Mathematics for Economists
Aalto University – Fall 2022

Problem Set 1
Due date: Wednesday 21.9 at 13.15

Exercise 1

Consider a firm whose profits prior to taxes are 50000. The firm donates amount x to Red Cross. This amount equals 10% of firms' after tax profits. However, the donation is deducible from taxable profits. The taxes are composed of federal tax of 5%, and the profits after federal taxes are subject to state taxation of 40%. Let y be the amount of federal taxes paid and z be the amount of state level taxes paid. Formulate three equations for x, y, and z. Express the equations in a matrix form "Ax=b".

Exercise 2

Use Gauss-Jordan elimination to solve the following system of linear equations:

$$\begin{aligned} 2x + 3y - z &= 1 \\ 3x + 5y + 2z &= 8 \\ x - 2y - 3z &= -1. \end{aligned}$$

Exercise 3

Show that vectors $\mathbf{u}_1 = (1, 1, 1), \mathbf{u}_2 = (0, 1, 1)$ and $\mathbf{u}_3 = (0, 0, 1)$ are linearly independent. Present $\mathbf{u} = (-1, 2, 4)$ as a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$.

Exercise 4

Reduce the following 3×4 matrix to its row echelon form and calculate its rank.

$$\begin{bmatrix} 1 & 2 & -3 & 0 \\ 2 & 4 & -2 & 2 \\ 3 & 6 & -4 & 3 \end{bmatrix}$$

Exercise 5

Assume that a data on five individuals is given in the rows of the below matrix

$$\begin{pmatrix} 1 & 45 & 23 & 1 & 0 \\ 1 & 78 & 40 & 0 & 1 \\ 1 & 54 & 24 & 1 & 0 \\ 1 & 57 & 29 & 1 & 0 \\ 1 & 65 & 38 & 0 & 1 \end{pmatrix},$$

where the first column is constant corresponding to the intercept term b_0 in the below regression model, the second is the yearly income in thousands of euros, the third column is the age, the fourth one is a dummy for single households (takes value 1 if single household), and the last one is a dummy for non-single households.

a) What is the rank of the matrix? (You may use some computer program to find out the rank, but you should also argue why you could "see" that rank is not full)

b) You are asked to fit a model $y = b_0 + b_1x_1 + b_2x_2 + b_3x_3$, where y is the income, x_1 is the age, x_2 is the single-household-dummy, x_3 is the nonsingle-household-dummy. How would you modify the model and data to avoid *multicollinearity* (Multicollinearity=columns of the above data matrix being linearly dependent)?

Exercise 6

Let us consider a simple model of a closed economy: Y is GDP, T is the amount of taxes, G is government spending, I is the amount investments. As usual GDP has a decomposition $Y = C+I+G$, taxes are a share t of GDP $T = tY$, private consumption is $C = a(Y-T)+b$. Investments I and government spending G are exogenous. Note that t, a, b are exogenous, too.

a) Form a system of three linear equations for the (three) endogenous variables

b) Compute the determinant of the multiplier matrix of the system of equations of part a.

c) Use Cramer's rule to find out the equilibrium.

Some tips on doing linear algebra with Python

NumPy has a linear algebra module, which offers various methods on any numpy array (so youe need import NumPy e.g. "numpy import numpy as np").

Examples: Defining a matrix

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

as a NumPy array:

```
A=np.array([[2,-1,-0,0],[-1,2,-1,0],[0,-1,2,-1],[0,0,-1,2]])
```

Note: the rows are given as lists. Some commands (determinant, rank, inversion):

```
np.linalg.det(A)
np.linalg.matrix_rank(A)
np.linalg.inv(A)
```

Matrix product of A and B is "np.matmul(A,B)" and inner product of vectors is "np.dot(A,B)". Solving a linear equation $A\boldsymbol{x} = b$ happens by command "np.linalg.solve(A, b)".