MS-E1050 Graph Theory

Ragnar Freij-Hollanti

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Teachers

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Schedule

- **Lectures:**
  Thursdays 16-18, M1 **and** Fridays 10-12, Y405.
- **Exercise sessions:**
  Mondays 10-12, Y307 **or** Zoom.

Grading

- Five homework sheets, due Mondays 19.9., 26.9., 3.10., 10.10., and Wednesday 19.10.
- Returned in the Assignments folder on MyCourses.
- Graded by two of your peers (randomly selected). Grades are due one week after the assignment deadline.
- Each homework sheet gives a maximum of $5 \cdot 2$ for exercises $+ 2 \cdot 5$ for problems $+ 5$ for grading $= 25$ points.
- The four best homeworks count towards the final grade.

Literature

- **Reinhard Diestel:** *Graph Theory*.
- **Matthias Beck and Rayman Sanyal:** *Combinatorial Reciprocity Theorems*.
- **Slides** Updated on course homepage after every lecture.

Course content

You will learn *about*:

- the twelve topics mentioned in the left hand menu.
- combinatorial, geometrical, algorithmic, probabilistic, and algebraic aspects of graph theory.

You will learn *to*:

- Solve combinatorial problems of different kinds.
- Relate different mathematical topics to each other.

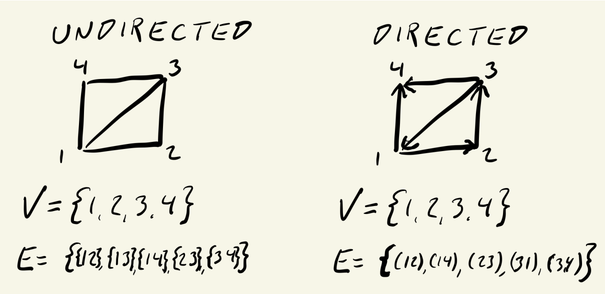
Discuss in small groups (10-15 minutes):

- What is your name?
- What is your quest?
- What is your favourite colour?
- Select a chairman! Preferably one who can share their screen and draw on it.
- How would you define what a graph is?

Definitions

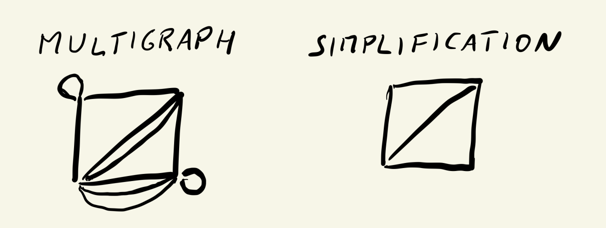
- A graph is a pair $G = (V, E)$.
- V is a set of *vertices* or *nodes*
    - $E \subseteq \{\{x, y\} : x, y \in V\}$ is a set of *edges* (undirected graph).

    or
    - $E \subseteq \{(x, y) : x, y \in V\}$ is a set of *directed edges* (digraph).
        - y is the *head* and x is the *tail* of the directed edge (x, y).
- $|G| = |V| = n$ and $\|G\| = |E| = m$



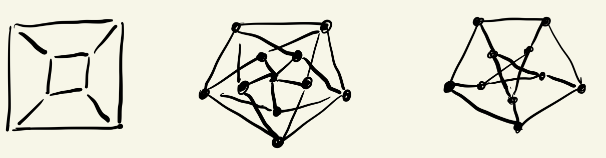
Definitions

- If $G = (V, E)$, then we also write (abusing notation) $V = V(G)$ and $E = E(G)$.
- If we allow $E(G)$ to be a *multiset* (i.e. repeated elements allowed), then G is a *multigraph*.
- A *loop* is an edge $\{x, x\}$ (or a directed edge (x, x)).
- If G is not a multigraph, and $x \neq y$ for all edges $\{x, y\} \in E(G)$, then G is a *simple* graph.



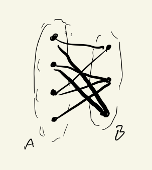
Definitions

- G is *finite* if $V(G)$ and $E(G)$ are finite sets (or multisets).
- In this course, unless explicitly mentioned, all graphs are simple and finite and undirected.



- G is *bipartite* if $V(G) = A \cup B$ where $A \cap B = \emptyset$ and

$$E(G) \subseteq \{xy : x \in A, y \in B\}.$$



Discuss in small groups (10-15 minutes):

- What are some use cases (examples from science or real life) of bipartite graphs?
- Does any (or all) of you know what a *cycle* in a graph is? Explain to the others!
- What can you say about the cycles in a bipartite graph?
- For a graph without an explicit bipartition of its vertices, can you think of an *efficient* way to see if it is bipartite or not?

Bipartite graphs

Complete graphs

Example

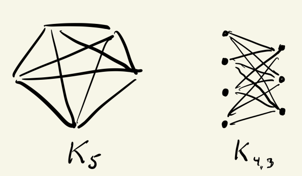
- The *complete graph* $K_n = (V, E)$ where

$$|V| = n \text{ and } E = \binom{V}{2} = \{e \subseteq V : |e| = 2\}.$$

- The *complete bipartite graph* $K_{m,n} = (A \cup B, E)$, where

$$|A| = m, |B| = n, A \cap B = \emptyset \text{ and } E = \{\{a, b\} : a \in A, b \in B\}.$$

- The *empty graph* $\overline{K_n} = (V, \emptyset)$ where $|V| = n$.



Substructures

- A *clique* in G is a set $Q \subseteq V(G)$ of pairwise adjacent nodes (so $G[Q]$ is complete).
- An *independent* (or *stable*) set in G is a set $S \subseteq V(G)$ of pairwise non-adjacent nodes (so $G[S]$ is empty).

- The size of the largest clique in G is $\omega(G)$.
- The size of the largest independent in G is $\alpha(G) = \omega(\overline{G})$.

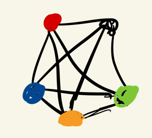
Definitions

- A (proper) *k-colouring* of $G = (V, E)$ is a map $\gamma : V \to \{1, 2, \ldots, k\}$ such that $\gamma(v) \neq \gamma(u)$ whenever $uv \in E$.

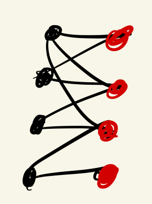
- The *chromatic number* of $G = (V, E)$ is the smallest $k \in \mathbb{N}$ such that there exists a k-colouring of G.
- In other words, $\chi(G) = k$ is the smallest number of independent sets into which $V(G)$ can be partitioned.

Examples

- $\chi(K_n) = n$.



- $\chi(G) = 2$ if and only if G is bipartite.



Discuss in small groups (10-15 minutes):

- Each colour class in a graph colouring is an independence set.
- The vertices of a clique have to all get different colours.
- Using this: Bound the chromatic number $\chi(G)$ from above in two different ways, in terms of $\alpha(G)$, $\omega(G)$, and n?
- Can you think of graphs for which these bounds are not tight?
- Do you think the bounds are tight for "most" graphs? And what does that even mean?

Chromatic numbers

Discuss in small groups (10-15 minutes):

- At any party, some pairs of people are friends, and others are not. Test your intuition. Are the following true or false?
- At a party with 5 guests, there are always either three mutual friends, or three mutual non-friends.
- What about a party with 6 guests?
- At a party with $a + b$ guests, there are always either a mutual friends or b mutual non-friends.
- At any *large enough* party, there are always either a mutual friends or b mutual non-friends.
  - How many guests $R(a, b)$ are needed, so that this holds for *all* parties?
  - How many guests are needed, so that this holds for *most* parties? And what does that even mean?

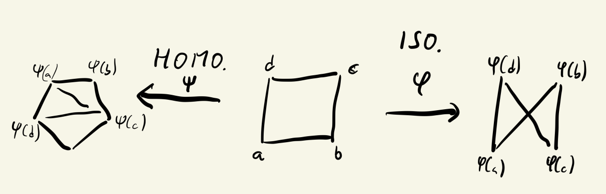
Ramsey Theory

Conclusion

- Today we have discussed some basic types of questions in graph theory.
- Some of these can be solved from first principles by clever high school students.
- Other questions require some sort of “theory”.
- Starting Thursday, we will develop combinatorial, probabilistic and algebraic tools to study graphs, and use those to solve problems.

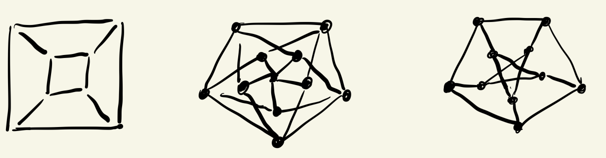
When are two graphs the same?

- A *homomorphism* $G \to G'$ is a map $\varphi : V(G) \to V(G')$ such that
$$\{u, v\} \in E(G) \Rightarrow \{\varphi(u), \varphi(v)\} \in E(G').$$
- An *isomorphism* $G \to G'$ is a *bijection* $\varphi : V(G) \to V(G')$ such that
$$\{u, v\} \in E(G) \Leftrightarrow \{\varphi(u), \varphi(v)\} \in E(G').$$



When are two graphs the same?

- An *isomorphism* $G \to G'$ is a *bijection* $\varphi : V(G) \to V(G')$ such that
$$\{u, v\} \in E(G) \Leftrightarrow \{\varphi(u), \varphi(v)\} \in E(G').$$
- If there is an isomorphism $G \to G'$, then G and G' are *isomorphic*.
- This is an equivalence relation on graphs.
- An "unlabelled graph" is an equivalence class of graphs under this isomorphism relation.



Terminology

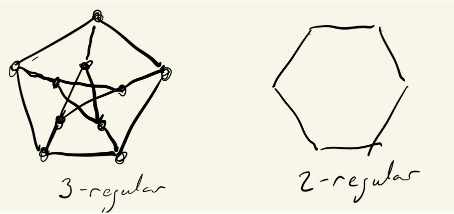
- Vertices x and y are *adjacent* if $\{e, y\} \in E$.
- The vertex x is *incident* to the edge e if $x \in e$.
- The edges e and e' are adjacent if $e \cap e' \neq \emptyset$.
- The (open) *neighbourhood* $N(v) = \{u \in V : \{v, u\} \in E\}$.
- The *degree* $d(v) = |N(v)|$ is the number of neighbours of v (in a simple graph).

Degrees

- Minimal degree $\delta(G) = \min_{v \in V(G)} d(v)$.
- Maximal degree $\Delta(G) = \max_{v \in V(G)} d(v)$.
- Average degree

$$d(G) = \frac{1}{|V(G)|} \sum_{v \in V(G)} d(v) = \frac{2\|G\|}{|G|}.$$

- If all vertices have the same degree k, so $\delta(G) = \Delta(G)$, then G is *k-regular*.



Degrees

Proposition

In any graph G, the number of vertices with odd degree is even.

Proof.

- Blackboard □

Degrees

- The integer sequence $(d_1, \ldots, d_n) \in \mathbb{N}^n$ is *graphical* if there exists a graph G with $V(G) = \{v_1, \ldots, v_n\}$ and $d(v_i) = d_i$.

Theorem (Havel, Hakimi 1955)

Assume $d_1 \geq d_2 \geq \cdots \geq d_n \geq 0$. Then the sequence $(d_1, \ldots, d_n)$ is graphical if and only if

- *$n = 1$ and $d_1 = 0$, or*
- *$(d_2 - 1, d_3 - 1, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n)$ is a graphical sequence.*

Proof.

- Blackboard □

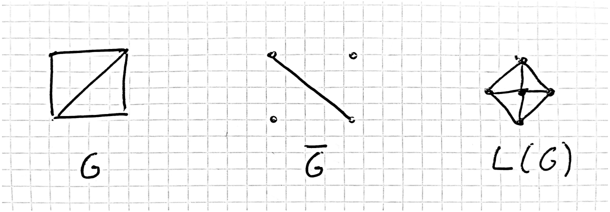
Definitions

- The complement graph of G is

$$\overline{G} = \left(V(G), \binom{V(G)}{2} \setminus E(G) \right).$$

- The line graph of G is

$$L(G) = (E(G), \{\{e, e'\} : e \cap e' \neq \emptyset\}).$$

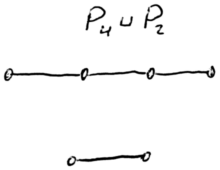


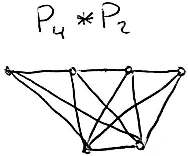
Definitions

- The disjoint union of two graphs G and H is

$$G \sqcup H = (V(G) \sqcup V(H), E(G) \sqcup E(H)).$$

- The *join* of G and H has $G \sqcup H$ as a subgraph, and in addition an edge xy for all $x \in V(G)$, $y \in V(H)$.





Definitions

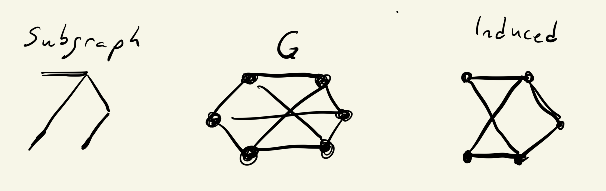
- The disjoint union of two non-empty graphs is always disconnected.
- The join of two non-empty graphs is always connected.
- $K_n \star K_m \cong K_{n+m}$ and $\overline{K_n} \star \overline{K_m} \cong K_{n,m}$

Substructures

- H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.
- H is an *induced subgraph* of G if $V(H) \subseteq V(G)$ and

$$E(H) = E(G) \cap \binom{V(H)}{2}.$$

  - If H is an induced subgraph of G, with $V(H) = U$, then we say that H is *induced on* U, and write $H = G[U]$.



Walks and paths

- A *walk* of length n in $G = (V, E)$ is a sequence $(v_0, v_1, \ldots, v_n)$ of nodes $v_i \in V$ where $\{v_{i-1}, v_i\} \in E(G)$ for $i = 1, \ldots, n$.
- A walk $(v_0, v_1, \ldots, v_n)$ is *closed* if $v_0 = v_n$.
- A *path* of length n in G is a subgraph

$$(\{v_1, v_2, \ldots, v_n\}, \{v_1 v_2, \ldots, v_{n-1}, v_n\}) \subseteq G$$

  with all vertices distinct.
  - So $(v_1, v_2, \ldots, v_n)$ is a non-revisiting walk of length $n - 1$.
- A path $(x, v_1, \ldots, v_{n-1}, y)$ is an *x-y-path*, often denoted x—y.

Closed walks and cycles

- A walk $(v_0, v_1, \ldots, v_n)$ is *closed* if $v_0 = v_n$.
- A *cycle* of length n in G is a subgraph

$$(\{v_1, v_2, \ldots, v_n\}, \{v_1v_2, \ldots, v_{n-1}v_n, v_nv_0\}) \subseteq G$$

  with all vertices distinct.
  - So $(v_1, v_2, \ldots, v_n, v_1)$ is a minimal closed walk of length n.

Paths

- Let $A, B \subseteq V(G)$, and let H be a subgraph of G.
- An $A--B$-path is a path

$$(\{v_1, v_2, \dots, v_n\}, \{v_1 v_2, \dots, v_{n-1}, v_n\}) \subseteq G$$

  where $\{v_1, v_2, \dots, v_n\} \cap A = \{v_1\}$ and $\{v_1, v_2, \dots, v_n\} \cap B = \{v_n\}$
- Note: If $A \cap B \neq \emptyset$, then there exist $A--B$-paths of length 1.
- A H-path is a path

$$(\{v_1, v_2, \dots, v_n\}, \{v_1 v_2, \dots, v_{n-1}, v_n\}) \subseteq G$$

  where $\{v_1, v_2, \dots, v_n\} \cap V(H) = \{v_1, v_n\}$.

Paths and cycles

- The *girth* $g(G)$ is the minimum length of a cycle in G.
- The *circumference* of G is the maximum length of a cycle in G.
- The *distance* $d_G(x, y)$ is the length of the shortest $x - y$-path in G
  - This notion of distance is a *metric*:

$$d_G(x, y) = 0 \Leftrightarrow x = y$$

$$d_G(x, z) \leq d_G(x, y) + d_G(y, z)$$

Paths and cycles

- The *diameter* of G is

$$\max_{x,y\in V(G)} d_G(x,y).$$

- The *radius* of G is

$$\min_{x\in V(G)} \max_{y\in V(G)} d(x,y).$$

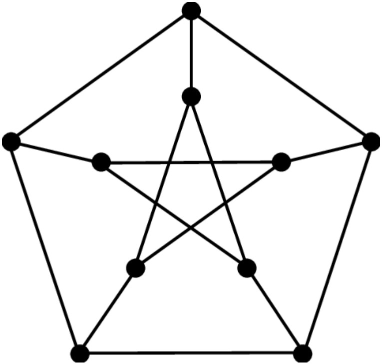
- A vertex $x \in V(G)$ that minimizes

$$\max_{y\in V(G)} d(x,y)$$

  is called a *central* vertex.

Exercise during the break

- Compute the girth, circumference, diameter and radius of the Petersen graph.



Paths and cycles

Proposition

Every graph G with $\delta(G) \geq 2$ contains a cycle of length at least $\delta(G) + 1$.

Proof.

- Blackboard □

Euler tours

- An Euler tour in a graph is a closed walk that traverses every edge in G exactly once.
- "Motivation": Can one take a walk across all the bridges in Königsberg without going over any bridge more than once?

Euler tours

Proposition

A connected graph G has an Euler tour if and only if every vertex in G has even degree.

Proof.

- $\Rightarrow$: Orient each edge according to which direction the Euler tour traverses it.
- Then every node has the same indegree as outdegree, so even total degree.
- $\Leftarrow$: Induction on the number of edges. (Blackboard.) □

Bipartite graphs

Lemma

G is bipartite if and only if G has no odd cycles.

Proof.

- $\Rightarrow$: Proved in an exercise last time.
- $\Leftarrow$: Suffices to prove it for connected graphs. Assume for a contradiction G is connected and has no odd cycles.
- A minimal odd length closed path is a cycle, so G has no odd length closed paths.

□

Bipartite graphs

Lemma

G is bipartite if and only if G has no odd cycles.

Proof.

- $\Leftarrow$: We assumed for a contradiction G is connected and has no odd length closed paths.
- Fix $v \in V(G)$, and define

$$A = \{y \in V(G) : d_G(x,y) \text{ is even}\}$$
$$B = \{y \in V(G) : d_G(x,y) \text{ is odd}\}.$$

- If there were an edge xy between two nodes in the same part, then v—x-y—v would be a closed walk of odd length.
- Contradiction, so (A,B) is a bipartition of $V(G)$. □

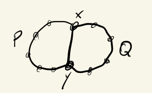
Bipartite graphs

Lemma

G is bipartite if and only if G has no induced odd cycles.

Proof.

- $\Rightarrow$: Follows from the previous lemma.
- $\Leftarrow$: Assume G is not bipartite, yet has no induced odd cycle.
- Consider a minimal odd cycle C in G. (Exists because G is not bipartite.) Let $e = \{x, y\}$ be an edge in $G[V(C)] \setminus C$.



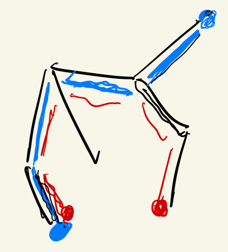
- The cycle C contains two x-y-paths P and Q.
- The cycles $P + e$ and $Q + e$ are shorter than C, and one of them is odd. Contradiction! □

Connectivity

- A graph is *connected* if there is a path between any pair of nodes.
- The maximal connected subgraphs are the *connected components* of the graph.
- The connected components form a partition of the graph.

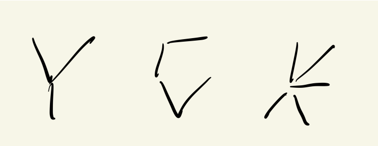
Trees

- A connected graph without cycles is a *tree*.
- A node is a *leaf* if it only has one neighbour.
- Every tree with $|T| \geq 2$ has at least two leaves. (endpoints on a maximal path).



Trees

- A connected graph without cycles is a *tree*.
- A graph without cycles is a *forest*
- Every forest is a disjoint union of trees. These are the connected components of the forest.



Trees

Theorem

The following are equivalent:

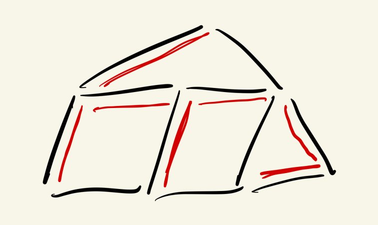
- $T = (V, E)$ *is a tree.*
- *For any* $u, v \in V$, *there s a* unique *u-v-path in* T.
- T *contains no cycle, and for any* $E \subsetneq F \subseteq \binom{V}{2}$, *the graph* (V, F) *contains a cycle.*
- T *is connected, and for any* $F \subsetneq E$, *the graph* (V, F) *is disconnected.*

Proof.

- Exercise □

Spanning trees

- A *spanning tree* in the connected graph (V, E) is a tree (V, E') that contains all the nodes and some of the edges $E' \subseteq E$ of the graph.
- A spanning tree exists in any connected graph:
- Start from one node. Add one edge at a time between a node in the tree and a node not yet in the tree.
  - Notice: the spanning tree is not unique.



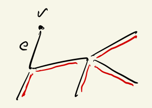
Trees

Lemma

A tree with n nodes has exactly $n-1$ edges.

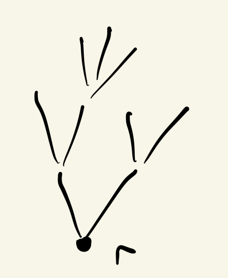
Proof.

- By induction on n. Trivial base case $n=1$.
- Assume $|V|\geq 2$, and let $v\in V(T)$ be a leaf, with only outgoing edge $e\in E(T)$.
- Then $(V\setminus\{v\}, E\setminus\{e\})$ is a tree with $n-1$ vertices and (by induction hypothesis) $n-2$ edges.
- So $|E|=n-1$. □



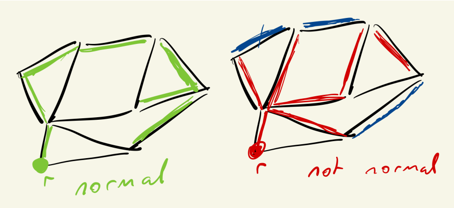
Rooted trees

- A *rooted tree* is a tree T with a distinguished node r. Then:
- The *level* of the node v is the length of the unique path $(r, \dots, v)$.
- The *tree order* associated to (T, r) is the partial order on $V(T)$ given by $v \leq u$ if the unique path from r to u goes through v.
- r is the unique minimal element in the tree order.



Normal trees

- A rooted spanning tree (T, r) in G is called a *normal* tree if all edges in G go between comparable elements in the tree order.
- Normal trees are also called *depth first search* trees.
- Normal trees exist in every connected graph for any prescribed root.
- Constructed via depth first search.



Edge spaces

MS-E1050

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- Consider $\mathbb{F}_2 = \{0, 1\}$ with addition $1 + 1 = 0$.
- Define the edge space (over $\mathbb{F}_2$) $\mathcal{E}(G) = \{f : E(G) \to \mathbb{F}_2\}$.
- Identify the elements in $\mathcal{E}(G)$ with subsets of $E(G)$.
- $$\langle f, f' \rangle = \sum_{e \in E} f(e)f'(e) = |f \cap f'| \mod 2.$$
- $f + f'$ corresponds to the symmetric difference of the sets f and f'.

Cycle spaces

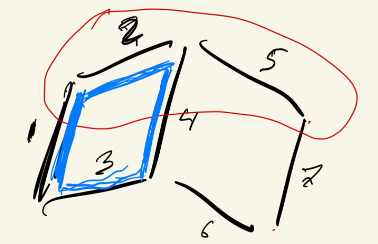
- The edge set of a cycle is called a *circuit*.
- The cycle space $\mathcal{C}(G) \subseteq \mathcal{E}(G)$ is generated (over $\mathbb{F}_2$) by the circuits in G.
- $F \in \mathcal{C}(G)$ iff and only if F is a *disjoint* union of circuits.
- $F \in \mathcal{C}(G)$ iff and only if every $v \in V(G)$ has even degree in F.
- Proofs of these equivalences: Exercise.

Cut spaces

- If $A \subseteq V(G)$, the set of edges between A and $\overline{A}$ is a *cut*.
- The cut space $\mathcal{B}(G) \subseteq \mathcal{E}(G)$ is generated (over $\mathbb{F}_2$) by the cuts in G.
- The symmetric difference of two cuts is a cut.
- So $F \in \mathcal{C}(G)$ iff and only if F is itself a cut.

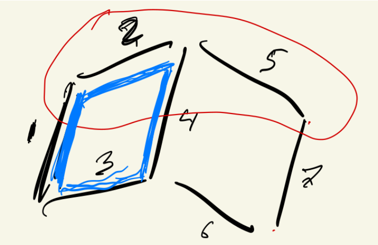
Cut and cycle spaces

Example



- $c = \{1,2,3,4\} \in \mathcal{C}(G)$.
- $b = \{1,4,7\} \in \mathcal{B}(G)$.
- $\langle b, c\rangle = |b \cap c| \mod 2 = |\{1,4\}| \mod 2 = 0$

Cut and cycle spaces

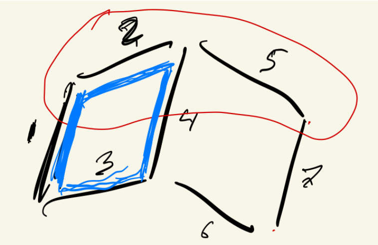


- A closed walk enters A equally many times as it leaves A.
- So a circuit c contains an even number of edges from every cut.
- So $\mathcal{B}(G) \perp \mathcal{C}(B)$ as vector spaces over $\mathbb{F}_2$.
- Standard linear algebra gives

$$\dim \mathcal{B}(G) + \dim \mathcal{C}(G) \leq \dim \mathcal{E}(G) = m.$$

- We will show that equality holds, so $\mathcal{B}(G) = \mathcal{C}(G)^{\perp}$.

Cut and cycle spaces



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- Standard linear algebra gives

$$\dim \mathcal{B}(G) + \dim \mathcal{C}(G) \leq \dim \mathcal{E}(G) = m.$$

- We will show that equality holds, so $\mathcal{B}(G) = \mathcal{C}(G)^{\perp}$.

Cycle space dimension

- Easy to check:

$$\mathcal{C}(G \sqcup H) = \mathcal{C}(G) \oplus \mathcal{C}(H) \text{ and } \mathcal{B}(G \sqcup H) = \mathcal{B}(G) \oplus \mathcal{B}(H).$$

- So assume G connected. Fix a spanning tree $T \subseteq G$.
- For each $e = \{u, v\} \in E - T$, consider the *fundamental cycle*

$$C_e = \{e\} \cup P_{uv},$$

  where P_{uv} is the unique $u - v$-path in T.
- For every $e \in E$, the collection $\{C_i : i \in E - T\}$ contains exactly one vector $C = C_e$ for which $C(e) = 1$.
- Therefore, the vectors $\{C_i : i \in E - T\}$ are linearly independent.
- So $\dim(\mathcal{C}(G)) \geq |E - T| = m - (n - 1)$.

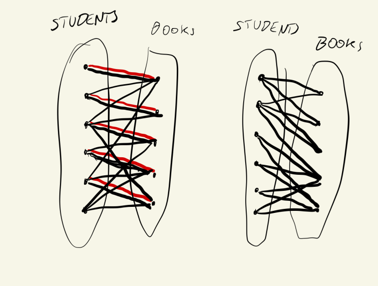
Cut space dimension

- Assume G connected. Fix a rooted spanning tree $T \subseteq G$.
- For each $e = \{u, v\} \in T$, with $v > u$ in the tree order, define $A_e = \{w \in V : w \geq v\}$.
- The cut B_e associated to A_e contains onely one edge from T, namely e.
- Therefore, the vectors $\{B_i : i \in T\}$ are linearly independent.
- So $\dim(\mathcal{B}(G)) \geq |T| = n - 1 = m - \dim(\mathcal{C}(G))$.
- It follows that $\mathcal{B}(G)^\perp = \dim(\mathcal{C}(G))$.

Matchings in bipartite graphs

Example

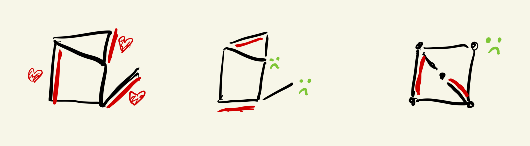
- Six students have to do a group task, where they have to read five different books.
- Nobody has time to read more than one book.
- Moreover, not all students have access to all of the books.
- Can they divide the task so that all the books get read?



Matchings in general graphs

Example

- In a collection of people, some pairs of people are willing to live happily ever after together, while some other pairs are not.
- Can we pair the population up so that everyone is together with someone they want to live with?



Matchings

Definition

- A *matching* in G is a collection $M \subseteq E(G)$ of pairwise disjoint edges.
- A matching M is *maximal* if it is not contained in any other matching on G
- A matching M is *complete on* $A \subseteq V(G)$ if every vertex in A is in some edge of M.
- A matching M is *perfect* (or *complete*) if it is complete on $V(G)$.

k-factors

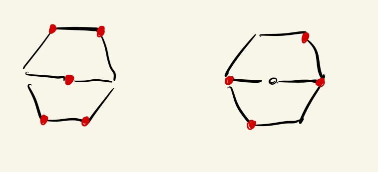
Definition

- A perfect matching on G is also called a 1-*factor*.
- More generally, a k-*factor* on G is a spanning k-regular subgraph of G.
- In particular, a 2-factor is a collection of pairwise disjoint cycles that cover the vertices of G.

Vertex cover

Definition

- A vertex cover is a collection $A \subseteq V(G)$ such that every edge in $E(G)$ contains at least one vertex in A.
- A vertex cover is *minimal* if it does not contain any other vertex cover.



Alternating paths

Definition

- Consider a bipartite graph $G = (A \sqcup B, E)$ with a matching M.
- An *alternating path* with respect to M is a path

$$P = a_0, b_1, a_1, b_2, \cdots v$$

  in G such that:
  - a_0 is not matched in M.
  - $\{a_i, b_i\} \in M$ for all $i \geq 1$.
- If the final vertex v of the path is unmatched, then P is an *augmenting path* with respect to M.

Augmenting paths

Definition

- An *augmenting path* with respect to M is a path

$$P = a_0, b_1, a_1, b_2, \cdots a_k, b_{k+1}$$

  in G such that:
  - $a_0 \notin e$ for every $e \in M$.
  - $\{a_i, b_i\} \in M$ for all $i \geq 1$.
  - $b_k \notin e$ for every $e \in M$.

Lemma

If P is an augmenting path with respect to a matching M, then

$$M' = M \setminus \{a_i b_i : 1 \leq i \leq k\} \cup \{a_i b_{i+1} : 0 \leq i \leq k\}$$

is a matching with $|M'| = |M| + 1$.

König's Theorem

Theorem

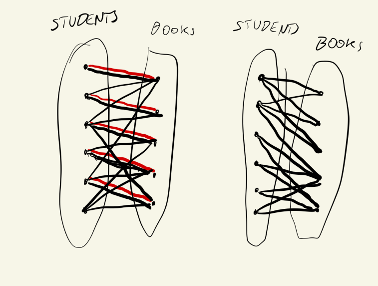
- *In any bipartite graph* $(A \sqcup B, E)$*, the size of the largest matching equals the size of the smallest vertex cover.*

Proof.

- $\leq$: Every vertex cover contains at least one end of each edge in the matching.
- These ends must all be different.
- $\geq$: Proof by alternating paths. (blackboard) $\square$

Another necessary condition

- Assume there is a set $S \in V(G)$ such that $|N(S)| < |S|$.
- Then there clearly can not be a complete matching on S, so not on G either.



Hall's Marriage Theorem

Theorem

- *A bipartite graph $(A \sqcup B, E)$ with $|A| \leq |B|$ contains a complete matching if and only if $|N(S)| \geq |S|$ for all $S \subseteq A$.*

Proof.

- $\Rightarrow$: Trivial.
- $\Leftarrow$: By induction on $|A|$. Obvious if $|A| = 1$.
- If $|N(S)| > |S|$ for all $S \subsetneq A$, $ab \in E(G)$ be an arbitrary edge.
- Hall's condition holds for the smaller graph $G' = G - \{a, b\}$, so there is a complete matching M' on G'.
- Then $M' \cup \{ab\}$ is a complete matching on G.

Hall's Marriage Theorem

$\Leftarrow$ continued.

- Remains to assume $|N(A')| = |A'|$ for some $A' \subsetneq A$.
- By induction, there is a complete matcing on $G' = G[A' \sqcup N(A')]$.
- Now if $S \subseteq A \setminus A'$, then

$$|N(S) \setminus N(A')| \geq |N(S \cup A')| - N(A') \\ = |N(S \cup A')| - |A'| \geq |S \cup A'| - |A'| = |S|.$$

- So $G - G'$ also satisfies Hall's condition and has a complete matching.
- These two matchings together form a complete matching on G. $\square$

1-factors

Corollary

- *Every non-empty bipartite regular graph has a 1-factor.*

Proof.

- $(A \sqcup B, E)$ regular $\Rightarrow |A| = |B|$.
- For any $S \subseteq A$,

$$k|S| = |E(S)| \leq |E(N(S))| = k|N(S)|,$$

so S satisfies Hall's criterion. □

2-factors

Corollary

- *Every regular graph of positive even degree has a 2-factor.*

Proof.

- Assume WLOG G connected.
- Positive even degree $\Rightarrow$ exists an *Euler tour* $v_0, v_1, v_2, \ldots, v_m = v_0$ with

$$E = \{v_i v_{i+1} : 0 \leq i < m\}.$$

- Construct a bipartite graph $G' = (V^+ \sqcup V^-, E)$, where

$$V^+ = \{v^+ : v \in V(G)\}, V' = \{v^- : v \in V(G)\}$$

and

$$E = \{\{v_i^+, v_{i+1}^-\} : 0 \leq i < m\}.$$

2-factors

Corollary

- *Every regular graph of positive even degree has a 2-factor.*

Continued.

- Construct a bipartite graph $G' = (V^+ \sqcup V^-, E)$, where

$$V^+ = \{v^+ : v \in V(G)\}, V' = \{v^- : v \in V(G)\}$$

and

$$E = \{\{v_i^+, v_{i+1}^-\} : 0 \le i < m\}.$$

- This graph is bipartite and regular, so has a perfect matching.
- This perfect matching projects to a 2-factor in G under the map $G' \to G$, $v^+ \mapsto v$, $v^- \mapsto v$.

□

Preference orders

- A *preference order* of a vertex $v \in V(G)$ is a linear order $\leq_v$ on $N(v)$.
- We say that v *prefers* u to w if $u \geq_v w$.
- A *preference ordered graph* is a graph G together with a preference order for every vertex $v \in V(G)$.

Preference orders

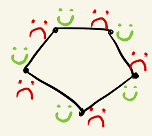
- Let M be a matching on a preference ordered graph.
- We say that a *desires* $b \in N(a)$ if $b \geq_a x$ for any x with $\{a, x\} \in M$.
- An edge $\{a, b\} \notin M$ is *critical* with respect to the matching M if a desires b and b desires a.
- A matching is *stable* if there is no critical pair with respect to M.
- We say that a is *satisfied* if a is unmatched **or** a is not contained in any critical edge.

Preference orders

- For some preference ordered graphs, there exist no stable matchings.
- Example: Consider the cyclic graph C_n with the preferences

$$i + 1 \geq_i i - 1$$

for all $i \in V(C_n) = \{0, 1, \ldots, n-1\}$, (addition mod n).



- If i is unmatched, then $\{i, i-1\}$ is always a critical edge.
- If n is odd, then some vertex is always unmatched, so no stable matching exists.

Gale's Marriage Theorem

Theorem

- *For any set of preferences* $\{\leq_x : x \in V(G)\}$ *on a bipartite graph* G*, there exists a stable matching.*

Proof.

- We will find such a matching by an algorithm that will terminate on a stable matching.
- The algorithm is (controversially?) not symmetric on the sets A and B.

Gale's Marriage Theorem

Continued.

- WHILE there exist desired unmatched $a \in A$:
  - Choose arbitrary desired unmatched $a \in A$.
  - Every element in B that desires a *proposes* to her.
  - a selects her favourite among the admirers (who leaves his previous partner if he was already matched).
- By construction, matched elements in A are never in a critical pair.
- The algorithm ends after at most $\sum_{b \in B} d(b)$ iterations.
- When the algorithm terminates, unmatched elements in A are also not in any critical pair. □

Gale's Marriage Theorem

Theorem

- *On a given graph G with given preference orders, all stable matchings have the same size.*

Proof.

- Exercise □

Tutte's condition

- If $|G|$ is odd, then G has no 1-factor. (duh!)
- Let $q(H)$ denote the number of odd components in the graph H.
- Assume S is a separator on G, A is a component of $G \setminus S$, and M is a 1-factor on G.
- Either M is a 1-factor on A, or M contains an AS-edge.
- So if G has a 1-factor, then $q(G \setminus S) \leq |S|$.

Tutte's condition

- Let $q(H)$ denote the number of odd components in the graph H.

Theorem

- *The graph G has a 1-factor if and only if $q(G \setminus S) \leq |S|$ for all $S \subseteq V(G)$.*

Proof.

- $\Rightarrow$: Trivial.
- $\Leftarrow$: Say that a set S is *bad* if $q(G \setminus S) > |S|$.
- If $G' \subseteq G$ is a spanning subgraph, and S is bad in G, then S is bad in G'.
- Assume G *edge-maximal* with no 1-factor. We want to find a bad set S.

Tutte's condition

Theorem

- *The graph G has a 1-factor if and only if $q(G \setminus S) \leq |S|$ for all $S \subseteq V(G)$.*

Continued.

- Assume G *edge-maximal* with no 1-factor.
- Consider
$$S = \{v \in V : \forall u \in V : vu \in E\}.$$
- Every component in $G \setminus S$ is complete by edge-maximality (technical lemma).
- If $|G|$ is even, then we would get a 1-factor unless if S is bad.
- If $|G|$ is odd, then $\emptyset$ is bad. □

3-regular graphs

Theorem

- *Every 3-regular bridgeless graph G has a 1-factor.*

Proof.

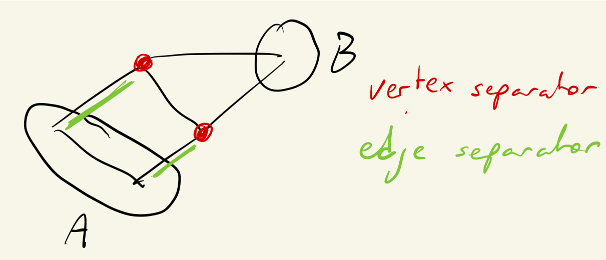
- We will show that G satisfies Tutte's criterion.
- Fix $S \subseteq V(G)$, and an odd component C of $G \setminus S$.
- $\sum_{c\in C} d(c)$ is odd, so there is an odd number of SC-edges.
- No bridge $\Rightarrow$ there are at least 3 SC-edges.
- So

$$3|S| \geq \sum \#SC\text{-edges} \geq 3q(G \setminus S),$$

where the sum is over all odd components of $G \setminus S$. □

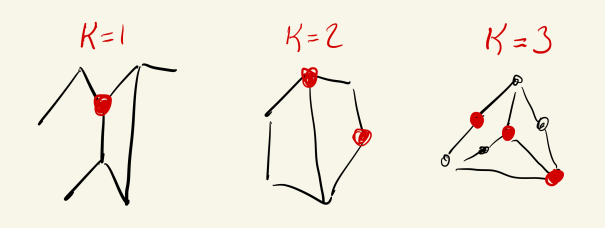
Connectivity

- $X \subseteq V(G) \cup E(G)$ is a *separator* of G if $G \setminus X$ is disconnected
- $X \subseteq V(G) \cup E(G)$ is an $A - B$-separator, for $A, B \subseteq V(G)$, if there is no path from A to B in $G \setminus X$.
  - If X consists only of vertices, it is a *vertex separator*
  - If X consists only of edges, it is a *edge separator*



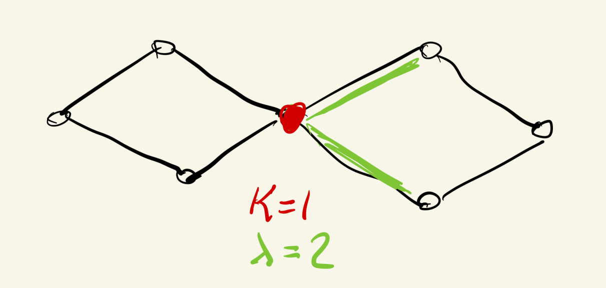
Connectivity

- The graph G is k-connected if $G \setminus X$ is connected for all $X \subseteq V(G)$ with $|X| < k$.
- The *connectivity* $\kappa(G)$ is the largest integer k such that G is k-connected.



Connectivity

- The graph G is edge-k-connected if $G \setminus X$ is k-connected for all $X \subseteq E(G)$ with $|X| < k$.
- The *edge connectivity* $\lambda(G)$ is the largest integer k such that G is k-edge-connected.



Connectivity

Theorem

- *For any non-complete graph G,*

$$\kappa(G) \leq \lambda(G) \leq \delta(G).$$

Proof.

$$\lambda(G) \leq \delta(G):$$

- The $d(v) = \delta(G)$ edges surrounding some vertex v separate v from the rest of the graph.

Connectivity

Proof.

$$\kappa(G) \leq \lambda(G):$$

- Consider a k element edge separator F.
- **Case one**: F covers all vertices of G.
- Consider v with $d(v) < n-1$, and let A be the connected component of v in $G \setminus F$.
- All edges v-y, $y \notin A$, are in F.
- All elements of $N(v) \cap A$ are in *different* edges of F.
- So $|N(v)| \leq k$, and so $N(v)$ is a separator of size $< k$.

Connectivity

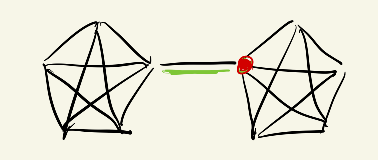
Proof.

$$\kappa(G) \leq \lambda(G):$$

- Consider a k element edge separator F.
- **Case two**: $v \in V$ in not incident to any edge in F.
- Let A be the connected component of v in $G \setminus F$.
- Let $A' \subseteq A$ be the set of vertices in A that are incident to an edge in F.
- So $A' \leq k$, A' separates v from $V \setminus A$ in $G \setminus F$. □

Connectivity

- So high connectivity implies high minimum degree.
- The opposite implication does not hold.



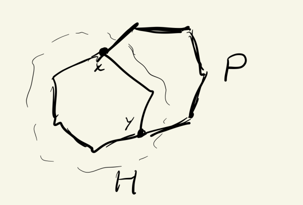
2-connected graphs

Theorem

G is 2-connected if and only if it can be inductively constructed by:

- *Starting from a cycle.*
- *Adding a H-path to H.*

- A H-path is a x-y-path P for some vertices $x, y \in H$, such that no internal vertex on P lies in H.



2-connected graphs

Theorem

G is 2-connected if and only if it can be inductively constructed by:

- *Starting from a cycle.*
- *Adding a H-path to H.*

Proof.

- $\Rightarrow$: Cycles are 2-connected, and 2-connectedness is preserved when adding H-paths
- $\Leftarrow$: By induction on $|G|$.

2-connected graphs

Theorem

G is 2-connected if and only if it can be inductively constructed by:

- *Starting from a cycle.*
- *Adding a H-path to H.*

Proof.

- For a contradiction, consider a *maximal* subgraph $H \subsetneq G$ constructed as in the theorem.
- By maximality, H is induced.
- G connected, so there is an edge $uv \in E(G)$ with $u \in H$, $v \notin H$.
- G 2-connected, so there is a H—v-path P in $G \setminus \{u\}$.
- $P + \{u, v\}$ is a H-path, contradicting the maximality of H. □

Paths and separators

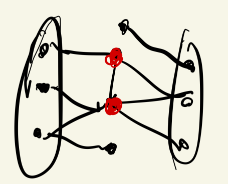
- $X \subseteq V(G) \cup E(G)$ is an $A - B$-separator, for $A, B \subseteq V(G)$, if there is no path from A to B in $G \setminus X$.
- A family of paths from A to B are
  - *Disjoint* if they have no vertices in common.
  - *Independent* if they have no *internal* vertices in common.
  - *Edge disjoint* if they have no vertices in common.
- Clearly,

$$\text{Edge disjoint} \Leftarrow \text{Independent} \Leftarrow \text{Disjoint}$$

Menger's local theorem

Theorem

- *Let G be a graph and $A, B \subseteq V(G)$.*
- *The minimum size of an (A, B)-vertex separator equals the maximum number of pairwise disjoint $A - B$-paths in G.*



- We allow the vertex separator to intersect $A \cup B$.
- Indeed, if $A \cap B \neq \emptyset$, then the vertex separator must contain $A \cap B$.

Menger's local theorem

Theorem

- *Let G be a graph and $A, B \subseteq V(G)$.*
- *The minimum size of an (A, B)-vertex separator equals the maximum number of pairwise disjoint $A - B$-paths in G.*

Proof.

- $\geq$: If there are k disjoint paths, then all of them must contain a vertex from the separator.
- So any separator has size at least k.

Menger's local theorem

Theorem

- *Let G be a graph and $A, B \subseteq V(G)$.*
- *The minimum size of an (A, B)-vertex separator equals the maximum number of pairwise disjoint $A - B$-paths in G.*

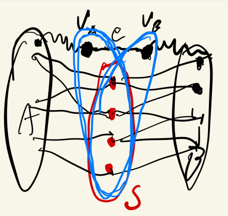
Proof.

- $\leq$: Assume there is no (A, B)-vertex separator of size $k - 1$.
- We claim that there are k pairwise disjoint $A - B$-paths in G.
- Proof by induction over $|E(G)|$.
- Base case: If $E = \emptyset$, then $A \cap B$ is a separator, so $|A \cap B| \geq k$.
- Then there are k trivial $A - B$-paths.

Menger's local theorem

Proof.

- $\leq$: Assume there is no (A, B)-vertex separator of size $k-1$.
- Assume for a contradiction that there are *not* k pairwise disjoint $A-B$-paths in G.
- Fix $e \in E(G)$. There are at most $k-1$ pairwise disjoint $A-B$-paths in $G \setminus e$.
- By induction hypothesis, $G \setminus e$ has an (A, B)-separator S with $|S| \leq k-1$.



Menger's local theorem

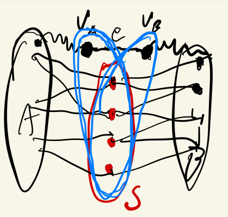
Proof.

- $\leq$: We assumed there were no (A, B)-vertex separator of size $k-1$ but also no k pairwise disjoint $A-B$-paths in G.
- By induction hypothesis, $G \setminus e$ has an (A, B)-separator S with $|S| \leq k-1$.
- There is an $A-B$-path in G that uses e and does not intersect S, because S is not a separator in G.

Menger's local theorem

Proof.

- $G \setminus e$ has neither a $(A, S \cup \{v_A\})$-separator nor any $(A, S \cup \{v_B\})$-separator of size $\leq k - 1$.
- By induction, there are k disjoint $(A, S \cup \{v_A\})$-paths and k disjoint $(B, S \cup \{v_A\})$-paths.
- These can be glued together with the edge e to form k disjoint (A, B)-paths. □



Menger's global theorem

Theorem

- *Let G be a graph. The following are equivalent:*
  - *G is k-connected.*
  - *For every $a, b \in V(G)$, there are k pairwise* independent *$a - b$-paths.*

Proof.

- The following are equivalent.
  - There are k pairwise *independent* $a - b$-paths.
  - There are k pairwise *disjoint* $N(a) - N(b)$-paths.
  - There is no $(N(a), N(b))$-separator of size $< k$.
- Every separator in the graph is an $(N(a), N(b))$-separator for some $a, b \in V(G)$.
- Thus, the conditions above hold for *all* vertices $a, b \in V(G)$ if and only if G is k-connected. □

Menger's edge-connectivity theorem

Corollary

- *Let G be a graph. The following are equivalent:*
  - *G is k-edge connected.*
  - *For every $a, b \in V(G)$, there are k pairwise* edge-disjoint *$a - b$-paths.*

Proof.

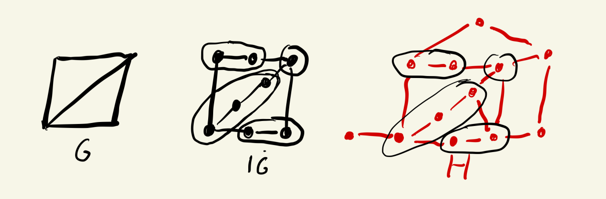
- Apply Menger's theorem to the *line graph* $L(G)$ of G.
- Edge disjoint (a, b)-paths in G are disjoint $(E(a), E(b))$-paths in $L(G)$. □

Minors

- If G is a graph, then IG ("inflated G") denotes any graph G' whose vertex set can be partitioned as a disjoint union $V(G') = \cup_{x \in V(G)} U_x$ where

$$xy \in E(G) \Leftrightarrow \exists v_x \in U_x, v_y \in U_y : v_x v_y \in E(G').$$

- If H has a subgraph isomorphic to an IG, then G is a *minor* of H.



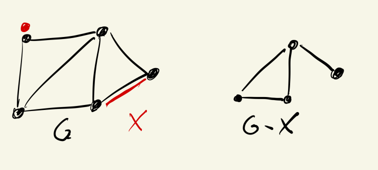
Minors

- The *deletion* of $X \subseteq E$ from $G = (V, E)$ is

$$G \setminus X = (V, E \setminus X).$$

- The *deletion* of $X \subseteq V \cup E$ from $G = (V, E)$ is

$$G \setminus X = (G \setminus (E \cap X))[V \setminus X].$$



Minors

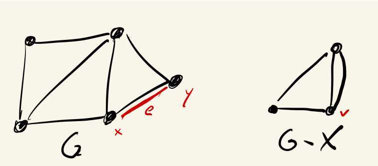
- The *contraction* of the edge $e = \{x, y\}$ from $G = (V, E)$ is $G/e = (V', E')$, where

$$V' = V \setminus \{x, y\} \cup \{v\}$$

and

$$E' = E \setminus \{xz\} \setminus \{yz\} \cup \{vz : xz \in E \text{ or } yz \in E\}.$$

- Observe that it is often (but not always) more natural to define the contraction as a multigraph.
- If $z \in N(x) \cap N(y)$, then we get two parallel edges vz.



Minors

- Contraction and deletion commute:

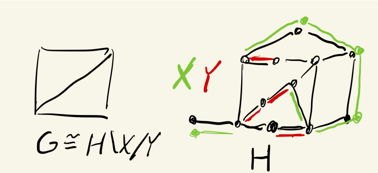
$$(G/e)/f \cong (G/f)/e,$$

and if $e \notin X$ then

$$(G \setminus X)/e \cong (G/e) \setminus X.$$

- So for $X \subseteq V \cup E$ and $Y \subseteq E \setminus X$, we can naturally define

$$G \setminus X/Y.$$

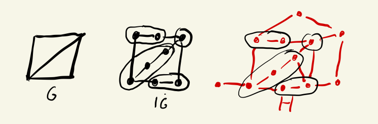


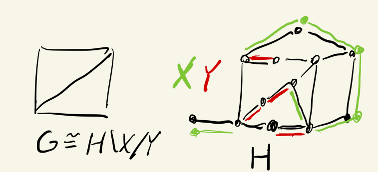
Minors

Proposition

- G *is a* minor *of* $H = (V, E)$ *if and only if there exists* $X \subseteq V \cup E$ *and* $Y \subseteq E \setminus X$*, such that*

$$G \cong H \setminus X / Y.$$





Building k-connected graphs

- Paraphrasing previous theorems:
  - Connected graphs can be obtained by glueing edges together along vertices.
  - 2-connected graphs can be constructed by glueing cycles together along paths.
- The family of connected 3-regular graphs is much more complicated than the families of connected 1- and 2-regular graphs
- The building blocks of the structure theorem for 3-connected graphs are K_4, and the operations are the inverse of contraction.

Two operations on multigraphs

Definition

The (v, N_x, N_y)-*vertex split* of G' is $G' \mapsto G = (V, E)$, where

$$V = V(G') \cup \{x, y\} \setminus \{v\}$$

and

$$E = E(G') \setminus \{e : v \in e\} \cup \{xz : z \in N_x\} \cup \{yz : z \in N_y\} \cup \{xy\},$$

where

$$v \in V(G'), \qquad N_x, N_y \subseteq N(v), \qquad N_x \cup N_y = N(v).$$

Proposition

$G' \cong G/e$ for some $e = xy \in E(G)$ if and only if G is a vertex split of G'.

Tutte's wheel theorem

- Our next goal is to prove the following theorem:

Theorem

A graph G is 3-connected if and only if there is a sequence of edges

$$e_1, \ldots, e_{m-6}$$

in G such that:

- $G/\{e_1, \ldots, e_k\}$ *is 3-connected for all* k.
- $G/\{e_1, \ldots, e_{m-6}\} \cong K_4$.

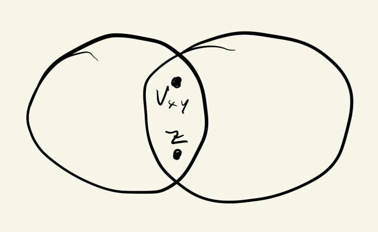
Two operations on multigraphs

Lemma

If G is 3-connected, then there is some edge $e \in E(G)$ such that G/e is also 3-connected.

Proof.

- Assume not.
- Then every edge is contained in a 3-separator.



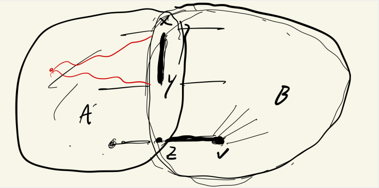
Two operations on multigraphs

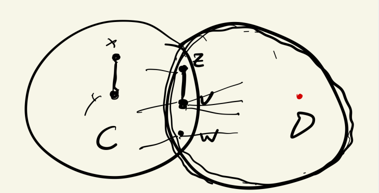
Lemma

If G is 3-connected, then there is some edge $e \in E(G)$ such that G/e is also 3-connected.

Proof.

- Assume B minimal.





$$D \subsetneq B \quad \square$$

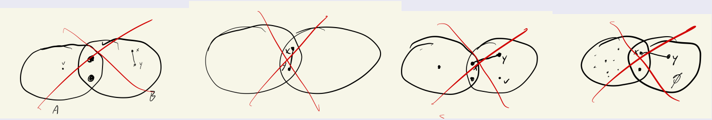
Two operations on multigraphs

Lemma

If G is 3-connected, $v \in V(G)$, and $N_x, N_y \subseteq N(v)$ satisfy $|N_x| \geq 3, |N_y| \geq 3$, then the (v, N_x, N_y)-vertex split of G is 3-connected.

Proof.

- Assume there were a 2-separator in the vertex split G'.
- Proof by contradiction by case separation



Tutte's wheel theorem

Theorem

A graph G is 3-connected if and only if there is a sequence of edges

$$e_1, \dots, e_{m-6}$$

in G such that:

- $G/\{e_1, \dots, e_k\}$ *is 3-connected for all k.*
- $G/\{e_1, \dots, e_{m-6}\} \cong K_4$.

Plane graphs

Definition

- A plane graph is a pair (V, E) (notice the abuse of notation) where
  - V is a set of points in $\mathbb{R}^2$
  - Every edge is a curve between two points in V.
  - The interior of an edge does not intersect any other edge or contain any vertex $v \in V$.
- Plane graphs have a natural multigraph structure.

Plane graphs

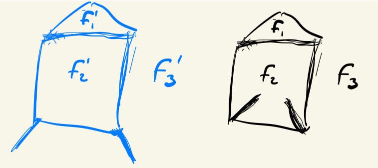
- The *graph drawing*

$$V \cup \bigcup_{e \in E} e \subseteq \mathbb{R}^2$$

  separates the plane into *faces*.
- Each face is topologically an open disc or a punctured open disc.
- If G is finite, then there is only one unbounded face, the *outer face*.
- If we want to remove the distinction between inner and outer faces, we draw our plane graphs on the sphere S^2 instead of in $\mathbb{R}^2$.
- If G is connected, then each face (except for the outer face) is an open disc, and is bounded by a closed walk in the graph G.

Planar graphs

- A *planar graph* is a graph that is isomorphic to the graph of some plane graph.
- In principle, two different plane graphs can yield the same planar graph.



Plane triangulations

Proposition

G is a maximally planar *graph if and only if every drawing of it is a triangulation of S^2.*

- A planar graph $G = (V, E)$ is maximally planar if $(V, E \cup \{e\})$ is nonplanar for any $e \notin E$.
- The implication $\Rightarrow$ is obvious, because if G can be drawn with a non-triangle face, then a chord can be added to this face without destroying planarity.
- The implication $\Leftarrow$ will follow shortly.

Euler's theorem

Proposition

- *A plane graph has only one face if and only if G is a forest.*

Proof.

- By induction on $|E|$. □

Euler's theorem

Proposition

- *If a plane graph has v vertices, e edges and f faces, then*

$$v - e + f = 2.$$

Proof.

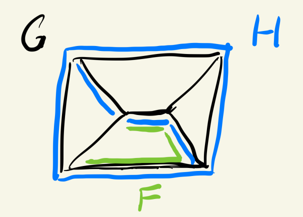
- By induction on $|E|$. □

Double counting

- $$2e = \sum_{\text{faces } F} |\partial F| \geq \begin{cases} 3f & \text{if } G \text{ simple} \\ 4f & \text{if } G \text{ simple bipartite} \end{cases}$$
- If G simple planar, then $e \leq 3v - 6$.
  - In particular, K_5 is not planar ($e = 10, v = 5$).
- If G simple bipartite and planar, then $e \leq 2v - 4$.
  - In particular, $K_{3,3}$ is not planar ($e = 9, v = 6$).

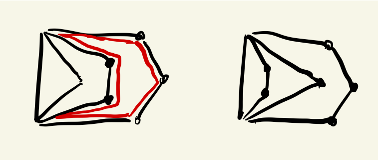
Faces

- If $H \subseteq G$, and the edges $F \subseteq E(G)$ are contained in a face of (some drawing of) G, then they are also contained in a face of (the induced drawing of) H.



2-connected planar graphs

- The faces of 2-connected plane graphs are bounded by cycles.
- By Euler's theorem, the *number* of face-bounding cycles in G does not depend on the drawing.
- However, the *set* of face-bounding cycles does.



Uniqueness of drawings

- The moral of the following theorem is that "3-connected planar graphs can essentially only be drawn in one way".

Theorem

- *Consider a fixed drawing of a 3-connected planar graph G.*
- *A cycle $C \subseteq E(G)$ bounds a face if and only if it is induced and separating.*

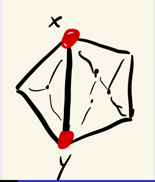
Uniqueness of drawings

Theorem

- *A cycle $C \subseteq E(G)$ bounds a face iff it is induced and non-separating.*

Proof.

- Face-bounding $\Rightarrow$ Induced:
- WLOG, assume C bounds the outer face and $x, y \in V(C)$.
- If $xy \in E(G)$, then $\{x, y\}$ is a 2-separator, contradicting 3-connectivity. □



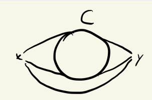
Uniqueness of drawings

Theorem

- *A cycle $C \subseteq E(G)$ bounds a face iff it is induced and non-separating.*

Proof.

- Face-bounding $\Rightarrow$ Non-separating:
- Assume C bounds a face, and let $x, y \in V(G) \setminus V(C)$.
- By 3-connecteivity and Mernger's theorem, there are 3 *independent* xy-paths.
- One of these paths must go outside of C (by topology).
- So C does not separate x from y. □



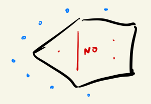
Uniqueness of drawings

Theorem

- *A cycle $C \subseteq E(G)$ bounds a face iff it is induced and non-separating.*

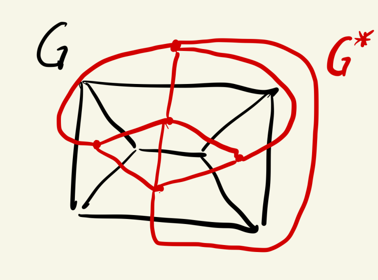
Proof.

- Induced and non-separating $\Rightarrow$ Face-bounding:
- C non-separating, so all vertices in $V(G) \setminus V(C)$ are in one of the two regions bounded by C.
- WLOG all vertices on the "outside" of C.
- C induced and no vertices inside of C $\Rightarrow$ no edges inside of C.
- Thus C bounds a face. □



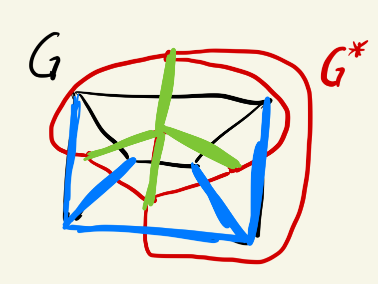
Plane duals

- Any plane graph $G = (V, E)$ has a plane dual $G^* = (F, E')$
- F is the set of faces of G, and there is a natural bijection $E \leftrightarrow E'$.
- Well defined up to topological equivalence.
- $(G^*)^* = G$.



Plane duals

- The complement of a spanning tree in G corresponds to a spanning tree in G^*.
- $T \subseteq E(G)$ acyclic $\Leftrightarrow$ $\bar{T}' \subseteq E(G^*)$ connected.

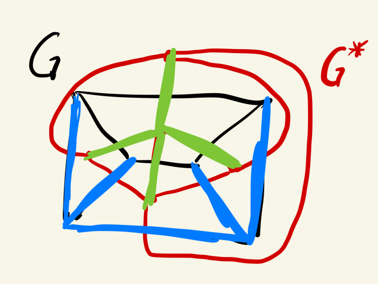


Plane duals

- The complement of a spanning tree in G corresponds to a spanning tree in G^*.
- This proves in a new way that

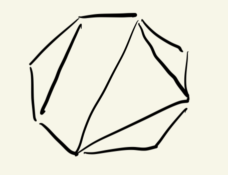
$$e = |E(G)| = (|V(G)| - 1) + (|V(G^*)| - 1) = (v - 1) + (f - 1),$$

so $v - e + f = 2$.

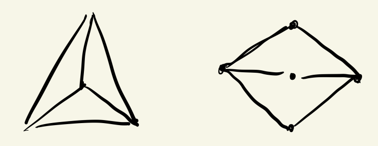


Outerplanar graphs

- A planar graph G is *outerplanar* if it has a drawing in which every vertex is on the outer face.



- Example: K_4 and $K_{3,2}$ are planar but not outerplanar.



Minors

- Assume G is (outer)planar and $e \in E(G)$.
- Then both G/e and $G \setminus e$ are (outer)planar.
- So the classes of (outer)planar graphs are closed under taking minors.
- In particular, no planar graph can have K_5 or $K_{3,3}$ as a minor.

Kuratowski's theorem

Theorem

- *A graph G is planar if and only if it does not contain K_5 or $K_{3,3}$ as a minor.*

- $\Rightarrow$ follows because minors of planar graphs are planar.
- $\Leftarrow$ Proof by contradiction, first reducing to the 3-connected case.

Lemma

- *An edge-minimal non-planar graph G that does not contain K_5 or $K_{3,3}$ as a minor is 3-connected.*

Kuratowski's theorem

Lemma (Reduction to 3-connected case)

- *An edge-minimal non-planar graph G that does not contain K_5 or $K_{3,3}$ as a minor is 3-connected.*

Proof.

- Blackboard □

Kuratowski's theorem

Lemma (Key Lemma)

- *A 3-connected graph G that does not contain K_5 or $K_{3,3}$ as a minor is planar.*

Proof.

- Blackboard □

- Kuratowski's Theorem follows from the reduction lemma and the key lemma.

Definitions

- A (proper) *k-colouring* of $G = (V, E)$ is a map $\gamma : V \to \{1, 2, \dots, k\}$ such that $\gamma(v) \neq \gamma(u)$ whenever $uv \in E$.

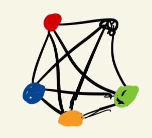
- In other words, a k-colouring is a graph homomorphism $G \to K_k$.
- The *chromatic number* of $G = (V, E)$ is the smallest $k \in \mathbb{N}$ such that there exists a k-colouring of G.
- In other words, $\chi(G) = k$ is the smallest number of independent sets into which $V(G)$ can be partitioned.

Definitions

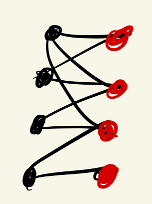
- The *chromatic number* of $G = (V, E)$ is the smallest $k \in \mathbb{N}$ such that there exists a k-colouring of G.
- In other words, $\chi(G) = k$ is the smallest number of independent sets into which $V(G)$ can be partitioned.

Examples

- $\chi(K_n) = n$.

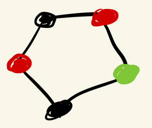


- $\chi(G) = 2$ if and only if G is bipartite.

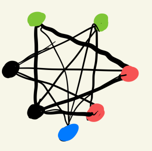


Examples

- $$\omega(C_n) = 2 \qquad \chi(C_n) = \begin{cases} 2 & n \text{ even} \\ 3 & n \text{ odd} \end{cases}.$$



- $$\omega(\bar{C}_n) = \left\lfloor \frac{n}{2} \right\rfloor \qquad \chi(\bar{C}_n) = \left\lceil \frac{n}{2} \right\rceil.$$



Lower bounds

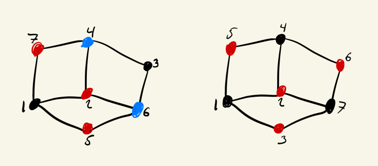
- $\omega(G) \leq \chi(G)$
  - Proof: Pairwise connected vertices need different colours.
  - Strict inequality for odd cycles and odd cocycles of length ≥ 5.
- $\chi(H) \leq \chi(G)$ if $H \subseteq G$ is a subgraph.
  - Proof: Any colouring of G restricts to a colouring of H.
- $\frac{|V(G)|}{\alpha(G)} \leq \chi(G)$.
  - Proof: $V(G)$ is the union of $\chi(G)$ colour classes of size $\leq \alpha(G)$.

Greedy colouring

- Order $V(G) = \{v_1, \ldots, v_n\}$ arbitrarily.
- For $i = 1, \ldots, n$: Let

$$\gamma(v_i) = \min\{c \in \mathbb{N} : \gamma(v_j) \neq c \text{ for all } 1 \leq j < i, v_j \in N(v_i)\}.$$

- Then γ is a proper colouring of G.



- For every vertex, there are at most $\Delta(G)$ forbidden colours.

Brooks' Theorem

- Any colouring gives an upper bound on $\chi(G)$.
- Greedy colouring shows $\chi(G) \leq \Delta(G) + 1$.

Theorem (Brooks, 1941)

If $\chi(G) = \Delta(G) + 1$ if and only if G is complete or an odd cycle.

- Proof: Clever vertex ordering + greedy colouring.

Greedy colouring

- Order $V(G)$ such that $d(v_n) = \delta(G)$, and recursively such that v_i has minimum degree in $G \setminus \{v_{i+1}, \dots, v_n\}$.
- Then the greedy colouring gives

$$\gamma(v_i) \leq \delta(G[v_1, \dots, v_i]) + 1.$$

- So

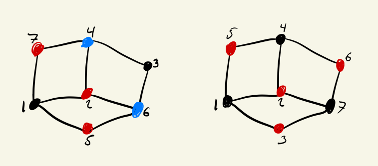
$$\chi(G) \leq \max_{H \subseteq G} \delta(H) + 1.$$

Upper bounds

- If $\chi(G) = k$, then for any k-colouring there must be at least one edge between every pair of colour classes.
- Thus
$$\binom{k}{2} \leq |E(G)| = m,$$
so
$$\chi(G) \leq \frac{1 + \sqrt{8m+1}}{2}.$$

Greedy colouring

- The greedy algorithm can be arbitrarily bad, depending on the vertex ordering.



- However, there exists a vertex ordering on which the greedy algorithm uses only $\chi(G)$ colours.
- So if we can perform the greedy algorithm for all possible orderings of V, we can compute the chromatic number *exactly*.
- But there are $n!$ possible ways to order V, so this is not an efficient algorithm.

Greedy algorithm

Theorem

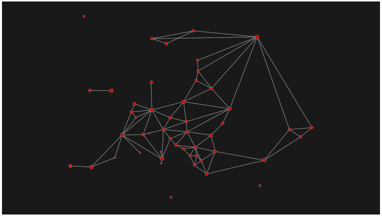
- *There exists a vertex ordering of $V(G)$ on which the greedy algorithm uses only $\chi(G)$ colours.*

Proof.

- Let $\gamma : V \to \{1, 2, \ldots, k\}$ be some k-colouring of G.
- Let V_i be the independent set $V_i = \{v \in V(G) : \gamma(v) = i\} \subseteq V$.
- Order the vertices such that all nodes in V_1 come first, then all nodes in V_2, and so on.
- Let $\delta : V \to \{1, 2, \ldots, k\}$ be a greedy graph colouring with respect to this ordering.
- By induction: $\delta(v) \leq i$ for all $v \in V_i$, so δ uses $\leq k$ colours. □

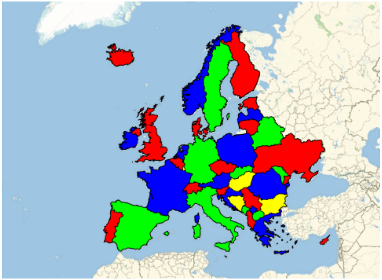
The Four Colour Theorem

- Colouring a plane graph $\leftrightsquigarrow$ Colouring a political map (with connected countries), such that neighbouring countries can be distinguished by their colours.



The Four Colour Theorem

- K_4 is planar (Luxembourg, Germany, France, Belgium), so at least four colours are needed to colour all planar maps.



- K_5 is not planar, but maybe we could need five colours anyway?

The Four Colour Theorem

Theorem (Apple, Haken, 1976)

- *Any planar graph G satisfies* $\chi(G) \leq 4$

- Proof by decomposition via extensive computer search.
- Enough to prove for 5-regular graphs.
- Computer aided colouring of > 1000 "reducible configurations" of > 100 vertices each.

The Four Colour Theorem

- Any planar graph G satisfies $\chi(G) \leq 4$
- Any graph that can be drawn without edge crossings on...
  - the torus satisfies $\chi(G) \leq 7$.
  - a Klein bottle satisfies $\chi(G) \leq 6$.
  - an orientable surface of genus g satsfies

$$\chi(G) \leq \left\lfloor \frac{7 + \sqrt{1 + 48g}}{2} \right\rfloor .$$

  - a non-orientable surface of genus k satisfies

$$\chi(G) \leq \left\lfloor \frac{7 + \sqrt{1 + 24k}}{2} \right\rfloor .$$

The Five Colour Theorem

- The following proof of the weaker *five colour theorem "almost" proves the four colour theorem.*
- *Remarkably (?) it uses geometric properties of plane graphs, rather than Kuratowski's theorem.*

Theorem (Heawood, 1890)

- *Any planar graph G satisfies* $\chi(G) \leq 5$.

The Five Colour Theorem

Theorem (Heawood, 1890)

- *Any planar graph G satisfies $\chi(G) \leq 5$.*

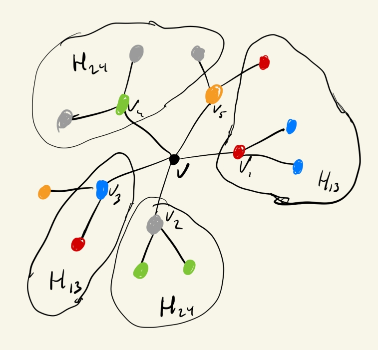
Proof.

- Average degree < 6, so choose a vertex v with degree ≤ 5.
- Enough to show that $G \setminus v$ can be 5-coloured such that only 4 colours are used on $N(v)$.
- Assume not, and fix a plane drawing of G and a 5-colouring of $G \setminus v$.

The Five Colour Theorem

Proof.

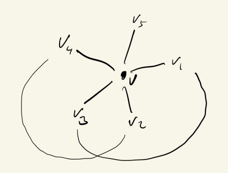
- WLOG, the neighbours of v are coloured $1, \ldots, 5$ in colockwise order.
- Let $H_{i,j}$ be the induced subgraph on the colour classes i, j.



The Five Colour Theorem

Proof.

- Every $v_1 - v_3$-path in $G \setminus v$ intersects with every $v_2 - v_4$-path in $G \setminus v$.

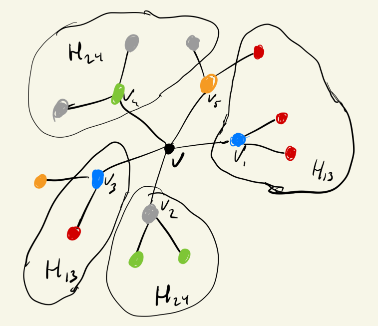


- But $H_{13} \cap H_{24} = \emptyset$.
- So either v_1 and v_3 are in different components of H_{13}, or v_2 and v_4 are in different components of H_{24}.

The Five Colour Theorem

Proof.

- Assume WLOG that v_1 and v_3 are in different components of H_{13}.
- We can swap the colours on the component of H_{13} containing v_1.
- After this, colour 1 is no longer used on $N(v)$. □



Motivation

- Some (many) graphs can not be k-coloured, although they have no $k - cliques$
- Erdös theorem (next week) says that χ is a "global" invariant.
- A graph can look like a tree within an arbitrarily large radius, but still have arbitrarily large chromatic number.
- We want to define a class of graphs where all obstacles to colouring are purely "local".

Perfect graphs

- G is *perfect* if $\chi(H) = \omega(H)$ for any induced subgraph $H \subseteq G$.

Example

- Complete graphs have $\omega(K_n) = n = \chi(K_n)$.
- Bipartite graphs have

$$\omega(G) = \chi(G) = \begin{cases} 2 & \text{if } E(G) \neq \emptyset \\ 1 & \text{if } E(G) = \emptyset \end{cases}$$

- Induced subgraphs of complete graphs are complete, and induced subgraphs of bipartite graphs are bipartite, so all such graphs are perfect.

Comparability graphs

- A poset is a (finite) set with an order relation $\leq$ (reflexive, antisymmetric, transitive).
- The *comparability graph* of a poset $(P, \leq)$ is

$$(P, E) \text{ where } xy \in E \text{ whenever } x \leq y.$$

- (In other words, it is the undirected version of the transitive closure of the Hasse diagram of P.)

Theorem

Comparability graphs of finite posets are perfect.

Proof.

Blackboard. □

Replicating

- If $v \in V(G)$, then G' is obtained from G by *replicating* v if

$$V(G') = V(G) \cup \{v'\}$$
$$E(G') = E(G) \cup \{uv' : uv \in E(G)\} \cup \{vv'\}.$$

Replicating

Theorem

- *Assume G is perfect and $v \in V(G)$.*
- *If G' is obtained from G by replicating v, then G' is also perfect.*

Combining perfect graphs

Theorem

- *Assume G and H are perfect graphs.*
- *If $G \cap H$ is a clique, then $G \cup H$ is perfect.*

Chordal graphs

- The class of *chordal graphs* is defined inductively as follows:
- Complete graphs are chordal.
- If G and H are chordal and $G \cap H$ is a clique, then $G \cup H$ is chordal.

Corollary

- *Chordal graphs are perfect.*

Strong Perfect Graph Theorem

- Graphs whose only induced cycles are C_3 are chordal, so perfect.
- Graphs whose only induced cycles are even are bipartite, so perfect.
- Graphs that have some odd induced cycle C_{2k+1}, $k \geq 2$, are *not* perfect.
- What about graphs that have induced even cycles *and* triangles?

Strong Perfect Graph Theorem

Theorem (Chudnovsky, Robertson, Seymour, Thomas, 2006)

- *G is perfect if and only if G has no induced subgraph C_n or $\bar{C}_n$ fr $n \geq 5$ odd.*

- $\Rightarrow$: Trivial, because $\omega(C_n) < \chi(C_n)$ and $\omega(\bar{C}_n) < \chi(\bar{C}_n)$ for odd $n \geq 5$.
- $\Leftarrow$: Extremely difficult. Proof uses technically complicated recursive constructions of all Berge graphs.
  - *Berge graphs* are the pre-SPGT name for graphs that have no induced subgraph C_n or $\bar{C}_n$ fr $n \geq 5$ odd.

Weak Perfect Graph Theorem

Theorem (Lovázs, 1972)

- *G is perfect if and only if $\bar{G}$ is perfect.*

- Clearly, SPGT implies WPGT.
- We prove WPGT as a corollary of the following characterization of perfect graphs.

Proposition

- *G is perfect if and only if*

$$\omega(H)\alpha(H) \geq n$$

for all induced subgraphs $H \subseteq G$.

Weak Perfect Graph Theorem

Proposition

- *G is perfect if and only if*

$$\omega(H)\alpha(H) \geq n$$

for all induced subgraphs $H \subseteq G$.

Proof.

- Blackboard. □

Random graphs

- Two reasons to study random graphs:
- To know what a "typical" graph looks like.
- Existence proofs (via The Probabilistic Method).

G(n,p): definition

- For fixed $n \in \mathbb{N}$, $p \in [0,1]$, we construct a probability space $G(n,p)$ of simple graphs with n vertices.
- $|V| = n$ fixed, $E \subseteq \binom{V}{2}$ random.
- For $S \subseteq \binom{V}{2}$, $\mathbb{P}(E = S) = p^{|S|}(1-p)^{\binom{n}{2}-|S|}$.
- Easy to check: The events $\{e \in E\}$ are independent for different edges e.

G(n,p): basic properties

- Sample G $G(n,p)$.
- By the union bound:

$$\mathbb{P}(\alpha(G) \geq k) \leq \binom{n}{k}(1-p)^{\binom{k}{2}}$$

and

$$\mathbb{P}(\omega(G) \geq k) \leq \binom{n}{k}p^{\binom{k}{2}}.$$

Basic probability theory

- Often it is easier to deal with expected values than with probabilities directly.
- Expected values of random variables can be manipulated by linearity.
- Example:

$$\mathbb{E}(\#K_k \subseteq G) = \sum_{K \in \binom{V}{k}} \mathbb{P}(K \text{ clique in } G) = \binom{n}{k} p^{\binom{k}{2}}$$

Counting cycles

- Let $k \geq 3$.
- $$\mathbb{E}(\#k\text{-cycles in } G) = \frac{n!}{2k(n-k)!}p^k$$
- Indeed, there are $\frac{n!}{2k(n-k)!}$ k-cycles in K_n.
- Each of these is a cycle in G with probability p^k.

Large random graphs

- Often, it makes sense to consider random graphs $G(n, p)$ where $n \to \infty$, and $p = p(n)$ is allowed to depend on n.
- Average degree $\approx \frac{p}{n-1}$.
- If p is (approximately) constant, we call the graph sequence *dense*, if $p = O(\frac{1}{n})$, then we call it *sparse*.
- Another frequently useful regime is $p \approx \frac{\log n}{n}$.

Erdös Theorem

- We are ready to prove Erdös's theorem.

Theorem

- *For all integers $k, \ell \in \mathbb{N}$, there exists a graph G with girth $> \ell$ and chromatic number k.*

- Moral: Chromatic number is a fundamentally global invariant.
- A graph can look like a tree within a radius $\frac{\ell}{2}$ from any vertex, and still have arbitrarily high chromatic number.

Erdös Theorem

Theorem

- *For all integers $k, \ell \in \mathbb{N}$, there exists a graph G with girth $> \ell$ and chromatic number k.*

- We will use random graphs to prove this, but the random graphs themselves do not have this property.
- Rather, random graphs with suitably chosen p have high chromatic number, and *not too many* cycles of length $< \ell$.
- So small modifications of random graphs yield the desired example.

Chromatic numbers of dense graphs

Theorem

- *Fix* $p \in (0,1)$, *and let* $G \sim G(n,p)$.
- *Let* $\chi_{n,p} = \frac{\log(1-p)^{-1}}{2} \frac{n}{\log n}$.
- *Fix* $\epsilon > 0$. *Then asymptotically almost surely,*

$$\mathbb{P}(\chi(G) \in [(1-\epsilon)\chi_{n,p}, (1+\epsilon)\chi_{n,p}]) \to 1 \text{ as } n \to \infty.$$

- We prove only the lower bound on the chromatic number: $\chi(G) > (1-\epsilon)\chi_{n,p}$ asymptotically almost surely.

Almost sure properties of dense graphs.

- For $i, j \in \mathbb{N}$, let $P_{i,j}$ be the following graph property:
- For every two sets $A, B \subseteq V$ with $A \cap B = \emptyset$, $|A| = i$, $B = j$, there exists $v \in V$ such that

$$A \subseteq N(v) \text{ and } B \cap N(v) = \emptyset.$$

- For example, $P_{1,1}$ is the property that no two vertices have the same neighbourhood.

Lemma

- *Fix $p \in (0, 1)$ and $i, j \in \mathbb{N}$.*
- *With probability $\to 1$ as $n \to \infty$, the graph $G \sim G(n, p)$ has property $P_{i,j}$.*

Almost sure properties of dense graphs.

Corollary

- *Fix* $p \in (0,1)$ *and* $k \in \mathbb{N}$.
- *With probability* $\to 1$ *as* $n \to \infty$, *the graph* $G \sim G(n,p)$ *is* k*-connected.*

$G(\aleph_0, p)$.

- We can construct a probability measure on graphs with a countable vertex set, just like we did for $G(n, p)$.
- This "countable random graph" has the property $P_{i,j}$ almost surely, for all i, j.
- But there is a *unique* countable graph (up to isomorphism) that has all these properties at once. This is the Rado graph.
- So this random graph is uniquely determined up to isomorphism, with probability one!

$G(\aleph_0, p)$.

Theorem

- *Let G and H be graphs with countable vertex sets.*
- *Assume that both G and H have property $P_{i,j}$ for all i, j.*
- *Then $G \cong H$.*

Proof.

- Let $V(G) = v_1, v_2, \dots$. Construct $\phi : G \to H$ recursively.
- Let $\phi(v_1) \in V(H)$ be arbitrary.
- Recursively, let $V_k = \{v_1, \dots, v_{k-1}\}$, and $N(v_k) \cap V_k = U_k$
- By property $P_{i,j}$, there exists $w \in V(H)$ such that

$$\forall x \in \phi(U_k) : xw \in E \text{ and } \forall x \in \phi(V_k \setminus U_k) : xw \notin E.$$

- Define $\phi(v_k) = w$. Then $\phi : G \to H$ is an isomorphism. □

Rado graph

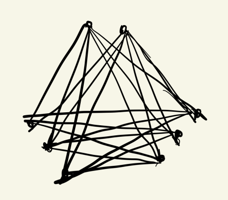
- Let $V = \mathbb{Z}_+$.
- For $x < y$, let $xy \in E$ if and only if the x:th last digit in the binary extension of y is 1.
- $N(1) = \{3, 5, 7, 9, \dots\}$.
- $N(2) = \{3, 6, 7, 10, 11, \dots\}$.
- $N(3) = \{4, 5, 6, 7, 12, 13, 14, 15, \dots\}$.
- Call $G = (V, E)$ the Rado graph.
- It has property $P_{i,j}$ for all i, j.
- So *up to isomorphism*, $G(\aleph_0, p)$ is the Rado graph with probability one.

Guiding questions

- How many edges can G have, if $|V(G)| = n$ and $\omega(G) < k$?
- How many edges can G have, if $|V(G)| = n$ and $\chi(G) < k$?
- How many edges can G have, if $|V(G)| = n$ and G has no subgraph isomorphic to H?
- How many vertices can G have, if G has no subgraph isomorphic to H_1 *and* $\bar{G}$ has no subgraph isomorphic to H_2?
- How many edges can G have, if $|V(G)| = n$ and G has no *minor* isomorphic to H?

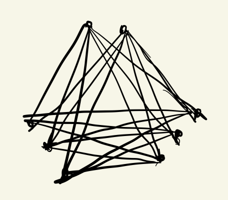
Turán's Theorem

- How many edges can G have, if $|V(G)| = n$ and $\omega(G) < k$?
- How many edges can G have, if $|V(G)| = n$ and $\chi(G) < k$?
- Remarkably (?) the answers to these two questions are the same.
- Let $T_r(n)$ be the complete r-partite graph with all parts the same size ± 1, and $t_r(n) = |E(T_r(n))|$.



Turán's Theorem

- Let $T_r(n)$ be the complete r-partite graph with all parts the same size ± 1, and $t_r(n) = |E(T_r(n))|$.



- Clearly, $\chi(T_r(n)) = \omega(T_r(n)) = r$.

Theorem

- *Any graph with n vertices and $> t_r(n)$ edges contains a clique of size $r + 1$.*

Turán's Theorem

Theorem

- *Any graph with n vertices and $> t_r(n)$ edges contains a clique of size $r + 1$.*

Proof.

- By induction on r.
- Consider an edge-maximal G without K_{r+1}.
- Consider $H = G \setminus Q$ where $Q \cong K_r$.

Turán's Theorem

Theorem

- *Any graph with n vertices and $> t_r(n)$ edges contains a clique of size $r+1$.*

Proof.

$$
\begin{aligned}
|E(G)| &= \#(Q-Q)\text{-edges} + \#(Q-H)\text{-edges} + \#(H-H)\text{-edges} \\
&\overset{I.H.}{\leq} \binom{r}{2} + (n-r)(r-1) + t_r(n-r) \\
&= t_r(n).
\end{aligned}
$$

□

Szemeredi's regularity lemma: paraphrazing

- "All really large graphs on M nodes, can be approximated by random graphs constructed as follows:
- Subdivide the M vertices into $k \leq M$ parts $V_1, \dots V_k$.
- For $v_i \in V_i$, $v_j \in V_j$, assign $v_i v_j \in E$ with probability p_{ij}."

ϵ-regular pairs

- For $A, B \subseteq V(G)$ with $A \cap B = \emptyset$, let

$$d(A,B) = \frac{\#A - B\text{-edges}}{|A||B|} \in [0,1].$$

- A, B is an ϵ-regular pair if, for all

$$X \subseteq A, Y \subseteq B \text{ with } |X| > \epsilon|A| \text{ and } |Y| > \epsilon|B|$$

it holds that

$$|d(X,Y) - d(A,B)| < \epsilon.$$

ϵ-regular partitions

- Fix $\epsilon > 0$. A partitioning $V(G) = V_0 \sqcup V_1 \sqcup \cdots \sqcup V_k$ is ϵ-regular if:
    - $|V_1| = |V_2| = \cdots = |V_k|$.
    - $|V_0| < \epsilon|V|$.
    - The number of *not* ϵ-regular pairs amonng $V_1, \ldots, V_k$ is $< \epsilon k^2$.

Szemeredi's regularity lemma

Theorem

- *For all $\epsilon > 0$ and all m, there exists M such that:*
- *Every graph G admits an ϵ-regular partition into k parts with $m < k < M$.*

- Proof strategy: start with an arbitrary partition into m parts.
- For each *not* ϵ-regular pair V, U in the partition, subdivide both U and V into two parts.
- Choose a common refinement of such subdivisions. We now have a partition into 2^{m-1} parts.

Szemeredi's regularity lemma

- Choose a common refinement of such subdivisions. We now have a partition into 2^{m-1} parts.
- Show that the *potential* q of the partition has now increased by at least ϵ^5, where

$$q(V_1, \ldots V_k) = \sum_{i,j} \frac{|V_i||V_j|}{|V|^2} d^2(V_i, V_j).$$

- The potential is increasing under refinement, and satisfies $0 < q < 1$, so the "algorithm" terminates after at most ϵ^{-5} refinements.
- The number of parts is thus bounded from above by $M = M(m, \epsilon)$.

Regularity graphs

■

Erdös-Stone's theorem

Theorem (Erdös, Stone, 1946)

- *For every $2 \leq r \leq m$, $\gamma > 0$, there exists an integer N such that every graph with $n \geq N$ vertices and at least $t_{r-1}(n) + \gamma n^2$ edges contains $T_r(m)$ as a subgraph.*

Lemma (Paraphrased)

- *If G contains R as a regularity graph with critical edge density $d > 0$ and $|G|/|R| \geq 2s/d^\Delta$, then every subgraph $H \subseteq R_s$ with maximal degree $< \Delta$ is also a subgraph of G.*

Erdös-Stone's theorem

Theorem (Erdös, Stone, 1946)

- *For every $2 \leq r \leq m$, $\gamma > 0$, there exists an integer N such that every graph with $n \geq N$ vertices and at least $t_{r-1}(n) + \gamma n^2$ edges contains $T_r(m)$ as a subgraph.*

Sketch.

- Consider an ϵ-regular partition into $> 1/\gamma$ parts, and a regularity graph with critical edge density γ.
- This regularity graph has n' vertices and $> t_{r-1}(n')$ edges, so contains a K_r subgraph.
- This yields a $T_r(m)$ subgraph in R_s (where $s = m/n'$), so also in G.

$\square$

Erdös-Stone's theorem

Theorem

- *For every* $2 \leq r \leq m$, $\gamma > 0$, *there exists an integer* N *such that every graph with* $n \geq N$ *vertices and at least* $t_{r-1}(n) + \gamma n^2$ *edges contains* $T_r(m)$ *as a subgraph.*

- So morally, *all* graphs with large enough size and edge density

$$> \frac{t_{r-1}(n)}{n} + \gamma \approx \frac{r-2}{1} + \gamma$$

contains *all* r-colourable graphs as subgraphs.

Exclusion numbers

- For $n \in \mathbb{N}$ and a graph H, let $\mathrm{ex}(n, H)$ be the largest number of edges in an n vertex graph with no H subgraph.
- In particular, $\mathrm{ex}(n, K_r) = t_{r-1}(n)$

Corollary

- *For every graph H,*

$$\lim_{n\to\infty} \frac{\mathrm{ex}(n, H)}{\binom{n}{2}} = \frac{\chi(H) - 2}{\chi(H) - 1}.$$

Exclusion numbers

Corollary

- *For every graph H,*

$$\lim_{n\to\infty} \frac{\mathrm{ex}(n,H)}{\binom{n}{2}} = \frac{\chi(H)-2}{\chi(H)-1}.$$

- So

$$\mathrm{ex}(n,H) = \begin{cases} \Theta(n^2) & \text{if } H \text{ not bipartite} \\ o(n^2) & \text{if } H \text{ bipartite} \end{cases}$$

Exclusion numbers

Corollary

- *For every graph H,*

$$\lim_{n\to\infty} \frac{\mathrm{ex}(n,H)}{\binom{n}{2}} = \frac{\chi(H)-2}{\chi(H)-1}.$$

Proof.

- Let $\chi(H) = r$.
- $H \not\subseteq T_{r-1}(n)$ for all n but $H \subseteq T_r(m)$ for large enough$m$.
- So

$$t_{r-1}(n) \le \mathrm{ex}(n,H) \le \mathrm{ex}(n, T_r(m)).$$

Exclusion numbers

Corollary

- *For every graph H,*

$$\lim_{n\to\infty} \frac{\mathrm{ex}(n,H)}{\binom{n}{2}} = \frac{\chi(H)-2}{\chi(H)-1}.$$

Continued.

- Erdös-Stone:

$$t_{r-1}(n) \leq \mathrm{ex}(n,H) \leq \mathrm{ex}(n,T_r(m)) = t_{r-1}(n) + o(n^2).$$

- $$\frac{r-2}{r-1} \leftarrow \frac{t_{r-1}(n)}{\binom{n}{2}} \leq \frac{\mathrm{ex}(n,H)}{\binom{n}{2}} \leq \frac{t_{r-1}(n)+o(n^2)}{\binom{n}{2}} \to \frac{r-2}{r-1}. \qquad \square$$

Exclusion numbers

- What is the growth rate of $\text{ex}(n, H)$ for bipartite graphs?

Theorem

- $$c_1 n^{2-\frac{2}{r-1}} \leq \text{ex}(n, K_{r,r}) \leq c_2 n^{2-\frac{1}{r}}$$

  for some universal constants c_1, c_2.

Conjecture (Erdös-Soós)

- *For any tree T with k edges,* $\text{ex}(n, T) = \frac{n(k-1)}{2}$.

Ramsey