Aalto University
School of Science

MS-E2114 Investment Science
Lecture 1: Cash flow analysis

Ahti Salo

Systems Analysis Laboratory
Department of Mathematics and System Analysis
Aalto University, School of Science

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Overview

Investment science

Investment types

Cash flow models of investments

Time value of money

Interest

Inflation

Present value and future value

Internal rate of return

Examples

Overview

What is investment science?

- **Investment** = Commitment of current resources with the aim of receiving later benefits
- Examples
  - Study this course to learn and to earn credits
  - Construct a new factory for profitable growth in the future
  - Purchase shares in company X to receive dividends and to profit from possibly increasing share price
- **Investor** = A person or an organization who is responsible for making the investment decision
  - Resources need not be owned by the investor - most fund managers manage other people's money
- **Investment science** = Consists of the scientific principles, theory and methods which assist investors in making investment decisions

Financial investments

- **Asset** = Any resource owned or controlled by a business or a legal economic entity
- **Financial asset** = A non-physical asset whose value is derived from a contractual claim
  - E.g., bank deposits, bonds, companies' share capital
  - Economic value of financial assets is measured in monetary terms (=money)
  - Money is a generally accepted medium of exchange, a measure of value, or a means of payment
- **Financial investment** = Purchase of an asset which either
  1. generates future income or
  2. can appreciate in value (so that it can possibly be sold at a profit later on)
- This course is focused on financial investments

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Some examples of investments

1. Bank deposit
   - Deposit 80 000 € for one year at the 0.25% interest rate
2. Saving for future pension
   - Pay an extra 200 € per month for 15 years
   - Get an extra 250 € addition to your monthly pension starting at the age of 65
3. Stock investment
   - Buy shares of a stock at the current price 2000 €
   - Sell once the stock has (hopefully) appreciated to 2400 €
   - Make a profit of 400 € per share

Different types of investments

- Single asset – Multiple assets
  - Buy a single stock vs. a portfolio of many stocks
  - Cf. actively managed mutual funds vs. passive index funds
- Deterministic value – Random value
  - Fixed interest rate investments vs. investments with uncertain future value
- Liquid market – Illiquid market
  - Exchange traded asset vs. illiquid assets (e.g., rare works of art)
- Single period – Multiperiod - Continuous
  - Either
    (i) decide now whether to invest or not
    vs.
    (ii) make an initial investment now and reserve an option to invest more either at 1) fixed time points or 2) continuously

Investing in markets

- Comparison principle
  - The attractiveness of an investment is determined by comparing it with other comparable investments
- Elimination of arbitrage opportunities
  - Type A: Positive initial cash flow followed by nonnegative payoff in all possible futures
  - Type B: No cost initially, nonnegative payoff in all futures and a strictly positive expected payoff
  - Competitive markets are usually assumed not to offer arbitrage opportunities (e.g, M. Friedman: *There's No Such Thing as a Free Lunch*, Open Court Publ. Company, 1975)
- Price dynamics
  - The price of an asset evolves dynamically according to a (stochastic) process
- Risk attitudes
  - Investors' have different risk attitudes: private persons tend to be more risk averse than institutional investors

Market price

- Assets are traded in financial markets
  - Freely $\approx$ Transaction costs and taxes are negligible
- Market price determined by supply and demand
  - **Bid price** = highest price that a buyer is willing to pay for an asset
  - **Ask price** = lowest price for which a seller is willing to sell an asset
- **No arbitrage** implies bid price $\leq$ ask price
- Comparison principle: If two assets have the same cash flows, then they must have the same price

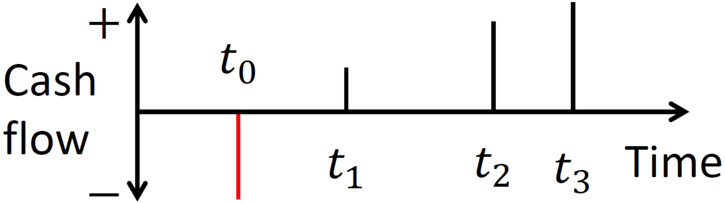
Important types of investment problems

1. Pricing
   - What is the right price of a financial asset in the light of all available information, based on the given assumptions?
2. Hedging
   - If the investor is exposed to financial risks, how should he or she invest to reduce these risks?
   - Insurance is one approach to hedging against risks
3. Portfolio optimization
   - Which portfolio consisting of available (financial) investment opportunities best matches the investor's risk-return preferences?

Overview

Cash flow models of investments

- The investment yields a **cash flow stream** $\mathbf{x} = (x_0, x_1, x_2, \dots, x_n)$
  - x_i = cash flow at time t_i, $t_i < t_j$ for all $i < j$
  - Negative cash flows $x_i < 0$ are <u>outflows</u>
    - E.g., expenses due to buying investments equipment
  - Positive cash flows $x_i > 0$ are <u>inflows</u>
    - E.g., revenues from selling assets



Overview

Time value of money

- In many cases, the sooner cash is available, the better
  - If the interest rate is positive, you can invest your cash now and get it back plus the accrued interest
  - Attractive unexpected investment opportunities may present themselves
  - Cash can be used for consumption
  - Even if you do nothing, you will not be worse off
    - Cf. normal savings accounts vs. fixed-term deposits: From the savings account one can usually freely withdraw cash anytime, but fixed-term deposits may involve a cancellation fee
- Consider alternatives A and B

  A. Receive $A = 100$ € now

  B. Receive $B = (100 + x)$ € after one year

  - The sum x for which your prefer these alternatives equally represents the time value of money

Time value of money

- In risk-free settings, the interest for risk-free investments typically represents the time value of money
- In general, the time value of money can be difficult to measure:
  - Inflation
  - Decreasing marginal utility
    - Preferences depend on one's current level of wealth
  - Risk aversion

Further reading: Frederick, S., Loewenstein, G., & O'Donoghue, T. (2002). Time discounting and time preference: A critical review. *Journal of Economic Literature*, 351-401.

Overview

Interest

- A simple loan for a single period:
  - Investor lends a **principal** A_0 (in €) for a single period at the end of which the investor receives $A_1 = A_0 + x$ (in €)
- **Interest** x can compensate, for example, for
  - lost time value of money to the investor
  - credit risk of the borrower
  - administrative expenses of lending
  - the profit margin required by the investor
- The quantity $r = x/A_0$ is the **interest rate**
- The interest rate of a loan with principal 100€ and interest 10€ is $r = 10\text{ €}/100\text{ €} = 10\%$

Compound interest

- Let A_n be the value of the loan in period $n = 0, 1, \ldots$
- **Simple interest** is paid only on the principal

$$A_n = A_0 + rnA_0 = (1 + rn)A_0$$

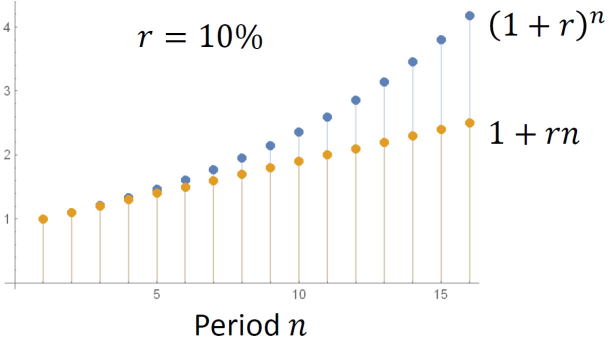
- **Compound interest** pays interest on interest as well

$$\begin{aligned} & A_n = (1 + r)A_{n-1} = (1 + r)^2 A_{n-2} = \cdots \\ \Rightarrow\ & A_n = (1 + r)^n A_0 \end{aligned}$$

  - The term comes from the fact that interest is *compounded* (=added to the principal) at the end of each period
  - If the interest is added to the principal of the loan rather than paid out to the investor, the interest is said to be *paid in kind* (PIK)

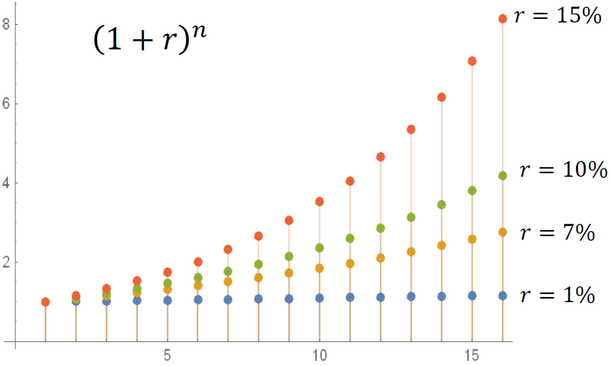
Compound interest

- Simple interest $\Rightarrow$ Linear growth
- Compound interest $\Rightarrow$ Geometric growth



Compound interest

- Rule of thumb: the value of the investment doubles in
  - about 10 periods when the interest rate is $r = 7\%$
  - about 7 periods when the interest rate is $r = 10\%$



Interest rates with different period lengths

- Interest rates are usually annualised
- $r' =$ interest rate for a single period of length $t = k/m,\ m, k \in \mathbb{N}$
  - In k years, the interest rate is r' compounded $m = k/t$ times
  - In k years, the one-year interest rate r is compounded k times
- The annual interest rate r which yields the same return as r' after k years is thus

$$(1 + r')^m = (1 + r)^k$$

- The rate $r > 0$ which solves this equation is the **annualised rate** of r', computed as the principal k-th root

$$r = (1 + r')^{m/k} - 1$$

Compounding frequency

- Interest is often stated as annual rates
  - Annual (=per year) nominal interest rate r (per annum, p.a.)
  - Compounded m times per year
  - Interest rate per each period is thus r/m
- **Effective rate** gives the equivalent interest with once-per-year compounding

$$1 + r_e = (1 + r/m)^m \Leftrightarrow r_e = (1 + r/m)^m - 1$$

- Example: One year with monthly interest compounded at r=6%

$$\Rightarrow k = 1, \quad t = \frac{1}{12}$$
$$\Rightarrow m = \frac{k}{t} = 12, \quad \frac{r}{m} = \frac{6\%}{12} = 0.5\%$$
$$\Rightarrow r_e = (1 + 0.005)^{12} - 1 \approx 6.17\%$$

Continuous compounding

- Continuous compounding $\equiv$ compounding frequency $m \to \infty$
- The **effective rate** for continuous compounding is

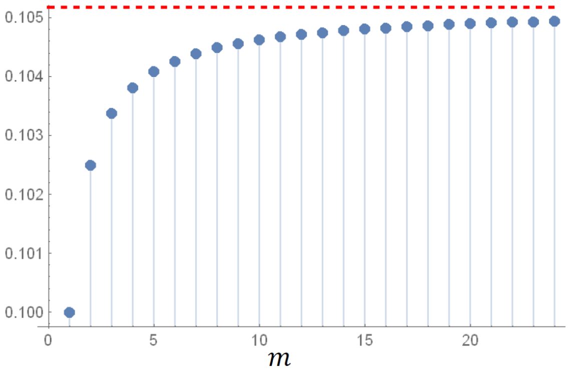
$$r_e = \lim_{m \to \infty} (1 + r/m)^m - 1$$
$$\Rightarrow r_e = e^r - 1$$

- **Example**: If the nominal rate with continuous compounding is 6%, the effective rate is

$$r_e = e^{0.06} - 1 \approx 6.18\%$$

Impact of compounding frequency

- Effective rate $r_e = (1 + r/m)^m$ with nominal rate $r = 10\%$ p.a. for various compounding frequencies
- Limit when $m \to \infty$ is $e^r - 1 \approx 0.105$



Growth of value with compound interest

- Nominal rate r, principal A_0
- Value growth when compounding m times per year
  - After k periods, the value is

$$A_k = \left(1 + \frac{r}{m}\right)^k A_0$$

- Value growth with continuous compounding
  - At time $t = k/m$, the value is $(1 + r/m)^{mt} A_0$, hence

$$A_t = \lim_{m \to \infty} \left(1 + \frac{r}{m}\right)^{mt} A_0 = e^{rt} A_0$$

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Inflation

- **Inflation** = Reduction of purchasing power due to higher prices
- **Inflation rate** f = %-annual price increase (for a given year)
  - $f = 2\% \Rightarrow$ an item that costs 1 € in 2022 will cost 1.02 € in 2023
  - Hence the **real value** of 1 € in 2022 is the same as $1/1.02 \approx 0.9804$ € in 2023
  - If you get a 5% interest on your investment, your purchasing power grows by the **real interest rate** r_0

$$
\begin{aligned}
1 + r_0 &= \frac{1+r}{1+f} \\
\Rightarrow r_0 &= \frac{r-f}{1+f} = \frac{3\%}{1.02} \approx 2.9\%
\end{aligned}
$$

Overview

Future value and ideal bank

- Suppose that the investor considers the investment and is able to deposit and borrow money at the risk-free interest rate
  - When the investor can deposit and borrow money at the same interest rate beyond limits, we say that there is an **ideal bank**
  - Interest rates for different maturities may differ from each other, but borrowing and lending have always the same interest rate
  - This convenient theoretical assumption greatly simplifies the analysis
- **Future value** = The amount of cash that the investment is worth to the investor at the chosen future time (normally the time of the last cash flow of the investment)

Future and present value in two periods

- **Future value (period 1) of a cash flow at period 0**: Value of a cash flow after accounting for interest

$$FV = (1 + r)A_0$$

- **Present value (period 0) of a cash flow at period 1**: Cash at present whose value after accounting for interest is A_1

$$PV = \frac{1}{1 + r}A_1 = dA_1$$

  - Discount factor $d = 1/(1 + r)$
  - Let $r = 8\%$, then $d = 1/1.08 \approx 0.93$
  - Let $A_1 = 100$ €, then $PV = dA_1 \approx 93$ €
  - Let $A_0 = 100$ €, then $FV = A_0/d \approx 108$ €

Present value and net present value

- Consider the cash flow stream $\mathbf{x} = (x_1, \ldots, x_n)$ with cash flows $x_1, x_2, \ldots, x_n$ at periods $1, 2, \ldots, n$
- **Present value of a cash flow stream x** is

$$
\begin{aligned}
PV &= x_0 + \frac{x_1}{1+r} + \cdots + \frac{x_n}{(1+r)^n} \\
&= x_0 + dx_1 + \cdots + d^n x_n
\end{aligned}
$$

  - $r =$ interest rate of a single period = discount rate
  - $d =$ discount factor of a single period
- **Net present value** (NPV) accounts for both positive and negative cash flows

Future value of a cash flow stream

- **Future value of a cash flow stream $\mathbf{x}$** at period n is

$$FV = (1+r)^n x_0 + (1+r)^{n-1} x_1 + \cdots + x_n$$

- Example:

$$\mathbf{x} = (-2, 1, 1, 1), \quad \text{period 1 year, annual interest rate } r = 10\%$$
$$FV = -2 \cdot 1.1^3 + 1 \cdot 1.1^2 + 1 \cdot 1.1 + 1 = 0.648$$
$$PV = -2 + \frac{1}{1.1} + \frac{1}{1.1^2} + \frac{1}{1.1^3} = 0.487$$

Main theorem on present value

Theorem

*(**Main theorem on present value**) The cash flow streams* $\mathbf{x} = (x_0, x_1, \ldots, x_n)$ *and* $\mathbf{y} = (y_0, y_1, \ldots, y_n)$ *are equivalent for an investor who has access to an ideal bank employing a constant interest rate* r *if and only if the present values of the two streams, evaluated at this interest rate, are equal.*

Proof outline:The cash flow $\mathbf{x}$ is equivalent to $(PV_{\mathbf{x}}, 0, \ldots, 0)$ and $\mathbf{y}$ is equivalent to $(PV_{\mathbf{y}}, 0, \ldots, 0)$. These are equal only if $PV_{\mathbf{x}} = PV_{\mathbf{y}}$ (otherwise, there would be an arbitrage opportunity).

- Implication: Investor only needs to compute PV, as the timing of cash flows can be adjusted through borrowing and lending operations with the ideal bank

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Definition: Two cash flow streams are said to be equivalent if and only if one cash flow stream can be transformed to the other one using the ideal bank.

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Internal rate of return

- **Internal rate of return (IRR)** of the cash flow stream $\mathbf{x} = (x_0, \ldots, x_n)$ is the interest rate r such that the present value of $\mathbf{x}$ is 0
- This r solves the equation

$$0 = x_0 + \frac{x_1}{1+r} + \frac{x_2}{(1+r)^2} + \cdots + \frac{x_n}{(1+r)^n}$$

- Equivalently, IRR is a number r satisfying $1/(1+r) = c$, where c satisfies the polynomial equation

$$0 = x_0 + x_1 c + x_2 c^2 + \cdots x_n c^n$$

Main theorem of internal rate of return

Theorem

*(**Main theorem of internal rate of return**) Assume that the cash flow stream* $(x_0, x_1, \ldots, x_n)$ *is such that* $x_0 < 0$ *and* $x_k \geq 0, k = 1, 2, \ldots, n$ *with a strict inequality for some* $k \geq 1$. *Then the equation*

$$0 = x_0 + x_1 c + x_2 c^2 + \cdots + x_n c^n$$

has a unique positive solution $c^* > 0$. *If* $\sum_{k=0}^{n} x_k > 0$, *then the corresponding internal rate of return* $r^* = (1/c^*) - 1$ *is positive.*

Proof: Let $f(c) = x_0 + x_1 c + \cdots + x_n c^n$. By assumption $f(0) = x_0 < 0$. Since $x_k \geq 0 \ \forall \ k \geq 1$ with a strict inequality for some $k \geq 1$, it follows that $f'(c) > 0, c > 0$ and $f(c) > 0$ for some large enough c. Thus, $f(c) = 0$ has a unique solution $c^* > 0$. If $\sum_{k=0}^{n} x_k > 0$, then $f(1) > 0$ and thus $c^* = 1/(1 + r^*) < 1 \Rightarrow r^* > 0$ □

NPV vs IRR

	Selection	Pros	Cons
NPV	Higher NPV preferred to lower NPV	Easy to compute Aims to maximize value	Discount rate must be defined (e.g. weighted average cost of capital; WACC) Insensitive to the size of the investment (an investment of 10 k€ and 10 M€ can have the same NPV)
IRR	Higher IRR preferred to lower IRR	Ranks by rate of return	Assumes that all cash flows can be reinvested at the same IRR Computation of IRR non-linear. IRR for a portfolio of investments is not a linear function of the IRRs of the individual investments' IRRs

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Ostridge farm

- A farmer herds ostridges for meat
  - His interest rate is $r = 0.1 = 10\%$
  - He can slaughter after one year for double the initial investment $\Rightarrow A = (-1, 2)$
  - Alternatively, he can slaughter after 2 years in return for triple the investment $\Rightarrow B = (-1, 0, 3)$

$$NPV_A = -1 + \frac{2}{1.1} = 0.82$$
$$NPV_B = -1 + \frac{3}{1.1^2} = 1.48 > NPV_A$$
$$IRR_A = 1.0, \text{ because } -1 + \frac{2}{1+1.0} = 0$$
$$IRR_B \approx 0.7, \text{ because } -1 + \frac{3}{(1+0.7)^2} = 0$$

- $NPV_A < NPV_B, IRR_A > IRR_B \Rightarrow$ Recommendations differ!

Ostridge farm

- The possibility of reinvesting profits in strategy A after one year was not considered in the IRR calculation
- If this profit can be reinvested, a more meaningful evaluation can be made by repeating the investments so that the time horizons match

Year	0	1	2	3	4
A	-1	2			
		-2	4		
			-4	8	
				-8	16
	-1				**16**

Year	0	1	2	3	4
B	-1		3		
			-3		9
	-1				**9**

Production machine

- A production facility uses a machine whose
  - procurement price is 10 k€
  - operating expenses are 2k€ in the first year and then 1k€ higher in each consecutive year
  - salvage value is 0 €
- Interest rate $r = 0.1 = 10\%$
- How often should a new machine be purchased?
- This is an example of a **cycle problem** of comparing recurring investments with different periods

Production machine

- Let $\mathbf{x} = (x_1, \ldots, x_k)$ be the cash flow stream from a single cycle
  - k is the length of cycle
  - Present value of one cycle is PV_k
  - Present value from the repetition of these cycles is PV
- After one cycle, the identical decision recurs at time k and thus

$$
\begin{aligned}
PV &= PV_k + \frac{1}{(1+r)^k} PV \\
\Rightarrow PV &= \frac{(1+r)^k}{(1+r)^k - 1} PV_k \\
&= C_k PV_k, \text{ where } C_k = \frac{(1+r)^k}{(1+r)^k - 1}
\end{aligned}
$$

Production machine

- Cash flows for different cycles length k shown below
  - PVs computed using `SUMPRODUCT`-routine of Excel
- The optimal strategy is to replace the machine every 5 years

k	0	1	2	3	4	5	6	7	8	9	C_k	PV_k	PV
1	-10	-2									11	-12	-130
2	-10	-2	-3								6	-14	-82
3	-10	-2	-3	-4							4	-17	-70
4	-10	-2	-3	-4	-5						3	-21	-65
5	-10	-2	-3	-4	-5	-6					3	-24	-64
6	-10	-2	-3	-4	-5	-6	-7				2	-28	-65
7	-10	-2	-3	-4	-5	-6	-7	-8			2	-32	-67
8	-10	-2	-3	-4	-5	-6	-7	-8	-9		2	-37	-69
9	-10	-2	-3	-4	-5	-6	-7	-8	-9	-10	2	-41	-71

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