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Differential and Integral Calculus 1

Exercises, Week 2

Work on Warm-up 1–4 during the exercise sessions of Week 2.
Submit on MyCourses your solutions for Homework 1 and 2 by Sunday, September 18th.

Warm-up 1: For each of the following functions, find its derivative in all points where the derivative exists. Use the definition of derivative as a limit.

$$(a) \quad f(x) = x^n \qquad\qquad (b) \quad g(x) = \ln(x),$$

where n can be any natural number.

Hint: (a) Recall that $x^n - x_0^n = (x - x_0)(x^{n-1} + x^{n-2}x_0 + x^{n-3}x_0^2 + \cdots + x^2x_0^{n-3} + xx_0^{n-2} + x_0^{n-1})$.
(b) Reduce to a famous limit by substitution.

Warm-up 2: Give an example or explain why an example does not exist:

1. A function $f\colon \mathbb{R} \to \mathbb{R}$ with exactly two points where f is not continuous and exactly three points where f is not differentiable.
2. A function $f\colon \mathbb{R} \to \mathbb{R}$ with exactly three points where f is not continuous and exactly two points where f is not differentiable.
3. A function $f\colon \mathbb{R} \to \mathbb{R}$ with infinitely many points where the derivative is 0 and infinitely many points where the derivative does not exist.

Warm-up 3: For each of the following functions, find its derivative in all points where the derivative exists. Use the differentiation rules and famous derivatives given in the lectures.

$$(a) \quad f(x) = \frac{-2}{(x+1)(\ln(x+1)+2)^2} \qquad\qquad (b) \quad g(x) = \frac{\ln(x)-1}{\ln(x)^2}.$$

Warm-up 4: Consider the function f defined as follows:

$$f(x) = \frac{\ln(x+1)}{-2-\ln(x+1)}.$$

Where is f defined? Find the limits of $f(x)$ as x approaches the boundaries of the domain of definition. Where does the graph of f intersect the axes? Find f' and f'' and study their sign and where they are zero. Use this to find the extreme values (minima and maxima) of f. Draw by hand a sketch of the graph of f, based on all of the above.

Submit on MyCourses your solutions **only** for the following two problems. Explain the reasoning behind your solutions, do not just return the final result. If you make use of important results from the lectures, state what they are.

Homework 1: Recall that the set $\mathbb{Q}$ of rational numbers consists of the fractions of integers $\frac{a}{b}$ where $a \in \mathbb{Z}$ and $b \in \mathbb{Z} \setminus \{0\}$. The set $\mathbb{R} \setminus \mathbb{Q}$ consists of all the elements of $\mathbb{R}$ that are not elements of $\mathbb{Q}$. For instance, $\sqrt{2}$, π and e are all elements of $\mathbb{R} \setminus \mathbb{Q}$, as they cannot be written as fractions of integers. Each element of $\mathbb{R}$ is either in $\mathbb{Q}$ or in $\mathbb{R} \setminus \mathbb{Q}$. The following result may be useful to solve this exercise:

> **Theorem.** The set $\mathbb{Q}$ is dense in $\mathbb{R}$: for any two elements $x < y$ of $\mathbb{R}$, there exists some $z \in \mathbb{Q}$ with $x < z < y$. The set $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$: for any two elements $x < y$ of $\mathbb{R}$, there exists some $z \in \mathbb{R} \setminus \mathbb{Q}$ with $x < z < y$.

Consider the functions

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ -1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases} \qquad\qquad g(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ -x & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

1. Is f continuous in 0?
2. Is g continuous in 0?
3. Does the derivative of f at 0 exist?
4. Does the derivative of g at 0 exist?

Use the ε–δ definition of a limit and the definition of derivative as a limit as explicitly as you can. Except in one point, where you can answer quickly by a theorem from the lectures. [2 points]

Homework 2: Consider the function f defined as follows:

$$f(x) = \frac{x}{\ln(x)}.$$

1. For what values of x is $f(x)$ defined? Compute the limits of $f(x)$ as x approaches the boundaries of the domain of definition.
2. Find the points where the graph of f intersects the axes.
3. Compute f' and f'' and study their sign and where they are zero. Find the extreme values (minima and maxima) of f.
4. Draw by hand a sketch of the graph of f, based on all of the above.

(This kind of exercise is likely going to be in the exam, where you cannot use a calculator. So you really should try to do this by hand first, and then maybe check with a computer.) [2 points]