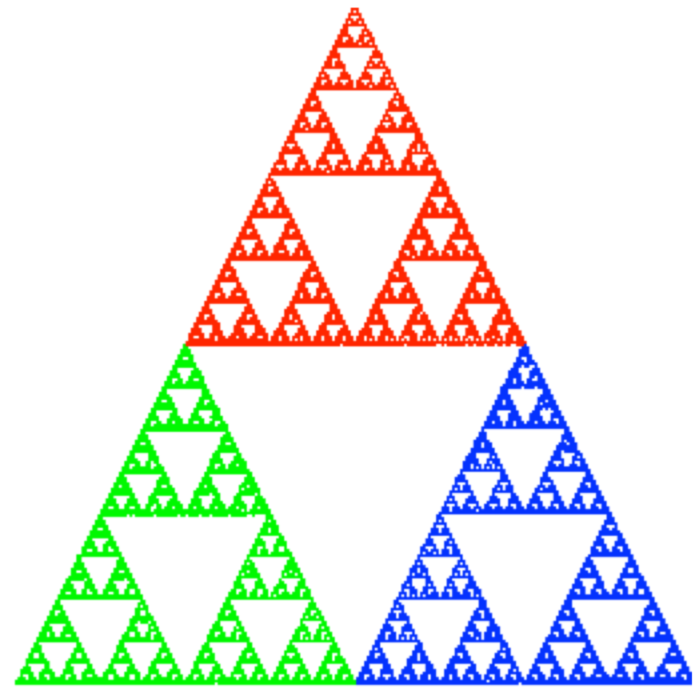


A self similar process in Sierpiński gasket (1916)



The Chaos Game (Barnsley)





Cathedral Anagni (Italy) 1104

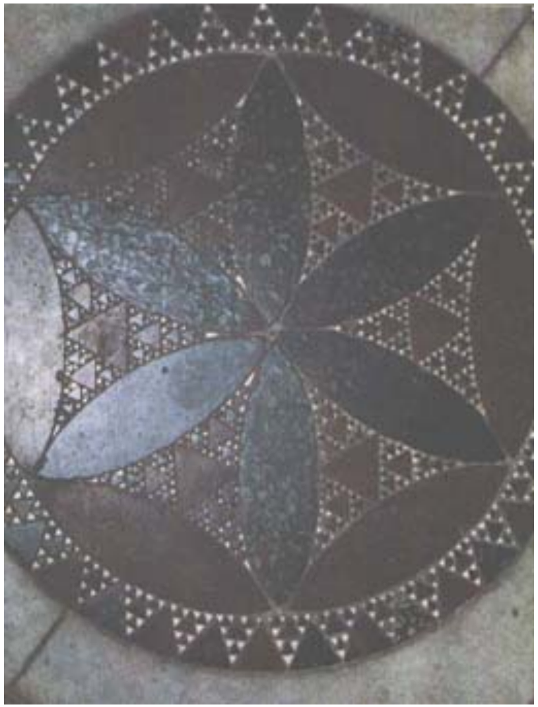
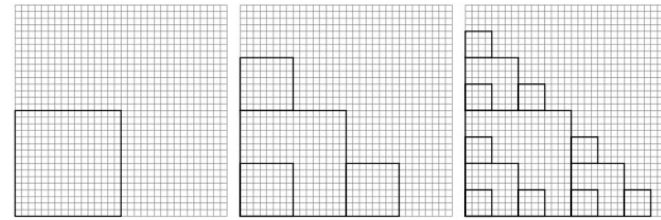
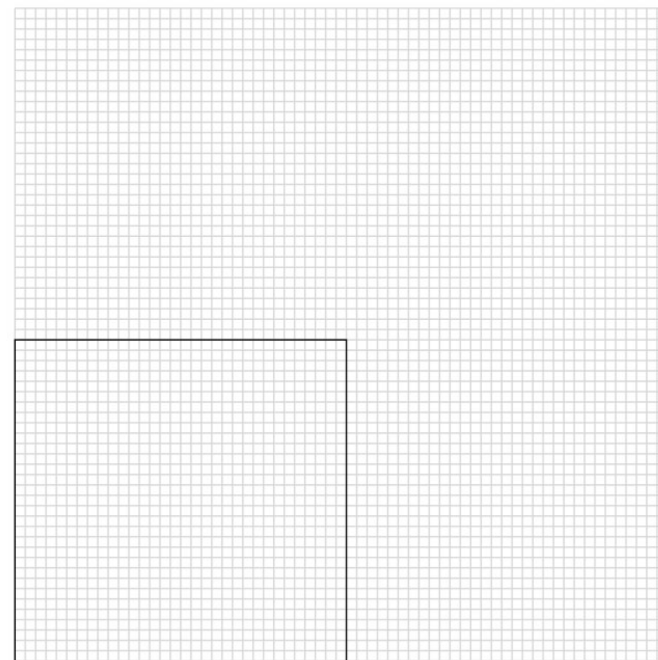
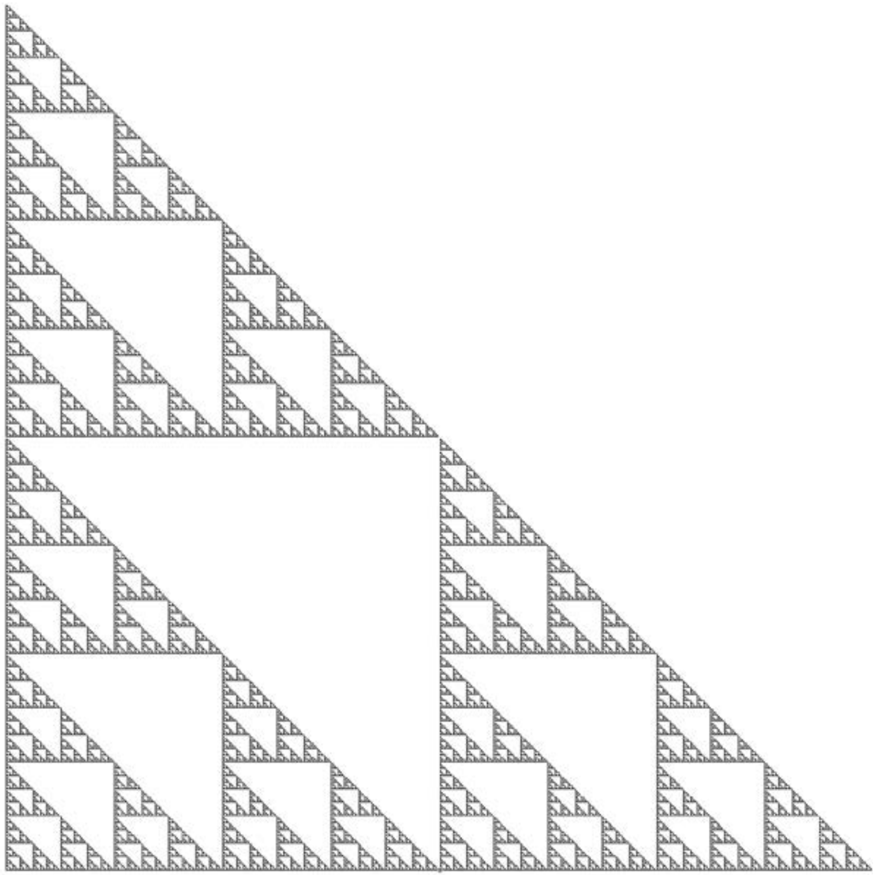


Fig. 6 SS. Giovanni e Paolo (13th century), Rome

Santa Maria in Cosmedin, Rome



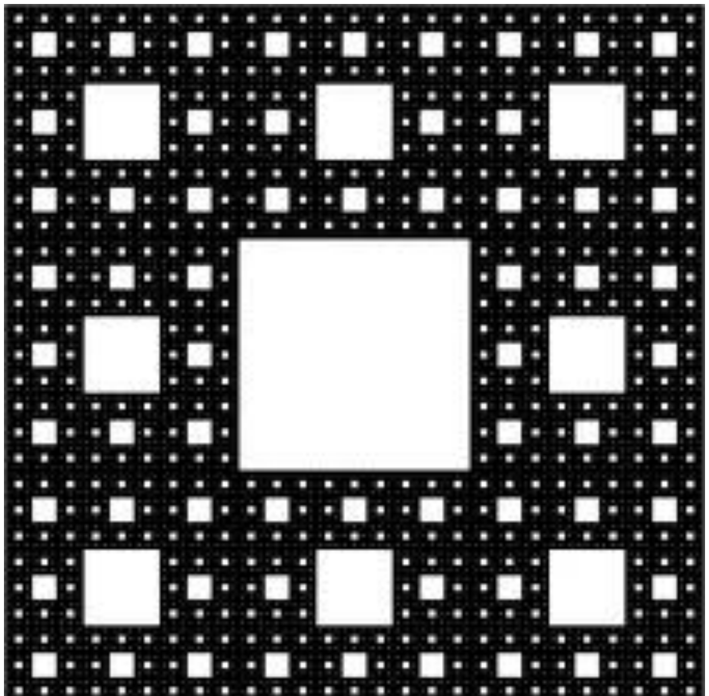


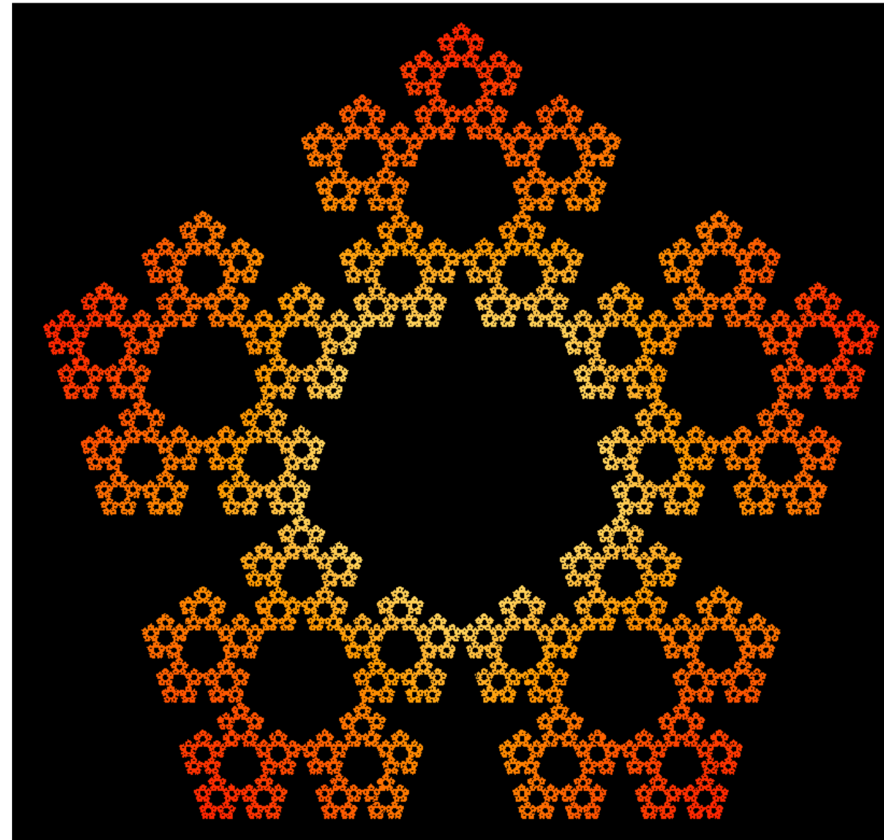


Escher's studies of Sierpinski gasket-type patterns

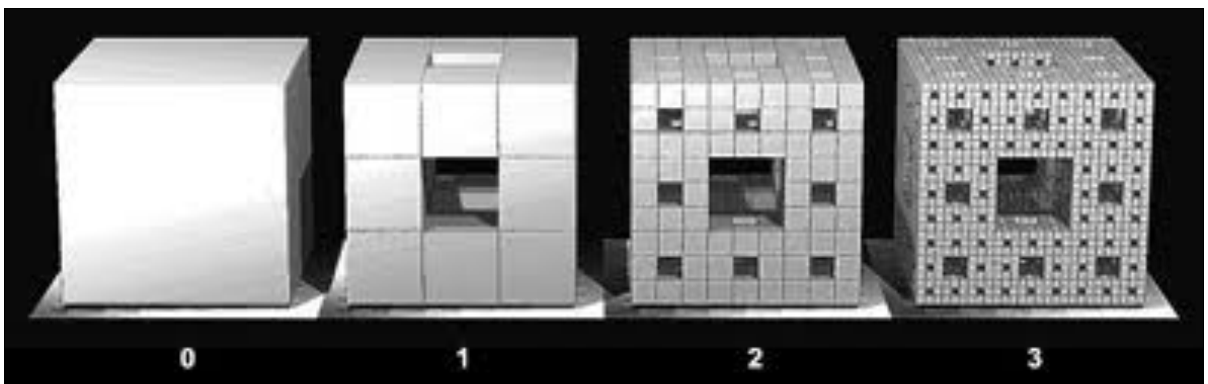
On twelfth-century pulpit of Ravello Cathedral, 1923

Sierpiński Carpet and generalisations

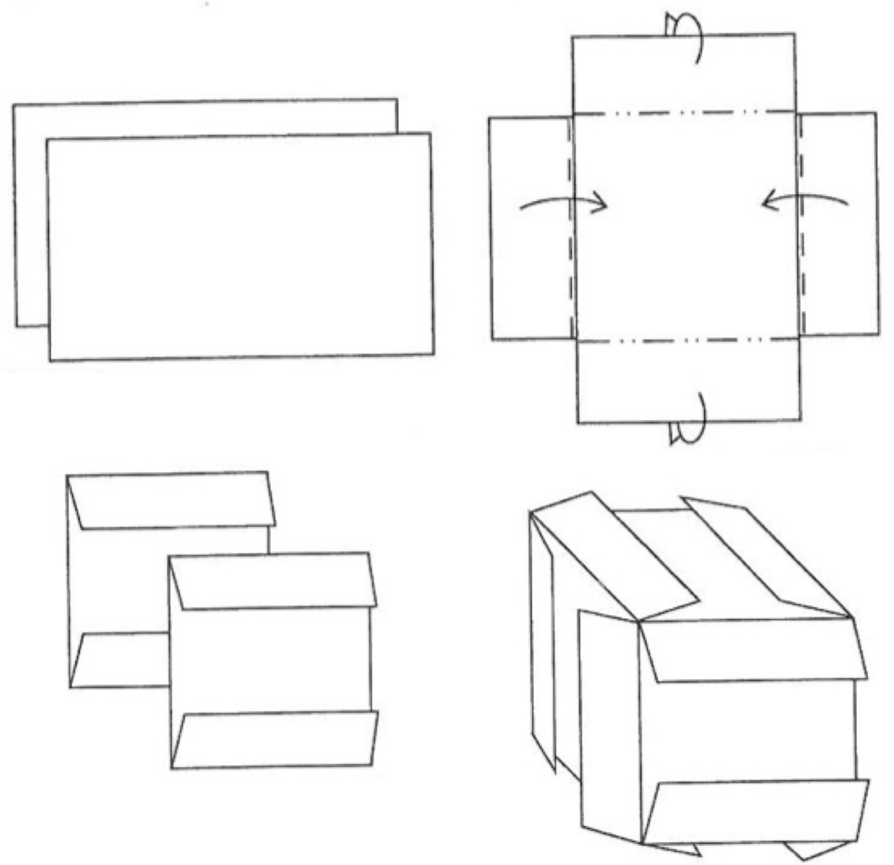


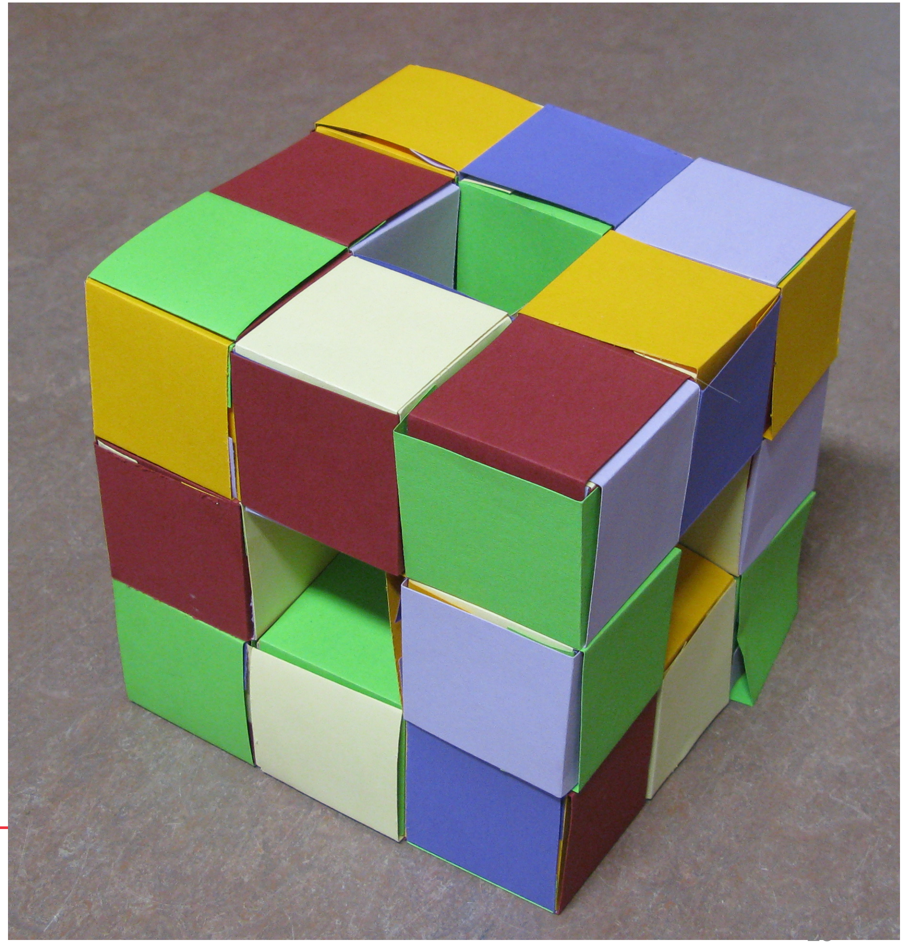


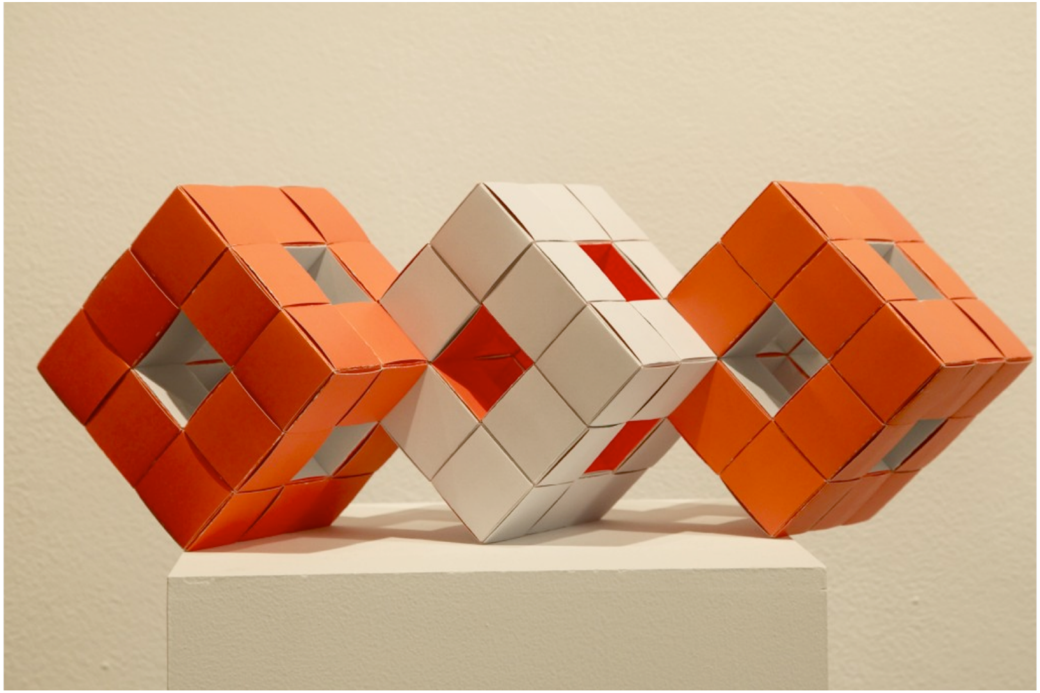
Karl Menger 1902-1985 and his sponge 1926



Menger sponge via business card origami



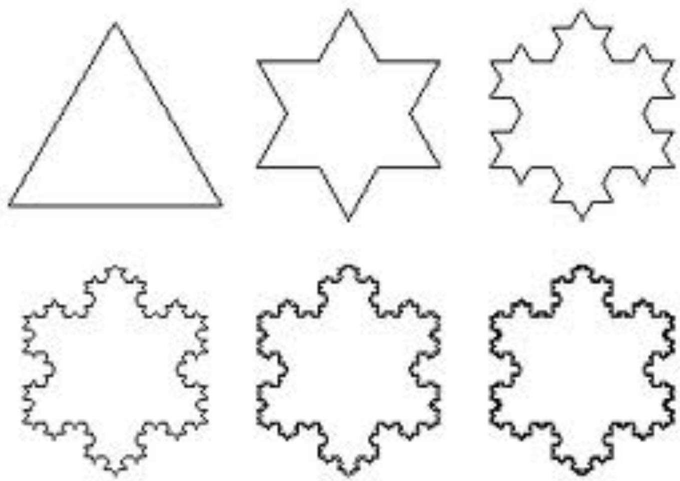




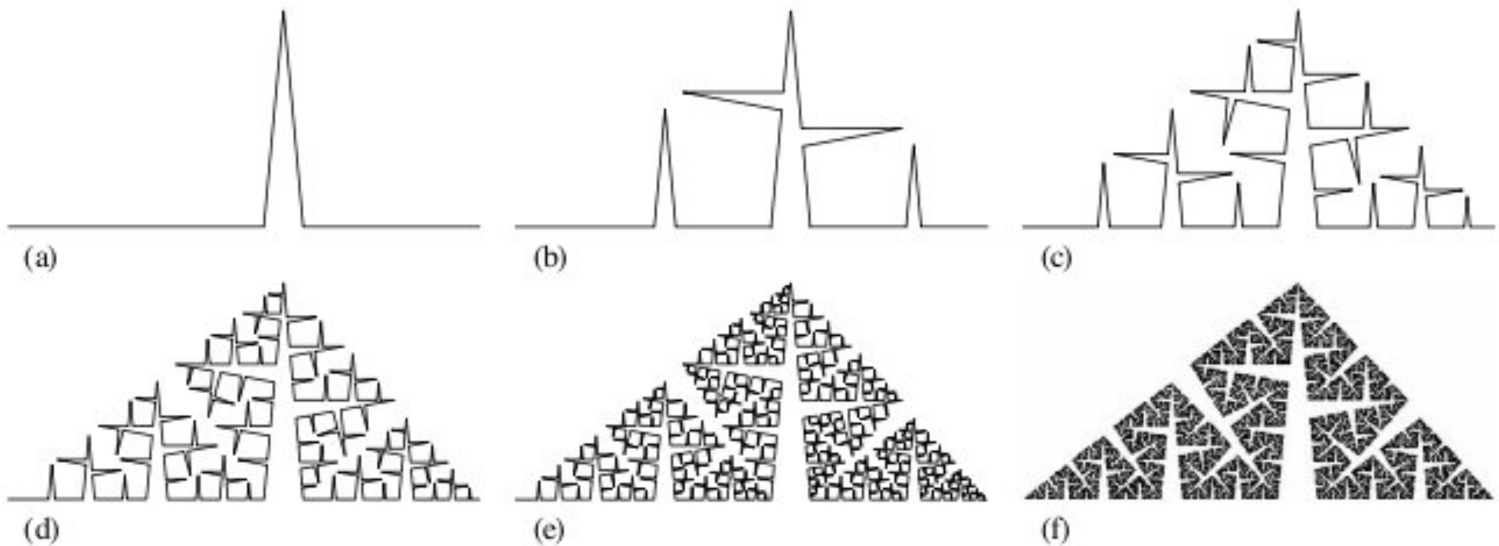
Three interlinked Level One Menger Sponges, by Margaret Wertheim.

Jeannine Mosely
66048 business cards

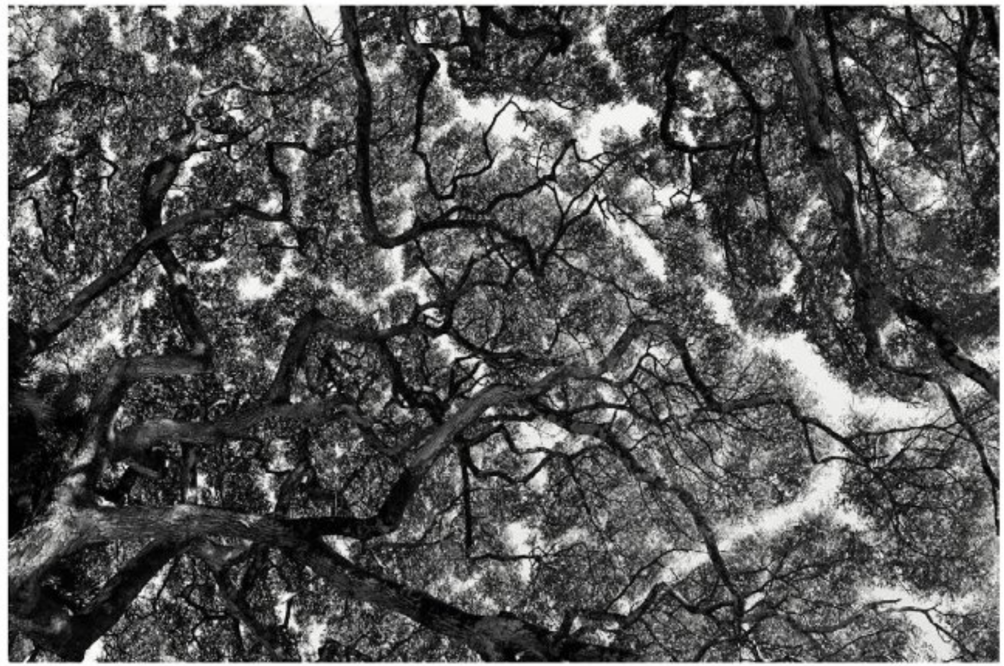
Niels Fabian Helge von Koch (1870-1924) and his snowflake (1904)



Evolution à la Mandelbrot



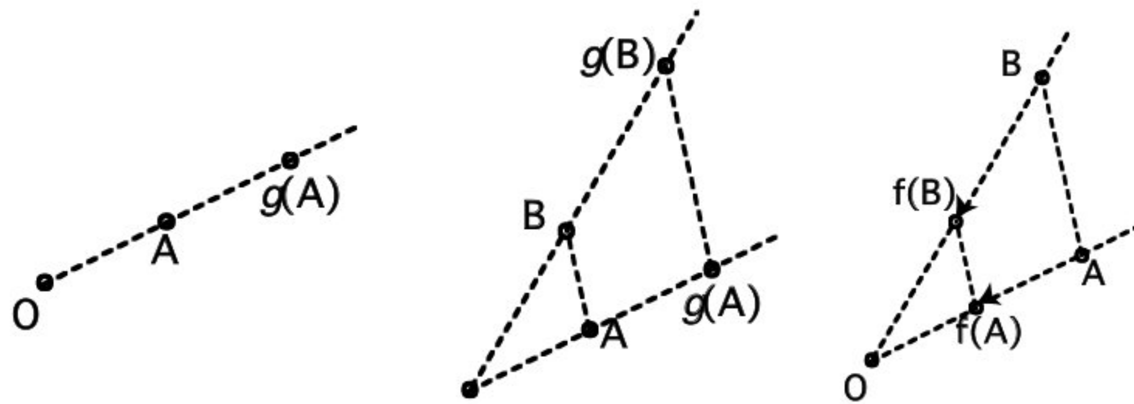
Canopy, by Craig Harris 2008



Similarity mapping

A plane transformation f is a *similarity* if there exists a positive number k such that for every point A and B, d(f(A),f(B))=kd(A,B). The number k Is a *stretching factor* of the similarity. Case k=1 gives a symmetry.

Similarities here are *central similarities* or *dilations*



Classification of similarities in the plane

Spiral symmetry: rotation composed with a central similarity (w.r.t same point)

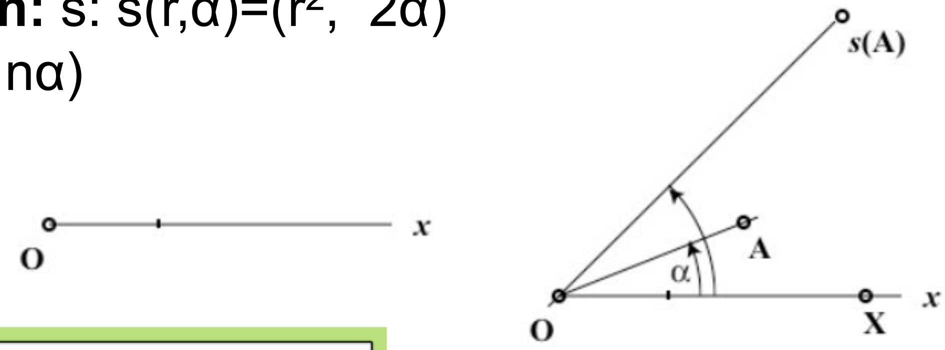
Dilative reflection: central similarity w.r.t. point O composed with a reflection w.r.t. a line going *through* O.

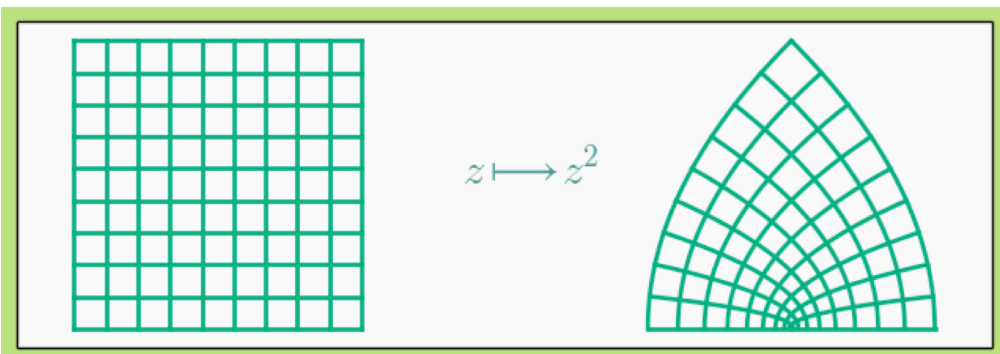
Can show: *Every similarity is a symmetry, a spiral similarity or a dilative reflection.*

Gaston Maurice Julia 1893-1978

Iteration of planar rational functions

Squaring transformation: s: $s(r,\alpha)=(r^2, 2\alpha)$
Power n: pn: $pn(r,\alpha)=(r^n, n\alpha)$

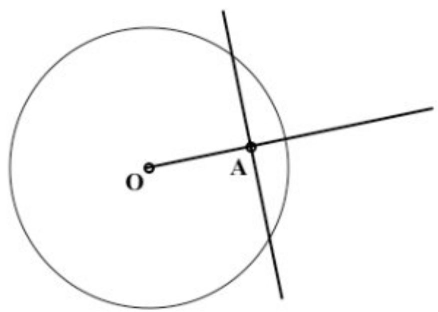


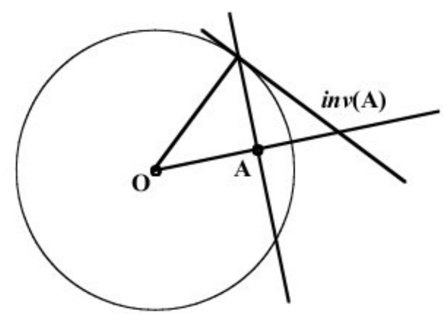


Preserves angles outside the origin !

… and (geometric) inversion in a circle

Planar rational maps are compositions of similarities, powers and inversions.





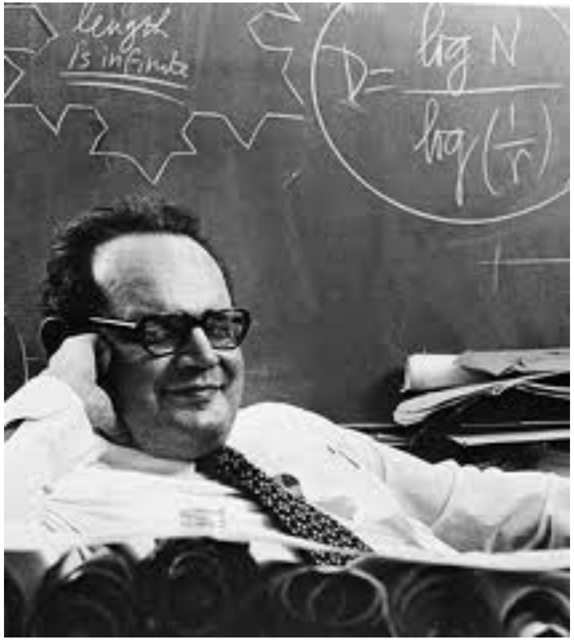
Pierre Joseph Louis Fatou 1878-1929

- ‘Fatou set’
- Holomorphic dynamics

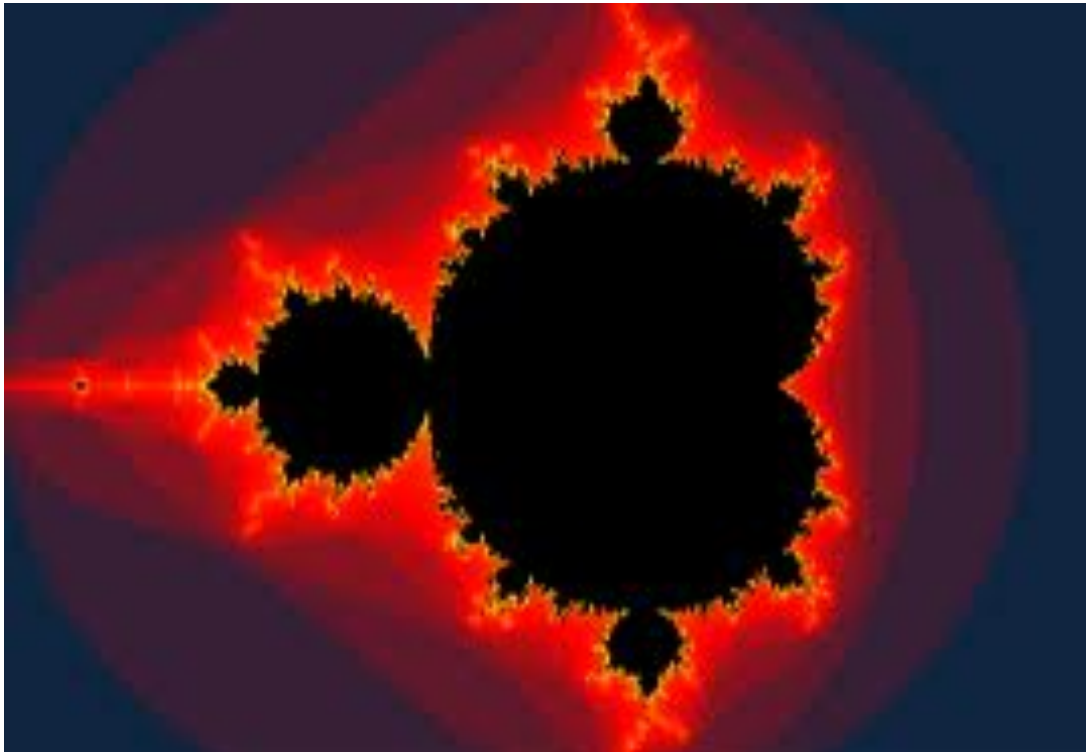
Benoit Mandelbrot 1924-2010

Mandelbrot coined (70's) the word 'fractal' to explain self similar objects

Fractus= fractured, broken



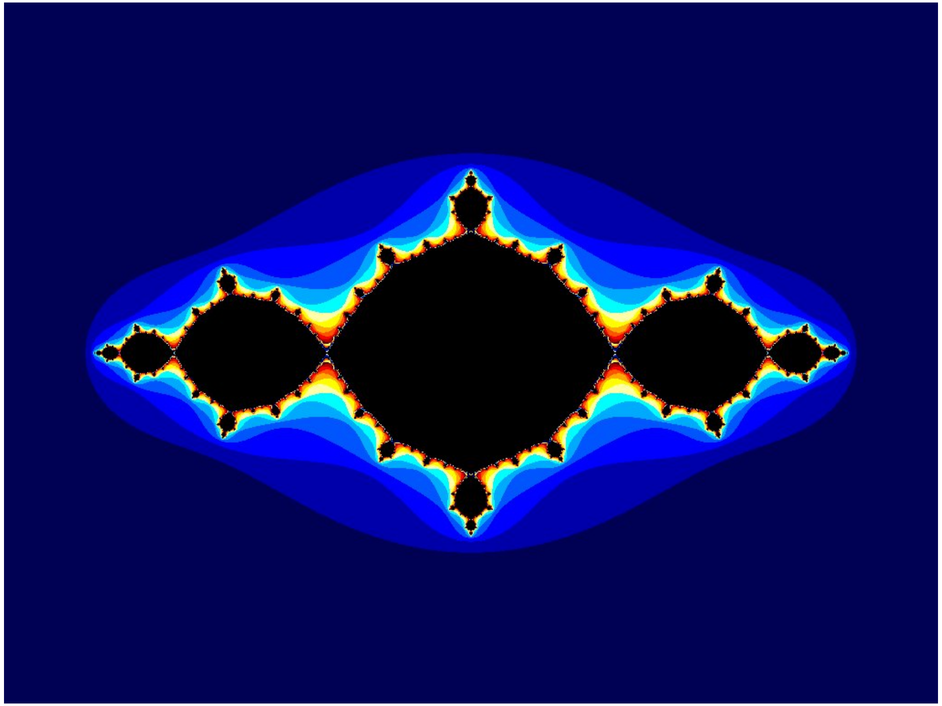
Mandelbrot set



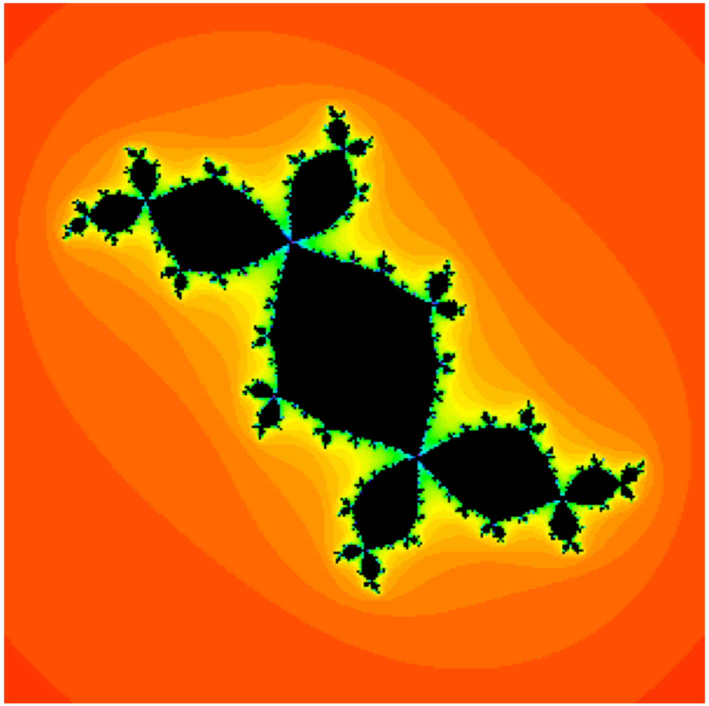
Parameter space for $C=(C_x,C_y)$ under f: $f(r,\alpha)=(r^2,2\alpha)+C$

Look at C=0 once more!

C=-1, Julia/Fatou set

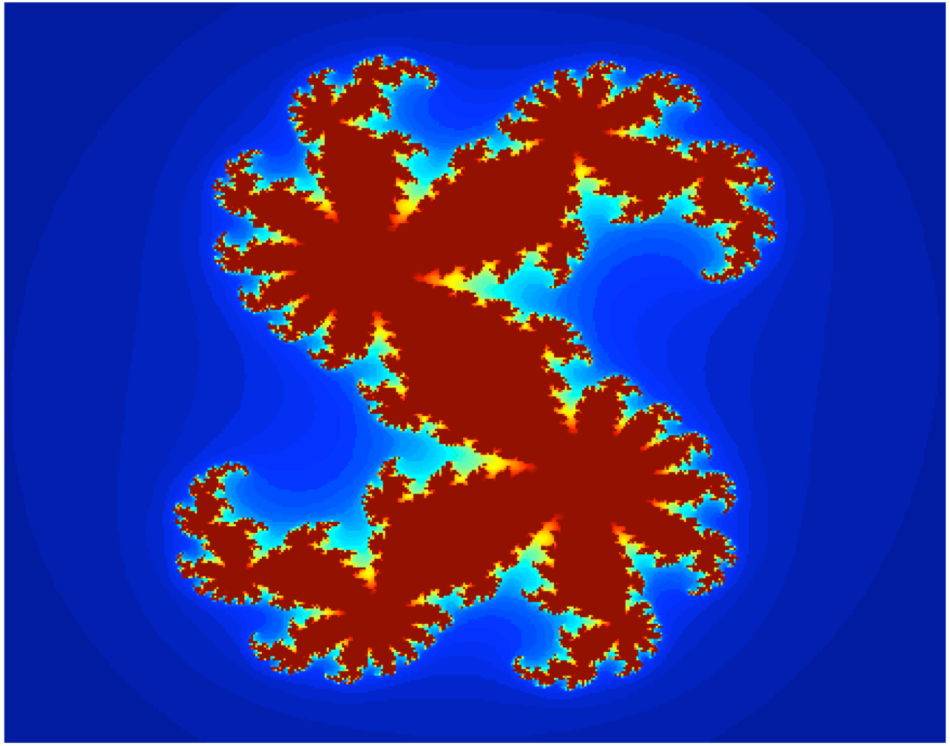


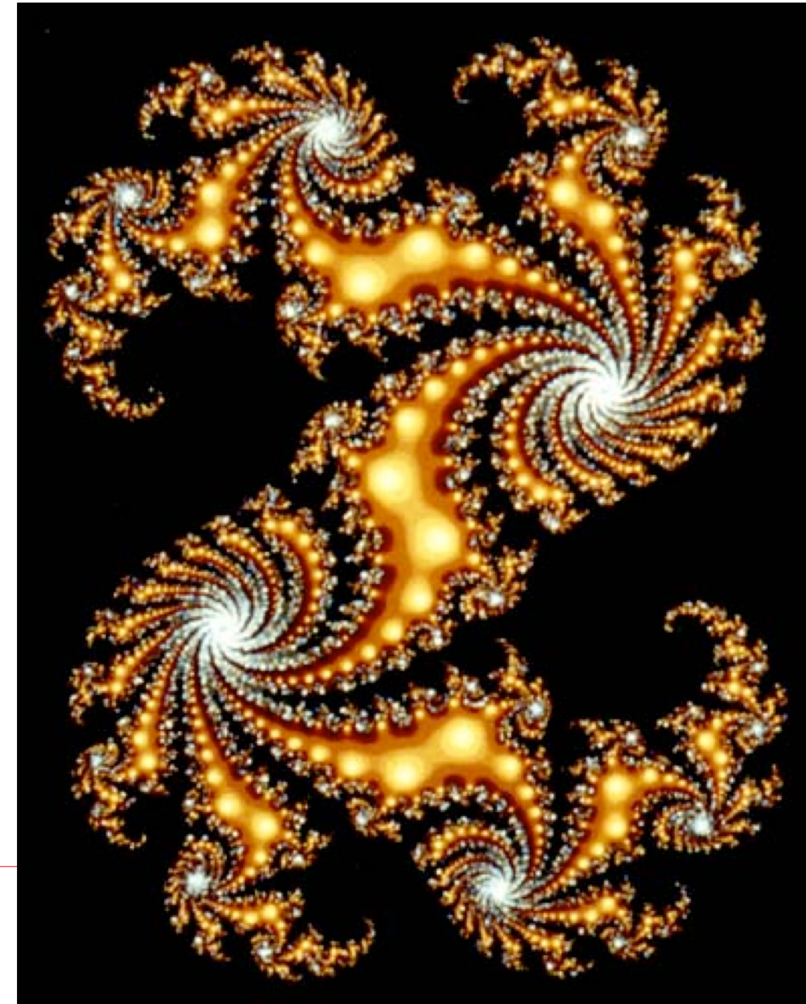
Douady rabbit (Adrien Douady 1935-2006)



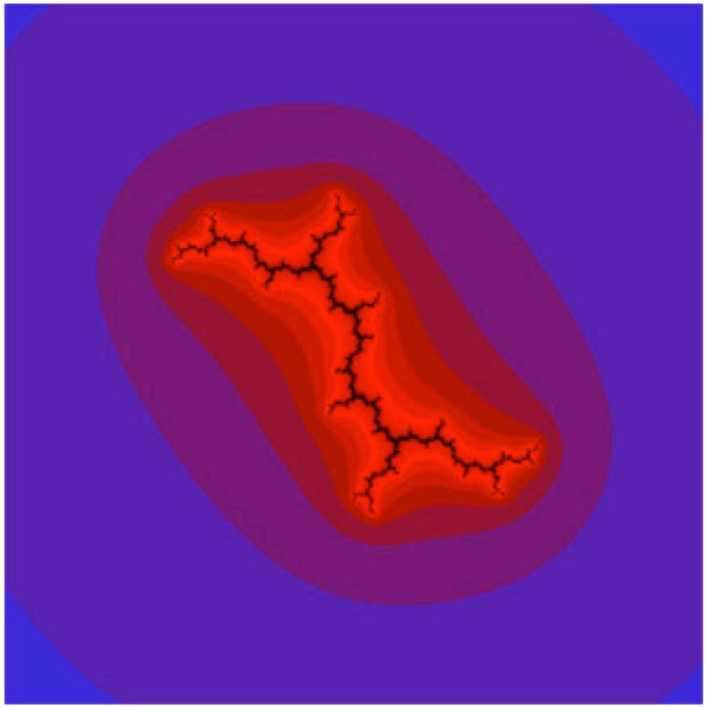
$C = -0.12256+0.74486i$.

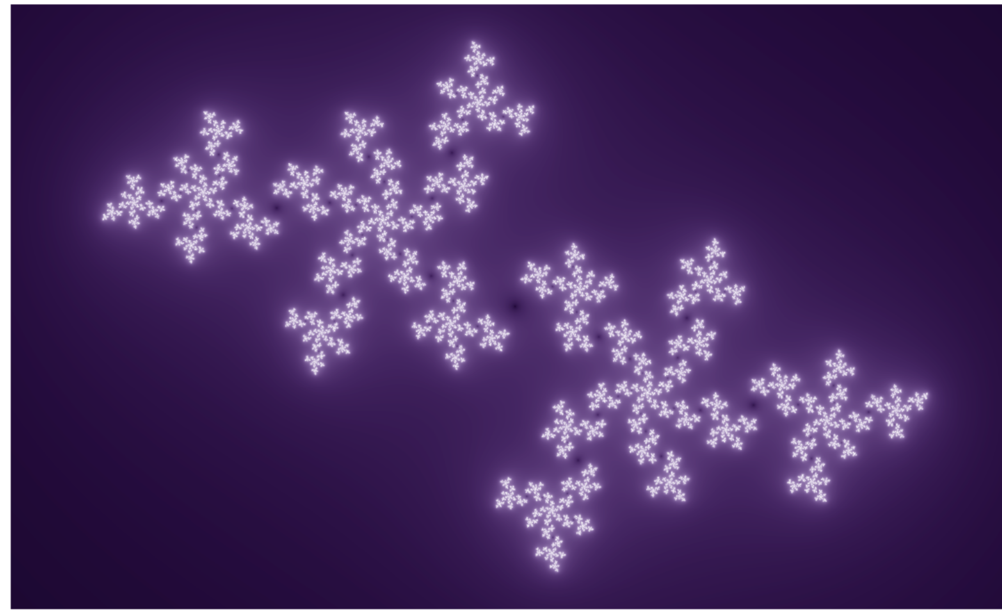
Dragon c=0.360284+0.100376i



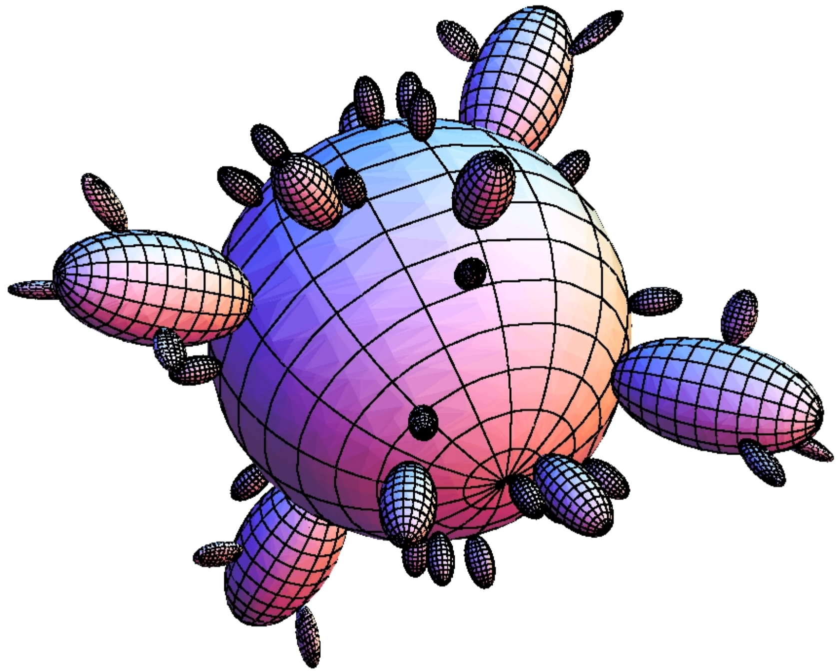


Dendrite and Cantor dust

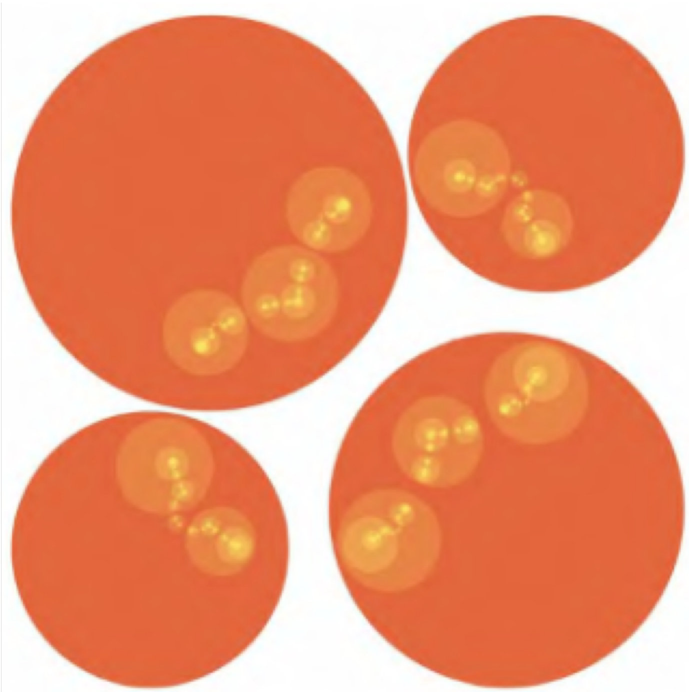




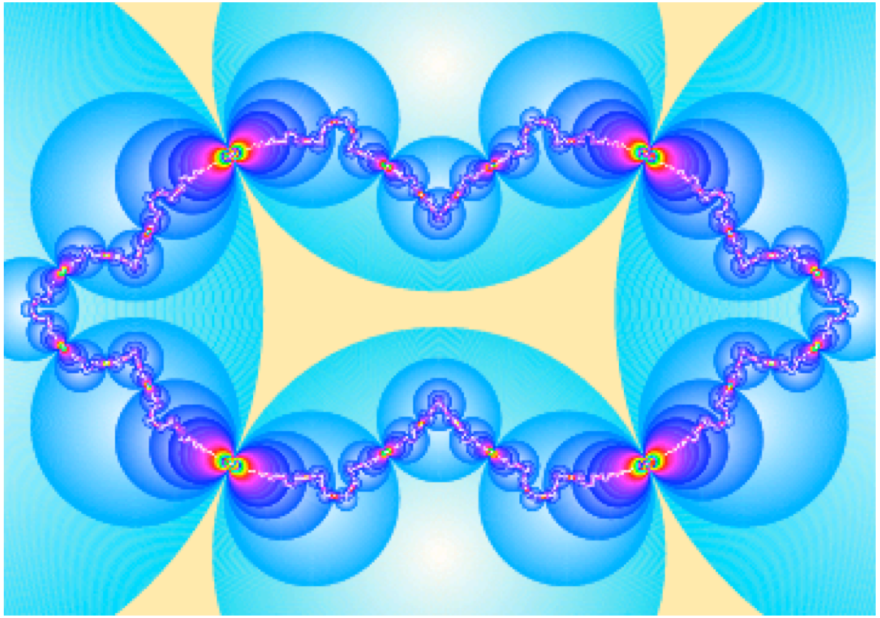
Higher dimensional analogues of complex polynomials (joint work in progress with G. Martin)



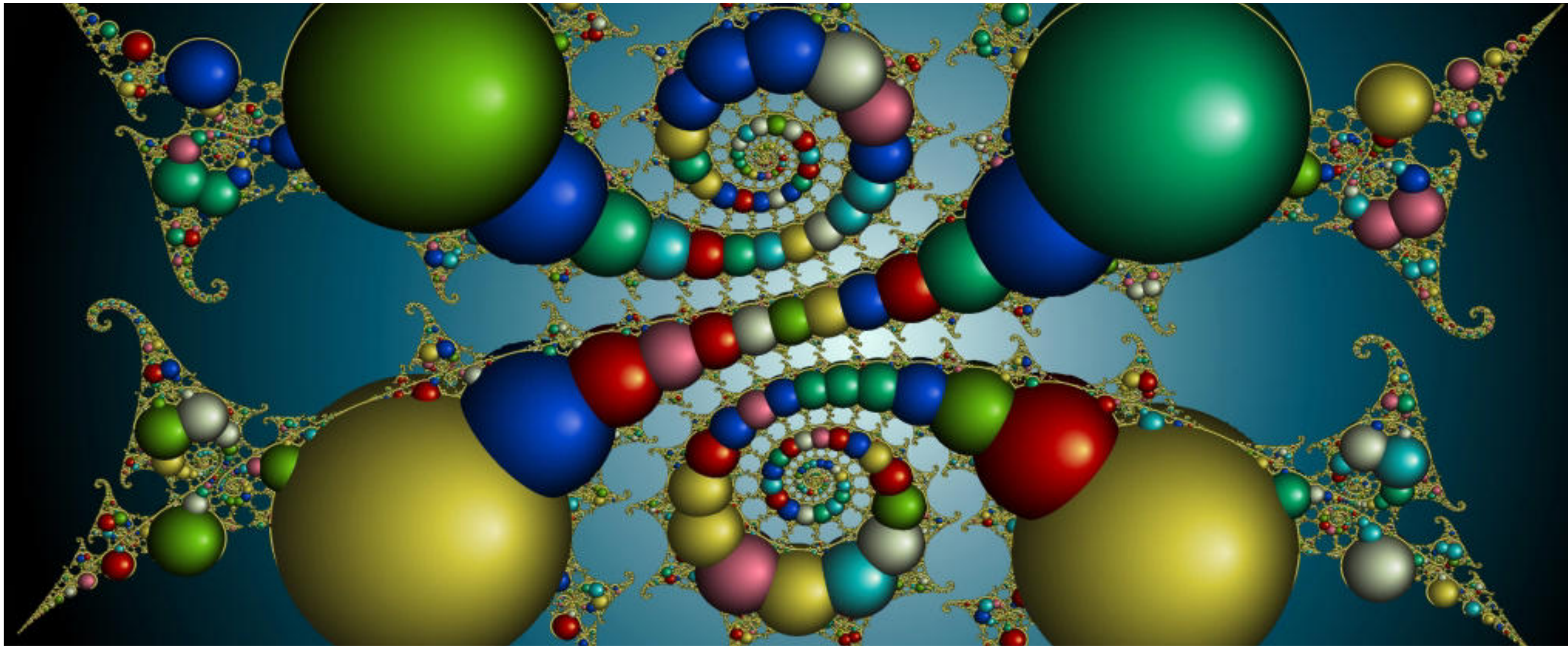
Kleinian groups



Ex: pairing of circles under Möbius transformations

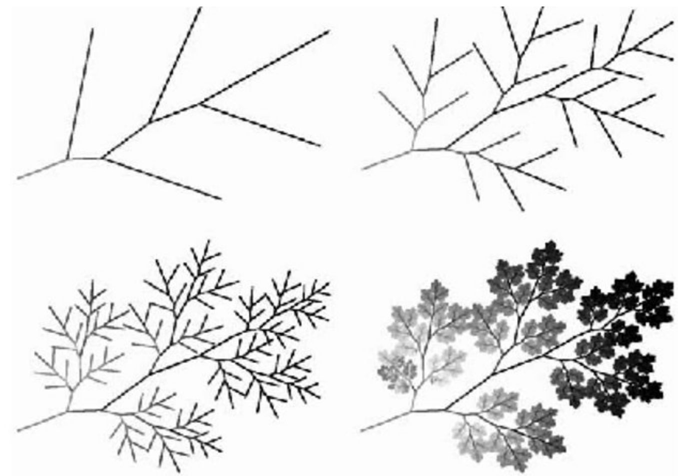


An artistic interpretation by Jos Leys



Fractals in approximating natural forms

Change from mechanical/geometrical to organic by using mathematical algorithm

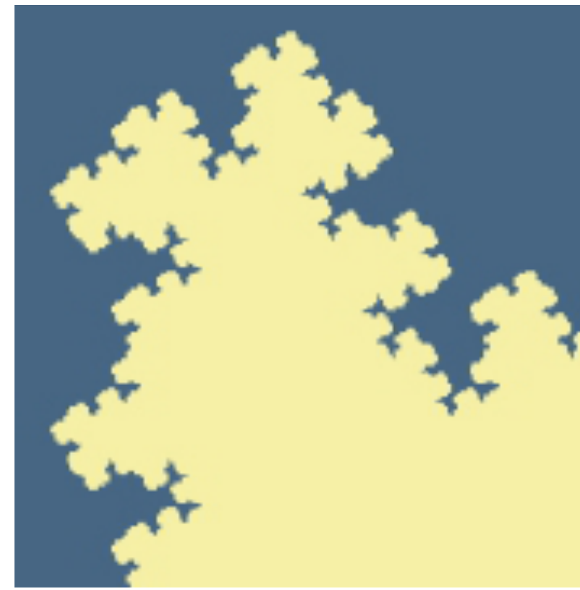


Aristid Lindenmayer 1925-1989 (L-systems) in plant biology

Artistic inventions of fractals a bit earlier and its reproduction by a process called Iterated Function System IFS.

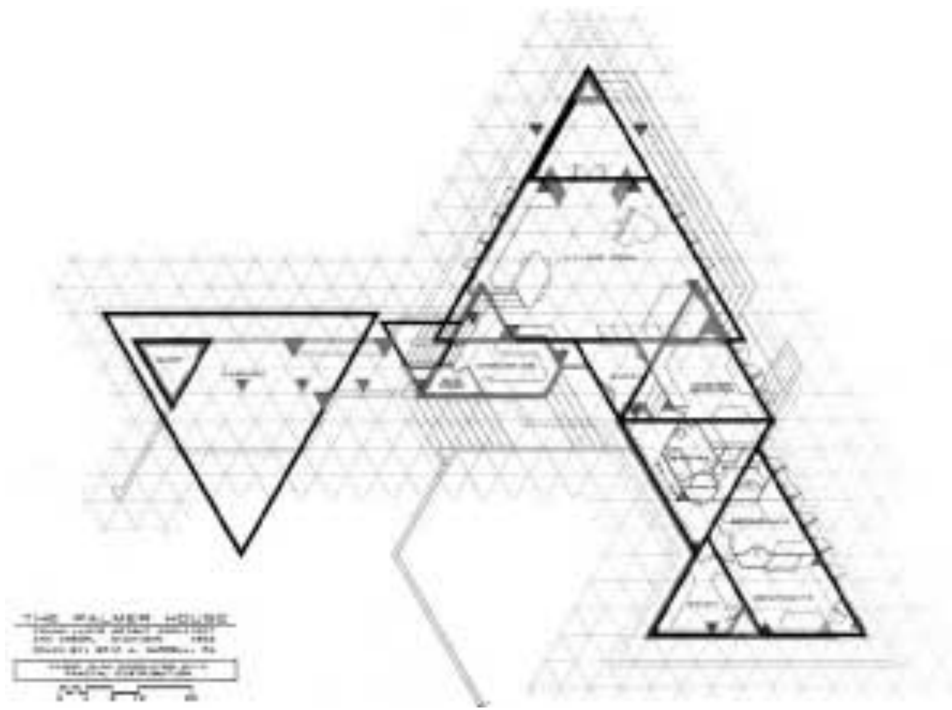
‘Driving Rain’ by Ando Hiroshige (1797-1858)

‘A Thousand Pictures of the Sea’ by Katsushika Hokusai (1817-1859) and IFS again

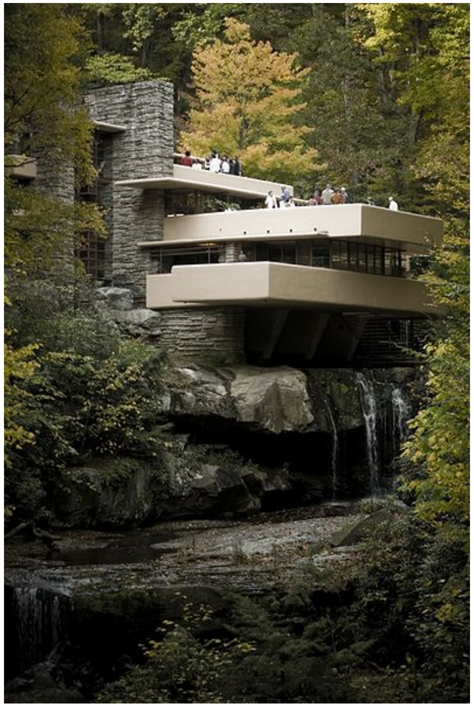


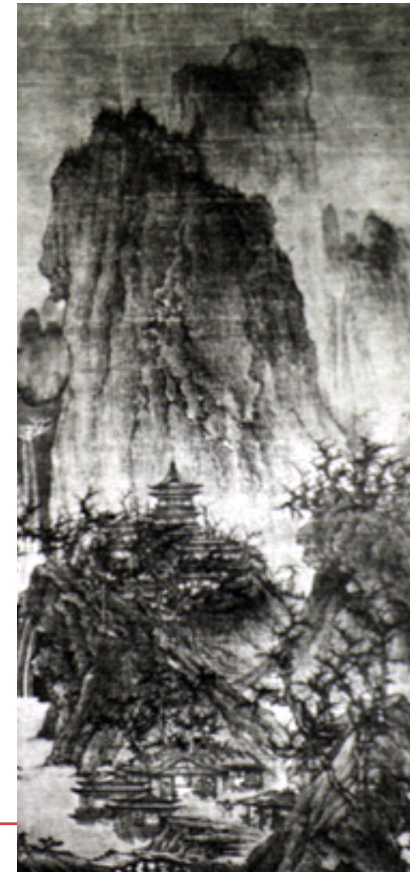
Frank Lloyd Wright (1867-1959)

Palmer house in Michigan (1950-51)



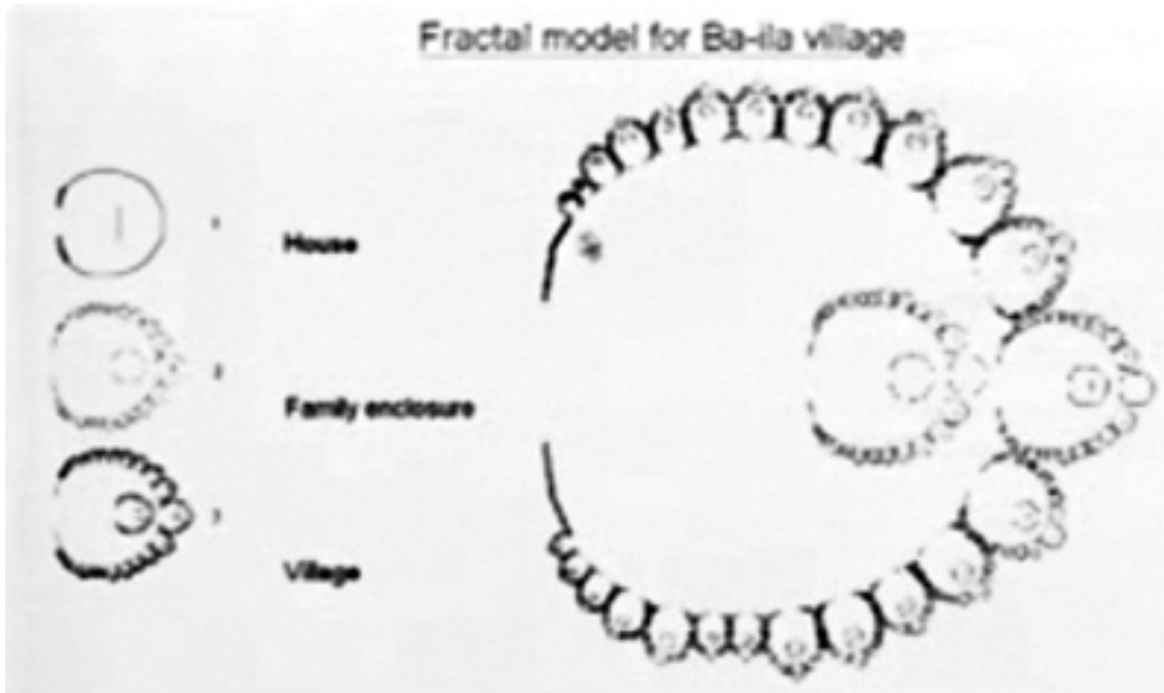
Fallingwater, Pennsylvania (1937) and Li Cheng (960-1127): Solitary Temple



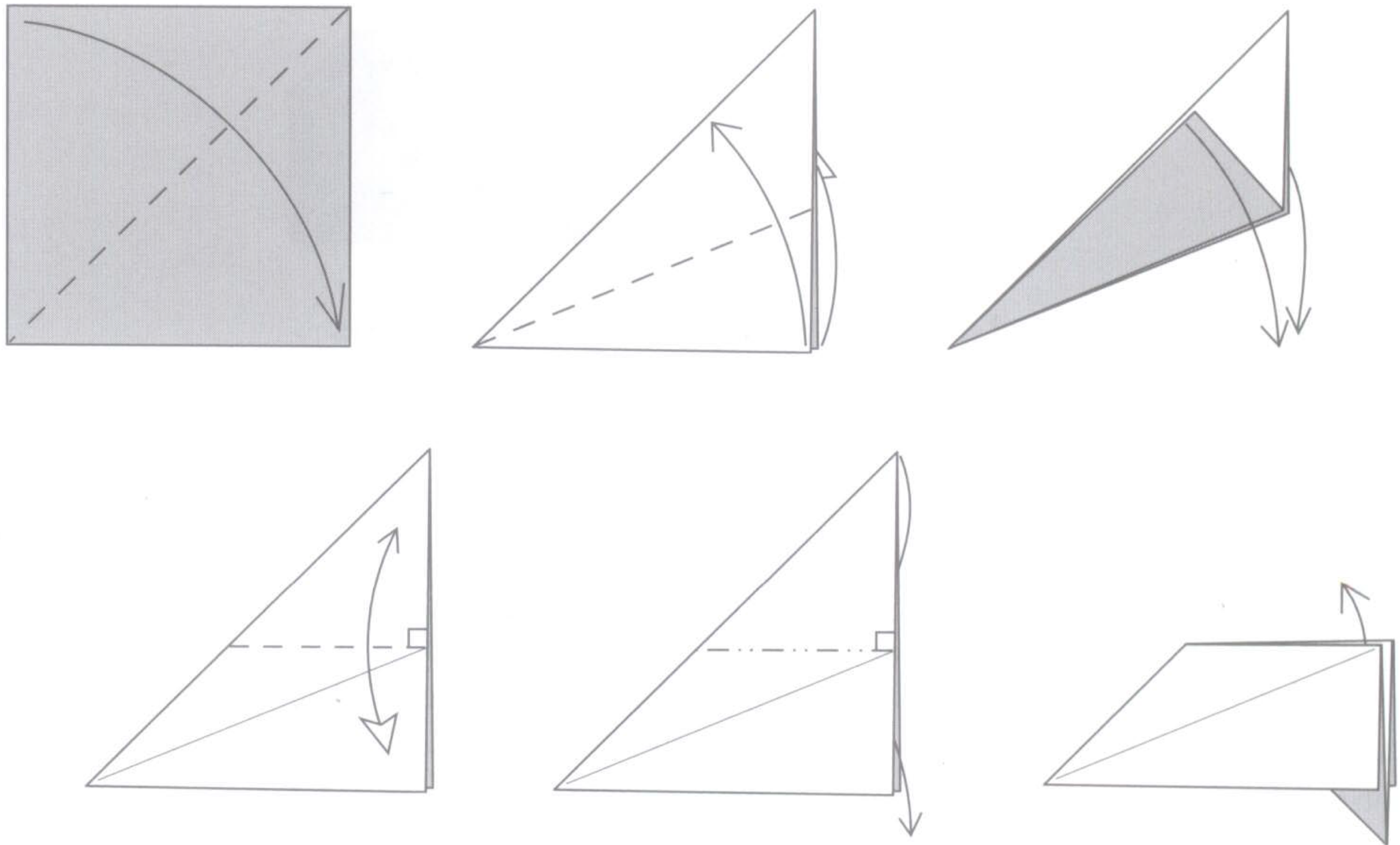


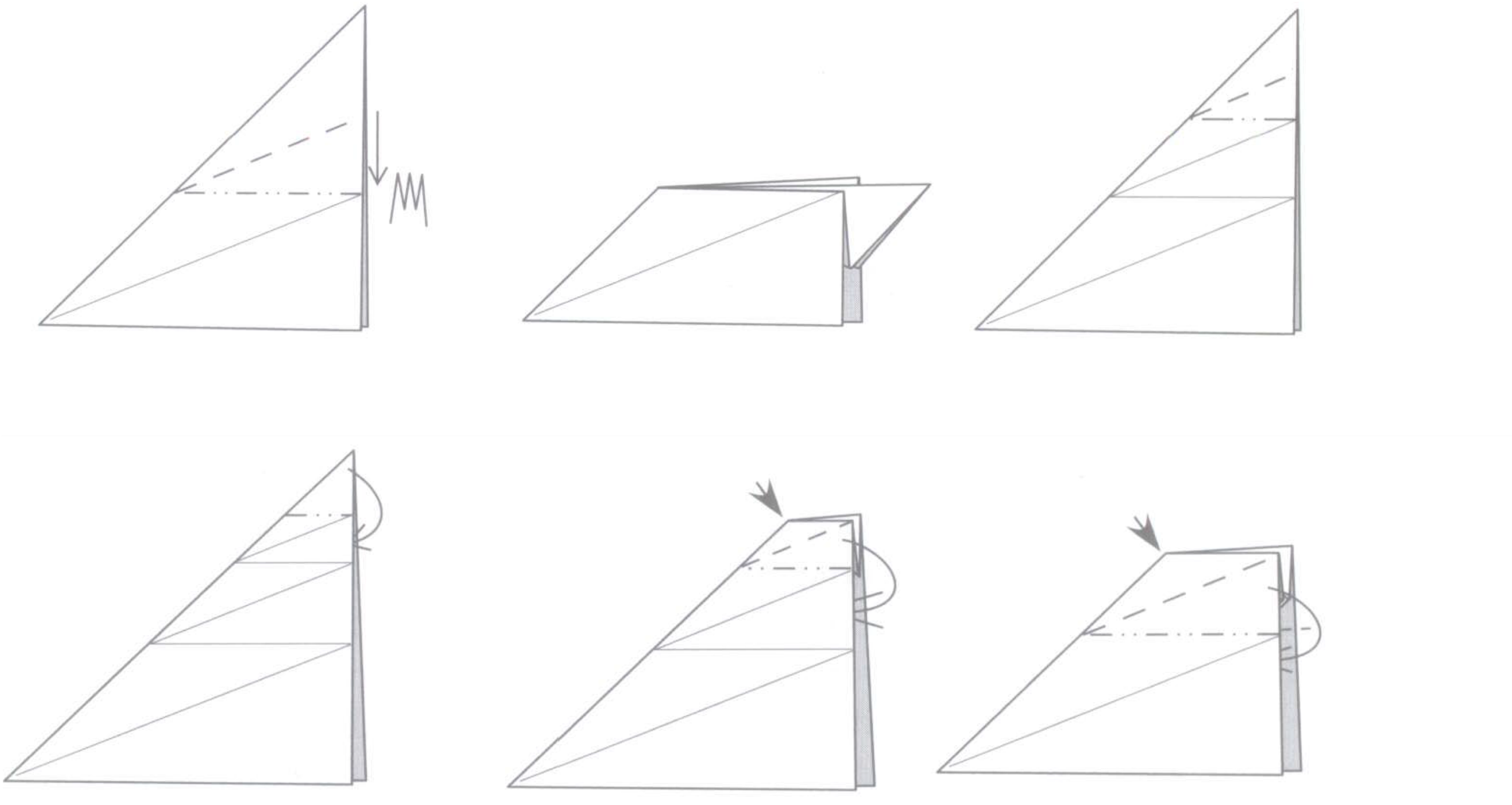
African fractals: Ron Eglash

http//www.ted.com/talks/ron_eglash_on_african_fractals.html

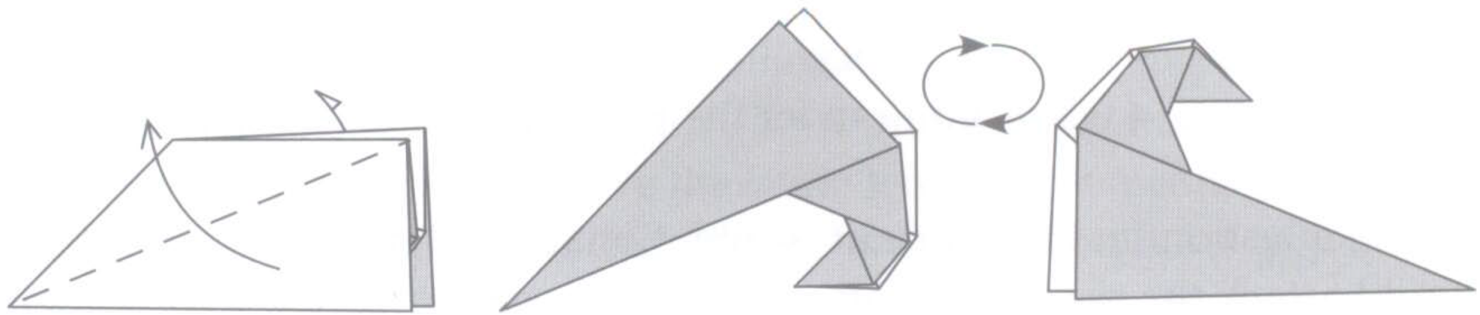


The self similar wave by Tom Hull



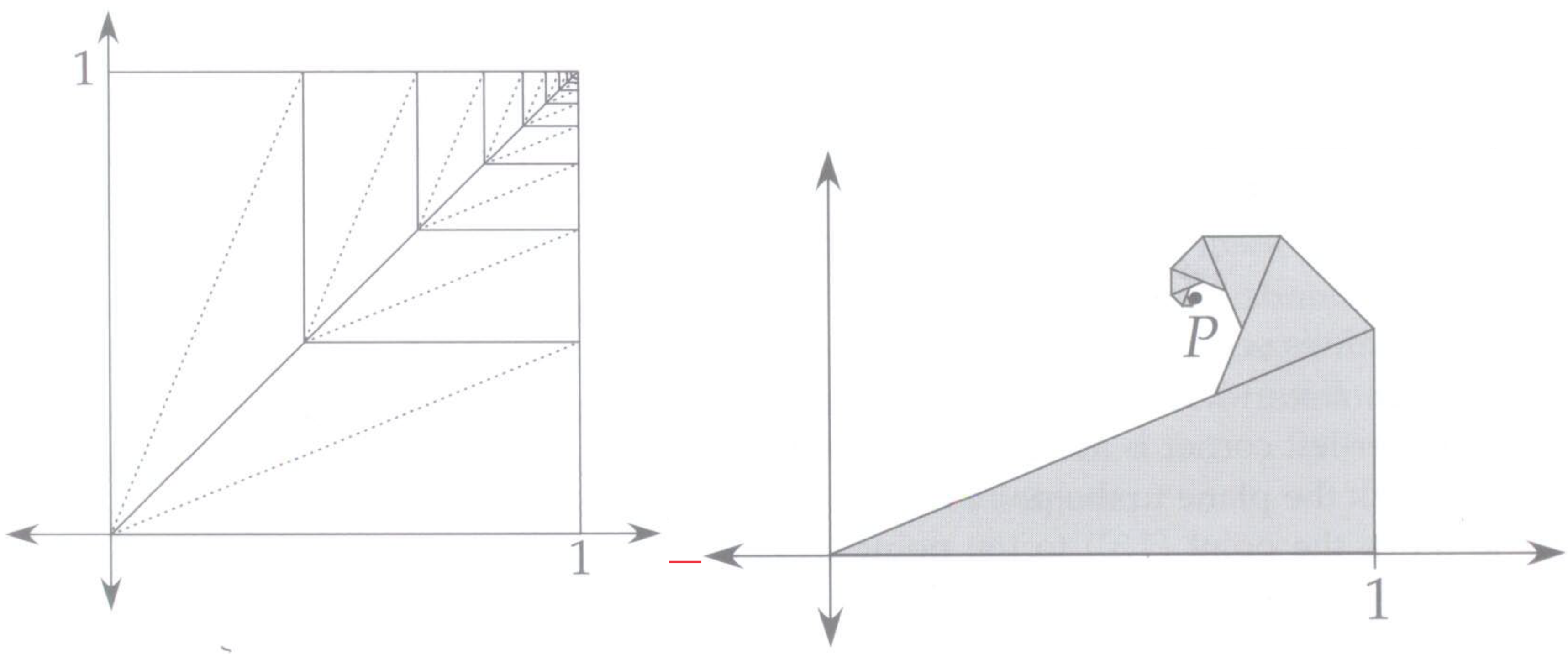


Is the outcome a fractal ?

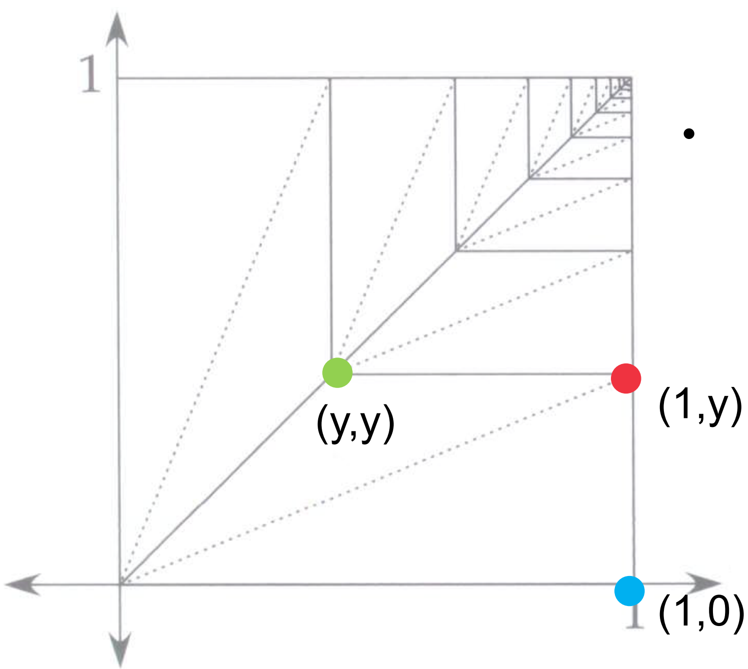


Question 1

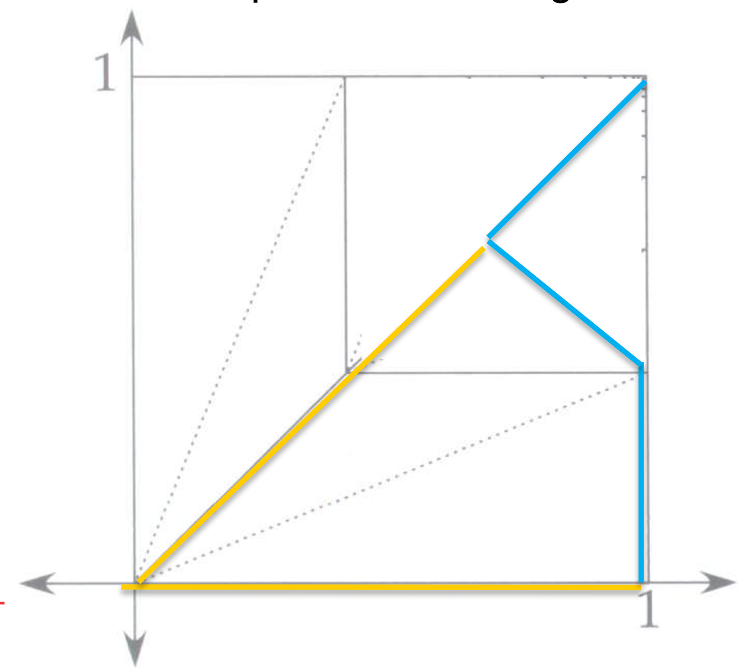
Suppose we started with a square paper with side length 1 and folded the wave with an infinite number of levels, what would the coordinates of the limit point P of the spiral be ?



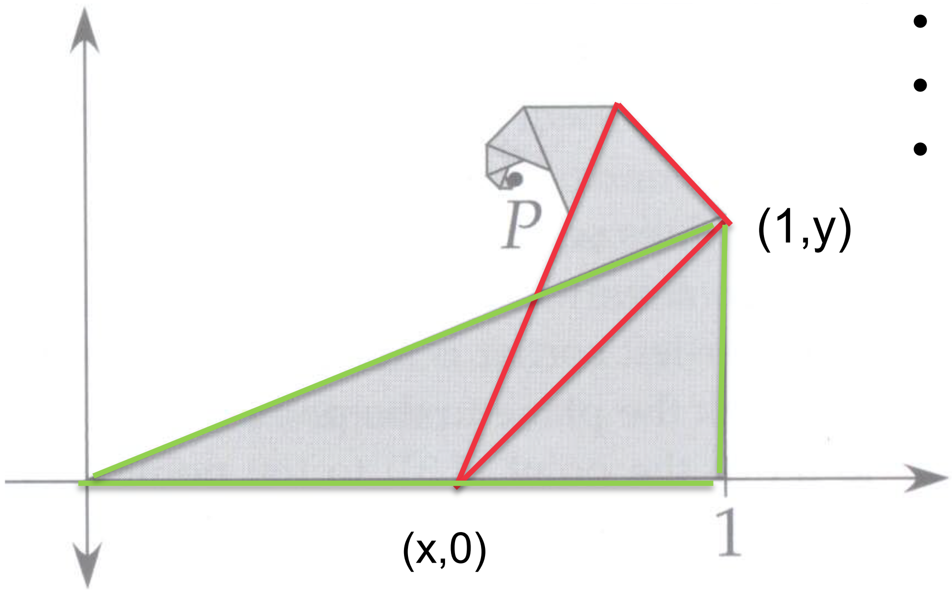
Geometric transformation solution



- Find affine mapping (scaling+ translation) taking (1,0) to (1,y) (and hence unit square to a square of side length y
- y = √2 -1 from the picture=> scaling factor =1-y =2- √2



Self similarity in the folded wave pattern

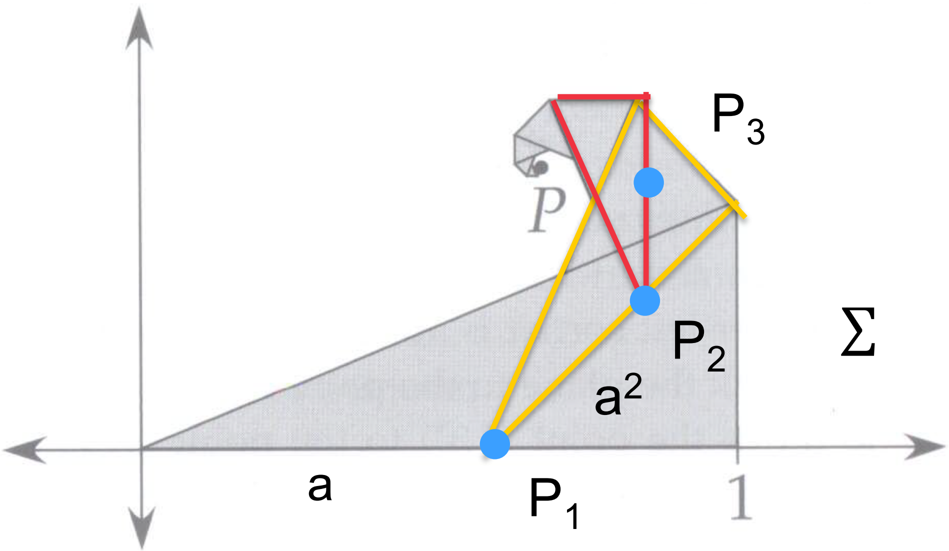


- (1,0) mapped (1,y) again
- (0,0) mapped to (x,0) x=1-y =2- √2
- P will become a fixed point of the mapping

⇒ Rotation of 45 ° counter clockwise wrt to the origin + scaling by factor 2- √2 + translation from origin to point (x,0)

⇒ P=(2/3, √2 /3)

Solution through complex (= nature's) numbers



$P_0 = 0,\ P_1 = a = 2 - \sqrt{2}$

$P_2 = P_1 + a^2 e^{i\theta},\ \theta = \frac{\pi}{4}$

$P_3 = P_2 + a^3 e^{i2\theta}$

…

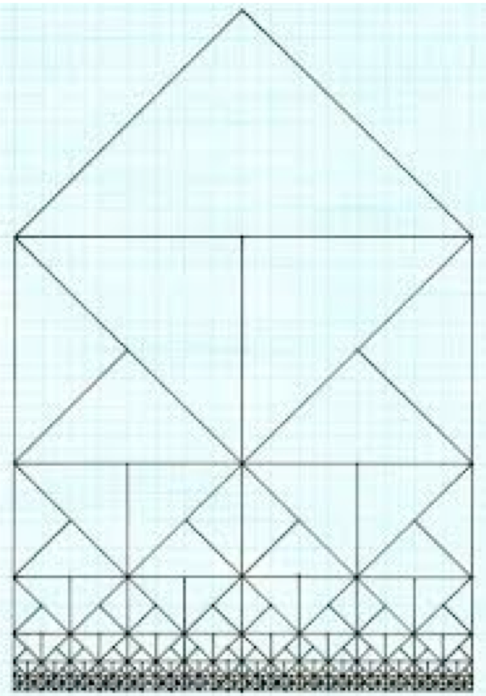
$P_n = P_{n-1} + a^n e^{i(n-1)\theta}$

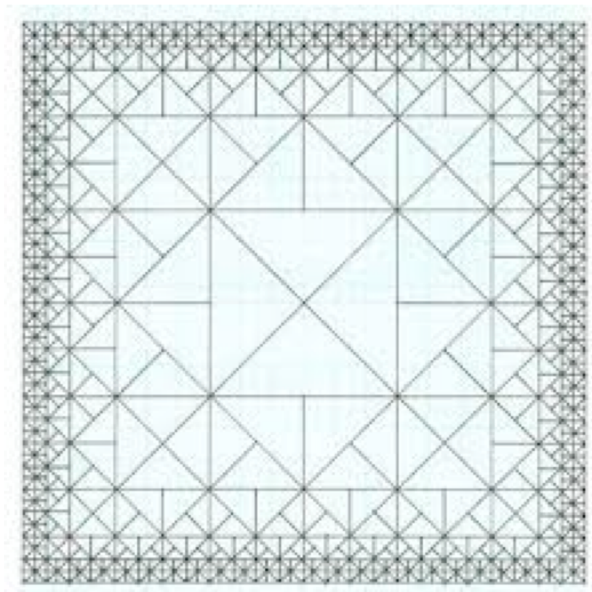
$\Rightarrow P = a\sum(ae^{i\theta})^n = 2/3 + \sqrt{2}/3\ i$

The spiral is logarithmic: $|P - P_n| = \sqrt{6}/3\ (2-\sqrt{2})^{n+1}$

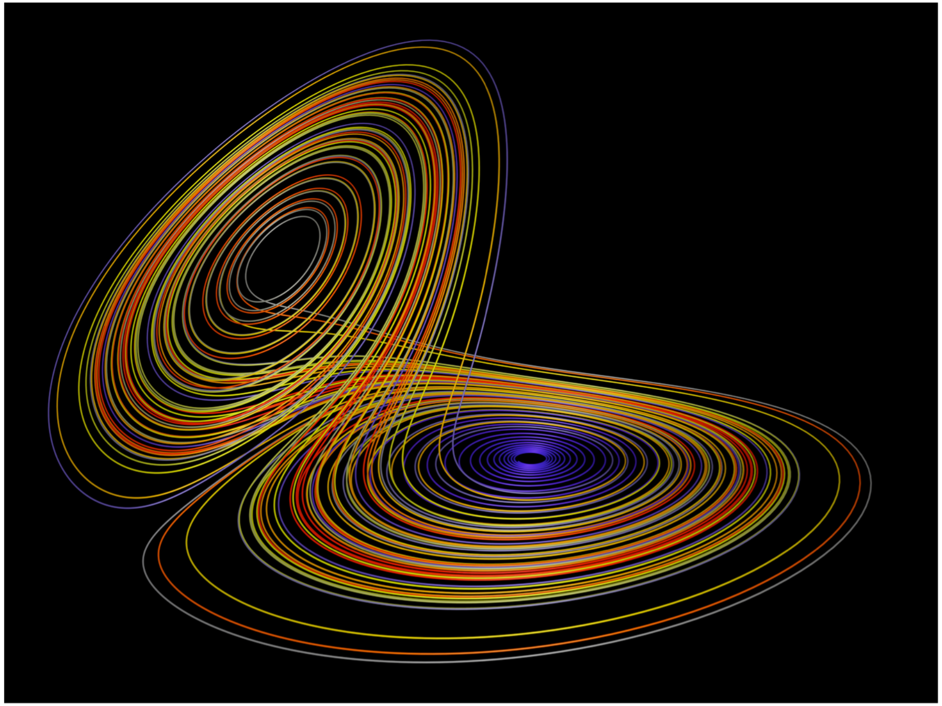
Some other self similar origami patterns

Sometimes called ‘fractal origami’ but are they ***fractals*** really ?





Why is Lorenz attractor called ‘fractal’ ?



Crocheting the Lorenz manifold

Hinke M. Osinga & Bernd Krauskopf
The Mathematical Intelligencer 26(4) (2004) 25-37