Mathematics for Economists

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Vectors

Some preliminaries

Cartesian product of sets

- sets $X_1, X_2 \ldots, X_n$
- cartesian product of sets
  $X_1 \times X_2 \times \cdots \times X_n = \{(x_1, x_2, \ldots, x_n) : x_1 \in X_1, x_2 \in X_2, \ldots, x_n \in X_n\}$
- if $X_i = X$ for all $i = 1, \ldots, n$, then X^n denotes the Cartesian product of the sets

Example: deck of cards, ranks $= \{A, K, Q, J, 10, 9, 8, 7, 6, 5, 4, 3, 2\}$ and suits$= \{\heartsuit, \spadesuit, \clubsuit, \diamondsuit\}$

- both ranks$\times$suits and suits$\times$ranks correspond to the entire card deck

Euclidean spaces

- The n-dimensional Euclidean space is the set

$$\mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{n \text{ times}},$$

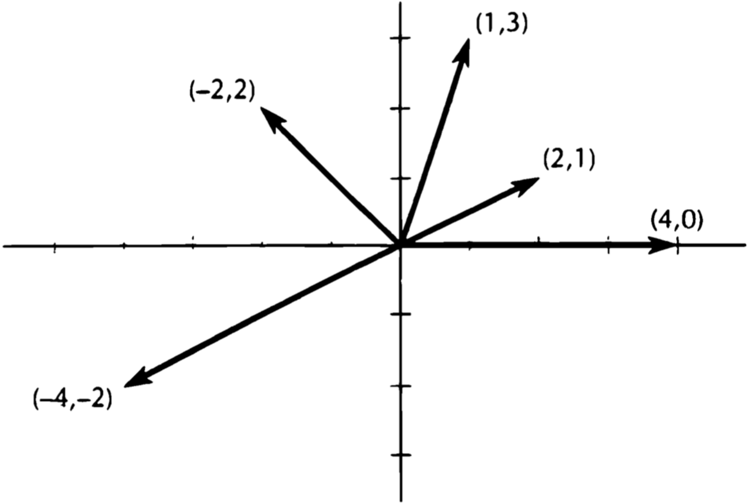
  where $n \geq 1$

- A point in $\mathbb{R}^n$ is an n-tuple $(x_1, \ldots, x_n)$ of real numbers (called coordinates)

- Note: A tuple is an *ordered* list. This means that, for example, $(1, 1, 2) \neq (2, 1, 1)$

Euclidean spaces

- Points in $\mathbb{R}^n$ can be interpreted as **vectors**



Notations for vectors

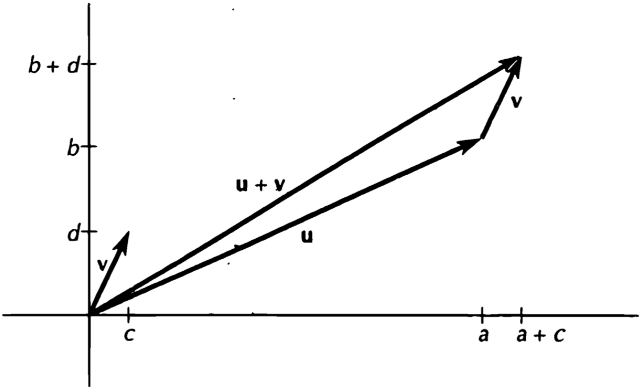
- $\boldsymbol{x} = \boldsymbol{y}$ means that $x_i = y_i$ for all $i = 1, \dots, n$
- $\boldsymbol{x} \geq \boldsymbol{y}$ means that $x_i \geq y_i$ for all $i = 1, \dots, n$
- but what about $>$ relation?
- during this course $\boldsymbol{x} \gg \boldsymbol{y}$ means that $x_i > y_i$ for all $i = 1, \dots, n$
- Example: $\boldsymbol{x} = (2a + 3b + 5c, a - 3c, 5b - 3c)$ and $\boldsymbol{y} = (10, -2, 2)$, $\boldsymbol{x} \geq \boldsymbol{y}$ corresponds to the system of inequalities

$$
\begin{array}{cccccc}
2a & +3b & +5c & \geq & 10 \\
a & & -3c & \geq & -2 \\
 & 5b & -3c & \geq & 2.
\end{array}
$$

Euclidean spaces

- **Addition** of vectors. If $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, \ldots, y_n)$ are vectors, then their sum is

$$\boldsymbol{x} + \boldsymbol{y} = (x_1 + y_1, \ldots, x_n + y_n)$$



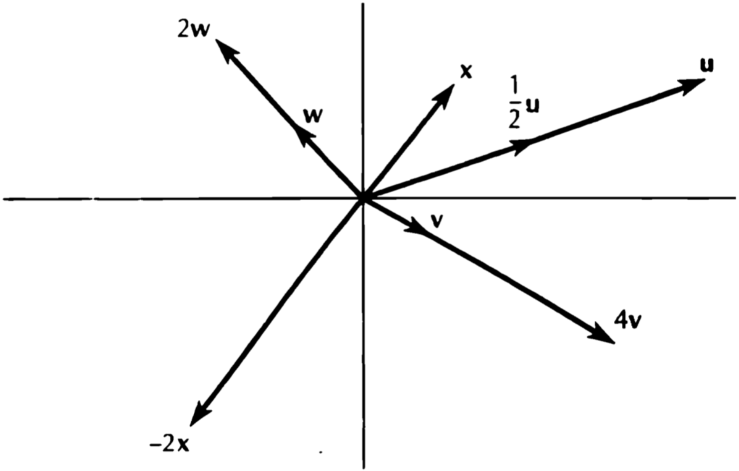
Euclidean spaces

- Vector addition satisfies:
  - Commutativity: $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{y} + \boldsymbol{x}$
  - Associativity: $\boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z}) = (\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z}$
  - Identity: $\boldsymbol{x} + \mathbf{0} = \boldsymbol{x}$, where $\mathbf{0}$ is the zero vector

Euclidean spaces

- **Scalar multiplication** of vectors. If r is a scalar and $\boldsymbol{x} = (x_1, \dots, x_n)$ is a vector, then their product is

$$r\boldsymbol{x} = (rx_1, \dots, rx_n)$$



Euclidean spaces

- The scalar multiplication of vectors satisfies the following distributive laws. For all scalars r, s and vectors $\boldsymbol{u}, \boldsymbol{v}$, we have

$$(r+s)\boldsymbol{u} = r\boldsymbol{u} + s\boldsymbol{u}$$
$$r(\boldsymbol{u}+\boldsymbol{v}) = r\boldsymbol{u} + r\boldsymbol{v}$$

Euclidean spaces

- The **norm** or **length** of a vector $\boldsymbol{x}$ is

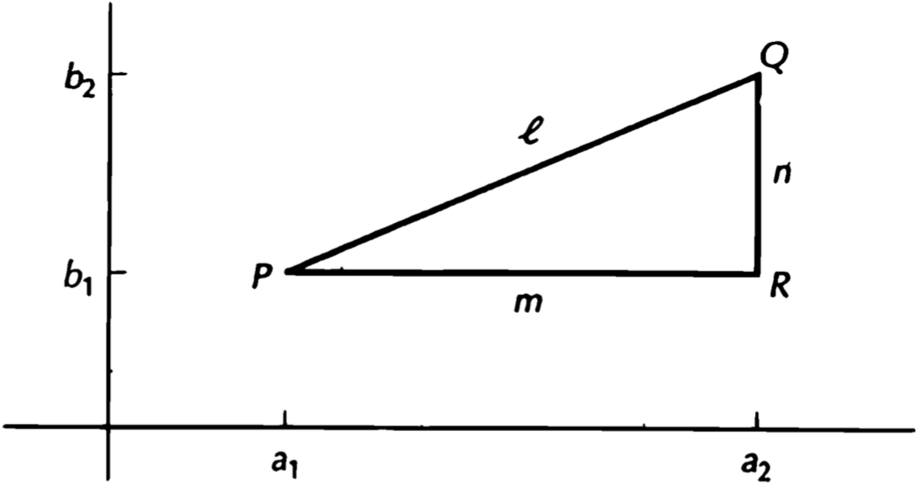
$$||\boldsymbol{x}|| = \sqrt{x_1^2 + x_2^2 + \cdots x_n^2}$$

- The Euclidean **distance** between two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ is

$$d(\boldsymbol{x}, \boldsymbol{y}) = ||\boldsymbol{x} - \boldsymbol{y}|| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}$$

Euclidean spaces

- Distance between P and Q in $\mathbb{R}^2$:



Euclidean spaces

- The **inner product** or **dot product** of two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ is

$$\boldsymbol{x} \cdot \boldsymbol{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

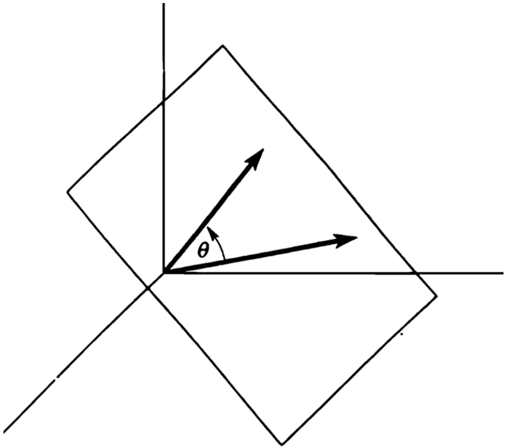
- Note that

$$||\boldsymbol{x}|| = \sqrt{\boldsymbol{x} \cdot \boldsymbol{x}}$$
$$||\boldsymbol{x} - \boldsymbol{y}|| = \sqrt{(\boldsymbol{x} - \boldsymbol{y}) \cdot (\boldsymbol{x} - \boldsymbol{y})}$$

Euclidean spaces

- For any two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, let θ be the angle between them. Then we have

$$\boldsymbol{x} \cdot \boldsymbol{y} = ||\boldsymbol{x}|| \, ||\boldsymbol{y}|| \;\cos\theta$$



- If $\boldsymbol{x} \cdot \boldsymbol{y} = 0$, the two vectors are orthogonal

Lines, planes, hyperplanes

- Line in $\mathbb{R}^2$ is a set of the form $\{\boldsymbol{x} \in \mathbb{R}^n : ax_1 + bx_2 = \alpha\}$, item note that in (x_1, x_2) plane we can write the equation of a line as $x_2 = -ax_1/b + \alpha/b$, slope is $-a/b$
- Plane in $\mathbb{R}^3$ is a set of the form $\{\boldsymbol{x} \in \mathbb{R}^3 : ax_1 + bx_2 + cx_3 = \alpha\}$
- What next?
- Hyperplane in $\mathbb{R}^n$ is a set of the form $\{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{p} \cdot \boldsymbol{x} = \alpha\}$, where $\boldsymbol{p}$ is the normal of the hyperplane
- Example: two planes $P_1 = \{\boldsymbol{x} \in \mathbb{R}^3 : x_3 = 0\}$ and $P_2 = \{\boldsymbol{x} \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\}$, What is the intersection $P_1 \cap P_2$?

Examples

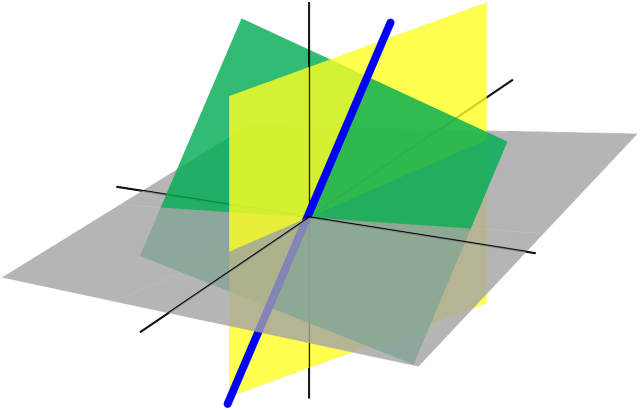
- What is the equation that determines the plane that has $(1, -1, 2)$ as its normal and passes through $(1, 2, 3)$?
- Assume that a line in x_1, x_2-plane has a parameterized presentation $x_1(t) = 1 + 2t$, $x_2(t) = -1 + 2t$, what would be the quation of the line in the form $ax_1 + bx_2 = c$?

Pairs of equations

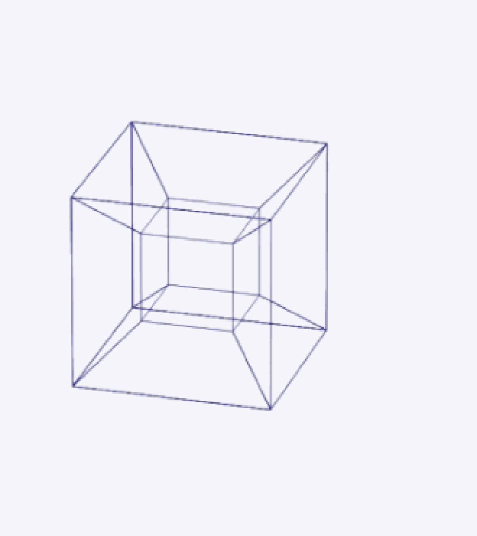
$$\begin{aligned} ax_1 + bx_2 &= e \\ cx_1 + dx_2 &= f \end{aligned}$$

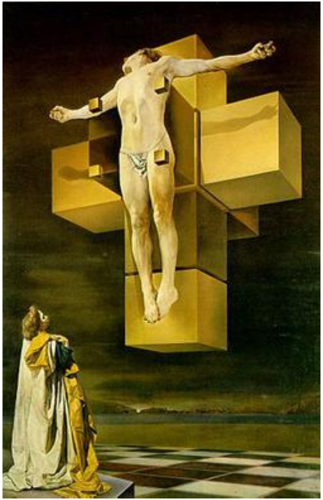
- Finding a solution means finding the point of intersection of two lines
- There is a solution whenever the lines are not parallel, or the two lines are the same line (in which case there is inifinitely many solutions)
- Can you formulate these conditions mathematically?
- What about higher dimensional cases?

Intersections of hyperplanes



Tesseract — hypercube in $\mathbb{R}^4$





Example

- Consumption bundle of n-goods, $\boldsymbol{x} = (x_1, \ldots, x_n)$, where x_i, $i = 1, \ldots, n$ are amounts consumed
- Price vector $\boldsymbol{p} = (p_1, \ldots, p_n)$
- Monetary value of the consumption bundle $p_1x_1 + p_2x_2 + \cdots + p_nx_n = \boldsymbol{p} \cdot \boldsymbol{x}$
- Monetary value of the consumption bundle $p_1x_1 + p_2x_2 + \cdots + p_nx_n = \boldsymbol{p} \cdot \boldsymbol{x}$
- Budget set $\{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{p} \cdot \boldsymbol{x} = w, \boldsymbol{x} \geq \mathbf{0}\}$, where w is the wealth
  - note: an intersection of hyperplanes

Linear independence

- Consider m vectors $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m$. We say that a vector $\boldsymbol{v}$ is a **linear combination** of $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m$ if there exist scalars $a_1, \ldots, a_m$ such that

$$\boldsymbol{v} = a_1 \boldsymbol{u}_1 + \cdots + a_m \boldsymbol{u}_m$$

Linear independence

- Suppose we want to express $\boldsymbol{v} = (3, 7, -4)$ as a linear combination of the three vectors

$$
\begin{aligned}
\boldsymbol{u}_1 &= (1, 2, 3) \\
\boldsymbol{u}_2 &= (2, 3, 7) \\
\boldsymbol{u}_3 &= (3, 5, 6)
\end{aligned}
$$

- We need to find three scalars a_1, a_2, a_3 such that $\boldsymbol{v} = a_1\boldsymbol{u}_1 + a_2\boldsymbol{u}_2 + a_3\boldsymbol{u}_3$
- That is, we need to solve the following system of linear equations:

$$
\begin{aligned}
a_1 + 2a_2 + 3a_3 &= 3 \\
2a_1 + 3a_2 + 5a_3 &= 7 \\
3a_1 + 7a_2 + 6a_3 &= -4
\end{aligned}
$$

- Verify that the unique solution is $a_1 = 2, a_2 = -4$, and $a_3 = 3$. Thus $\boldsymbol{v} = 2\boldsymbol{u}_1 - 4\boldsymbol{u}_2 + 3\boldsymbol{u}_3$

Linear independence

- Vectors $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m$ in $\mathbb{R}^n$ are **linearly dependent** if and only if there exist scalars $a_1, \ldots, a_m$, *not all zero*, such that

$$a_1 \boldsymbol{u}_1 + \cdots + a_m \boldsymbol{u}_m = \boldsymbol{0}.$$

- Vectors $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m$ in $\mathbb{R}^n$ are **linearly independent** if and only if

$$a_1 \boldsymbol{u}_1 + \cdots + a_m \boldsymbol{u}_m = \boldsymbol{0}$$

implies

$$a_1 = a_2 = \cdots = a_m = 0.$$

Linear independence

- Suppose one of the vectors $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m$ is equal to $\mathbf{0}$. Then the vectors must be linearly dependent
- Every nonzero vector u is, by itself, linearly independent
- If two of the vectors $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m$ are equal or one is a scalar multiple of the other, then the vectors must be linearly dependent
- Two vectors are linearly dependent if and only if one of them is a scalar multiple of the other
- If a set S of vectors is linearly independent, then any subset of S is linearly independent
- If a set S contains a subset of linearly dependent vectors, the S is linearly dependent
- The nonzero rows of a matrix in row echelon form are linearly independent