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Differential and Integral Calculus 1

Exercises, Week 1

Work on Warm-up 1–4 during the exercise sessions of Week 1.
Submit on MyCourses your solutions for Homework 1 and 2 by Sunday, September 11th.

Warm-up 1: For each point, give an example or explain why an example does not exist:

1. A sequence $a\colon \mathbb{N} \to \mathbb{R}$ that is strictly increasing and bounded. ("Strictly increasing" means that for all $n < m$ in $\mathbb{N}$ we have $a_n < a_m$. "Bounded" means that there exists some $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ we have $-N \le a_n \le N$.)
2. A function $f\colon \mathbb{R} \to \mathbb{R}$ that is strictly increasing and not injective. ("Strictly increasing" means that for all $x < y$ in $\mathbb{R}$ we have $f(x) < f(y)$. "Not injective" means that there exist some distinct a and b for which $f(a) = f(b)$.)
3. A function $f\colon \mathbb{R} \to \mathbb{R}$ that is strictly increasing, not bounded from below nor above, and not surjective. ("Not bounded from below nor above" means that for any $M > 0$ there is some $x \in \mathbb{R}$ with $f(x) > M$ and there is some $y \in \mathbb{R}$ with $f(y) < -M$. "Not surjective" means that there exist some $c \in \mathbb{R}$ for which no $x \in \mathbb{R}$ satisfies $f(x) = c$.)

Warm-up 2: Draw by hand, without the help of a computer, the following functions:

$$f(x) = |2x-1| - 1, \qquad g(x) = e^{|x|-1}, \qquad h(x) = x^2 \sin(\pi x).$$

Warm-up 3: Show explicitly, going through the limit definition, that

$$(a) \quad \lim_{x\to 2} \frac{2x^2+2x-12}{x-2} = 10 \qquad\qquad (b) \quad \lim_{n\to+\infty} \frac{n-1}{n} = 1.$$

Using the definition of a limit, explain why the following limits do not exist:

$$(c) \quad \lim_{x\to 0} \sin(1/x) \qquad\qquad (d) \quad \lim_{n\to\infty} (-1)^n.$$

Warm-up 4: Compute the following limits:

$$(a) \quad \lim_{x\to 1} \frac{x^2-x}{x-1} \qquad\qquad (c) \quad \lim_{n\to+\infty} \sqrt{n+1} - \sqrt{n}$$

$$(b) \quad \lim_{x\to 0} \frac{e^{x^2}-1}{\sin(2x^2)} \qquad\qquad (d) \quad \sum_{n=0}^{+\infty} \frac{1}{3^{2n}}.$$

Submit on MyCourses your solutions **only** for the following two problems. Explain the reasoning behind your solutions, do not just return the final result. If you make use of important results from the lectures, state what they are (for instance a theorem or a famous limit).

Homework 1: Find all real values of x satisfying both inequalities in the system

$$\begin{cases} |x+1| < |x-3| \\ |x-3| \geq 2|x|. \end{cases}$$

Hint: Draw. If you see x and a as points on the line, what is the number $|x-a|$? [1 point]

Homework 2: For each of the following limits, either compute it if it exists (without verifying the correctness through the definition, but explaining the steps in your calculation), or explain (by using the definition) why it does not exist:

(a) $\displaystyle\lim_{x\to 0} \frac{e^{2x}-1}{\ln(x+1)}$

(b) $\displaystyle\lim_{x\to 0} \frac{\sin(\ln(1+x^2))}{x}$

(c) $\displaystyle\lim_{x\to +\infty} \left(1+\frac{3}{2x}\right)^x$

(d) $\displaystyle\lim_{n\to +\infty} \cos(\pi n)$

(e) $\displaystyle\lim_{n\to +\infty} \frac{n^5+4n^4+e^n}{n^3+2n^2-e^n}$

(f) $\displaystyle\sum_{n=2}^{+\infty} \frac{1}{3^n}$

[3 points]