ELEC-A7200

Signals and Systems

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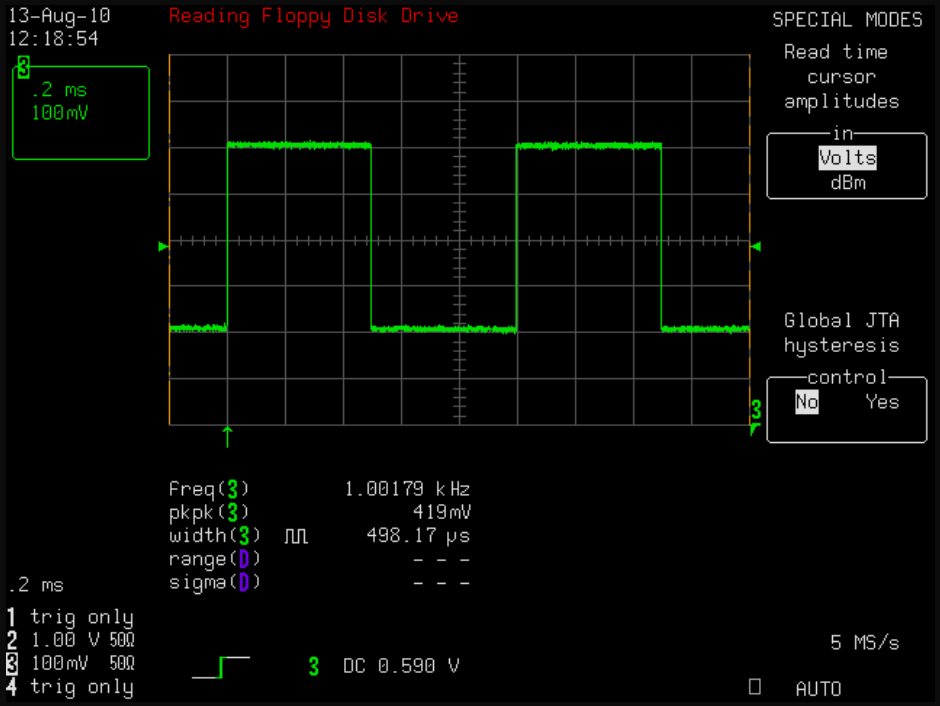
Fall 2022

Aalto University
School of Electrical
Engineering



Lecture 2
Special Signals
Convolution

Special signals and test functions

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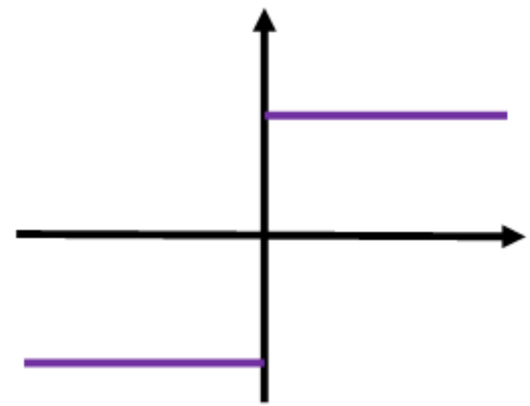
Signum function

Signum function

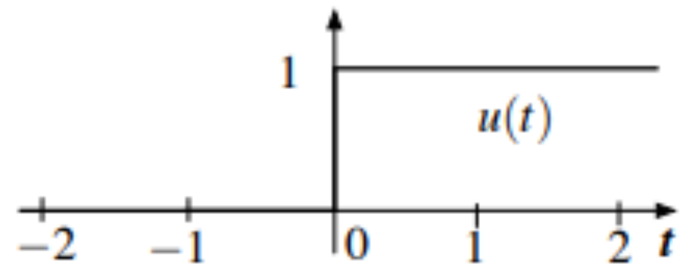
$$\operatorname{sgn}(t) = \begin{cases} -1 & t < 0 \\ 0 & t = 0ç \\ 1 & t > 1 \end{cases}$$



Step function

- **Unit step function** (a.k.a. unit step function, Heaviside step function)

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

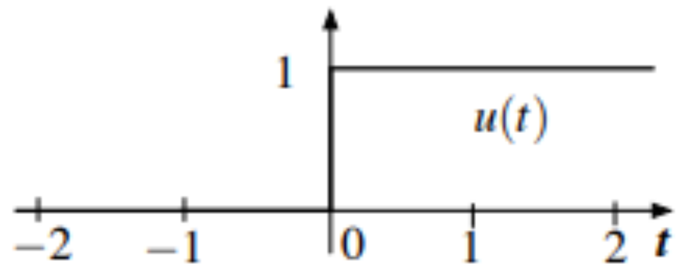


Note that sometimes is defined to be u(0)=1/2

Step function

Unit step function can also be defined as

$$u(t) = \frac{1 + \mathrm{sgn}(t)}{2}$$



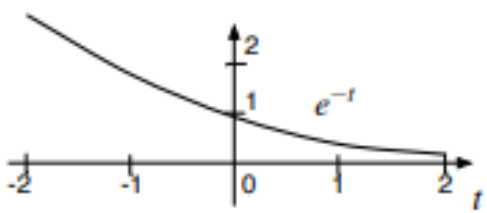
this leads to u(0)=1/2

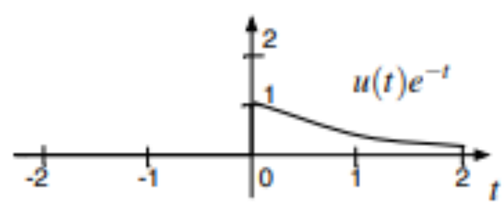
Uses of the unit step function

Extracting part of another signal or cutting the signal

Example:

$$x(t) = \begin{cases} e^{-t} & t \geq 0 \\ 0 & t < 0 \end{cases} \Leftrightarrow x(t) = e^{-t}u(t)$$



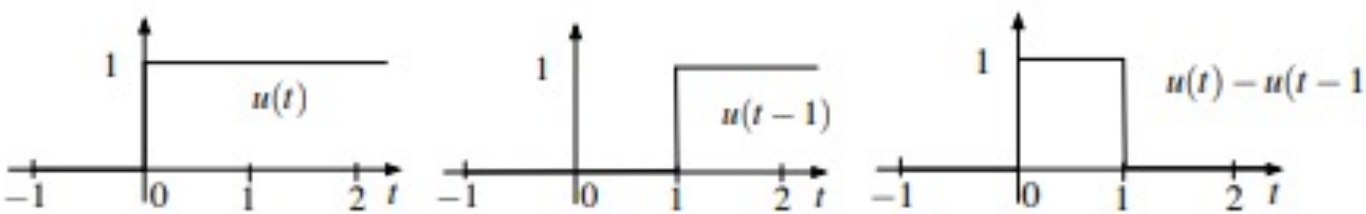


Uses of the unit step function

Combinations of unit steps to create other signals

Example:

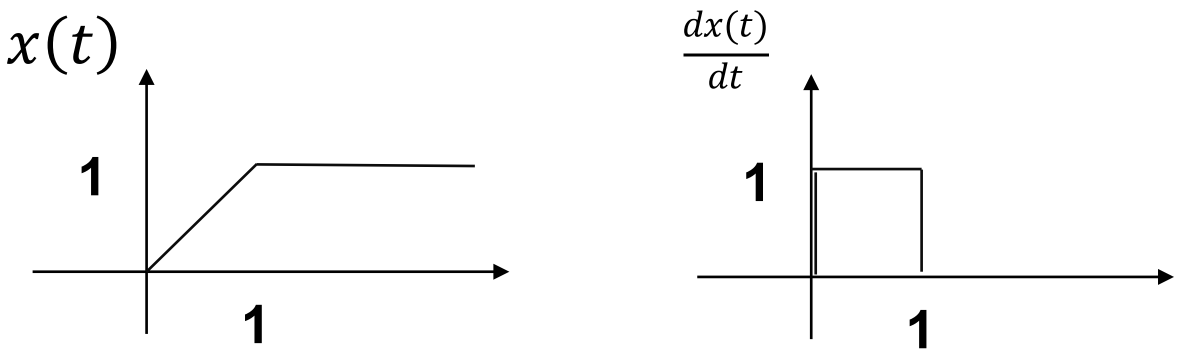
$$x(t) = \begin{cases} 0 & t < 0 \\ 1 & 0 \le t < 1 \\ 0 & t \ge 1 \end{cases} \Leftrightarrow x(t) = u(t) - u(t-1)$$



Uses of the unit step function

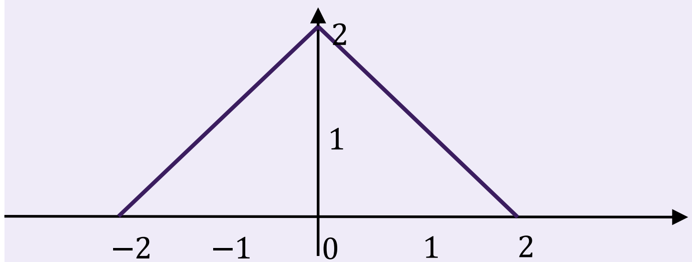
Derivatives of piecewise linear signals

$$x(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t < 1 \\ 1 & t \geq 1 \end{cases} \Leftrightarrow \frac{dx(t)}{dt} = u(t) - u(t-1)$$



Problem

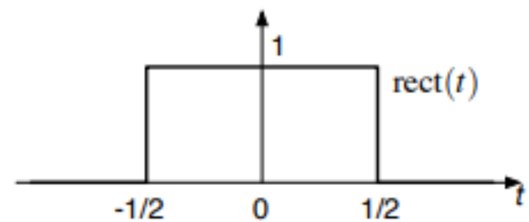
1. Express the derivative of the following triangular pulse using unit step function u(t)



Rectangular pulse

- **Unit rectangle pulse**

$$\mathrm{rect}(t) = \begin{cases} 1 & |t| \le \frac{1}{2} \\ 0 & |t| > \frac{1}{2} \end{cases}$$



  - has unit energy

$$E = \int_{-\infty}^{\infty} \mathrm{rect}^2(t)dt = \int_{-1/2}^{1/2} 1^2 dt = 1$$

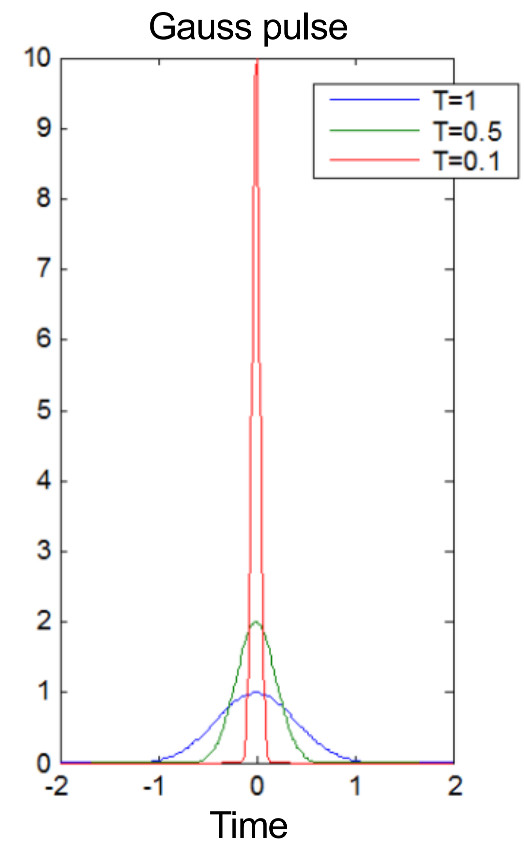
Gauss pulse

Gauss pulse (Gauss distribution)

$$\mathrm{gauss}(t;T) = \frac{1}{\sqrt{2\pi}T} e^{-\frac{1}{2}\left(\frac{t}{T}\right)^2}$$

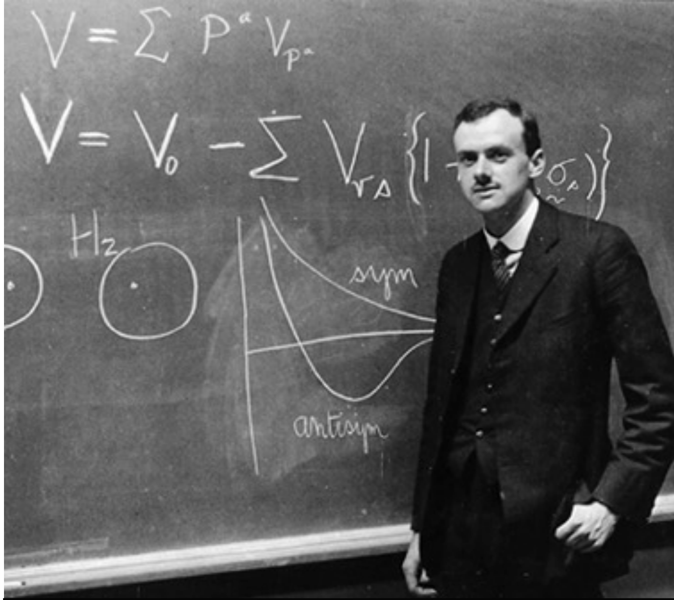
- Has unit integral

$$\int_{-\infty}^{\infty} \mathrm{gauss}(t;T)dt = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}T} e^{-\frac{1}{2}\left(\frac{t}{T}\right)^2} = 1$$



Dirac’s delta function

- Dirac delta function (δ function) is a generalized function or distribution, a function on the space of test functions.
- Dirac delta function belongs to the mathematical space of test functions and *distributions*.
- Dirac function allow us to differentiate functions whose derivatives do not exist in the classical sense.



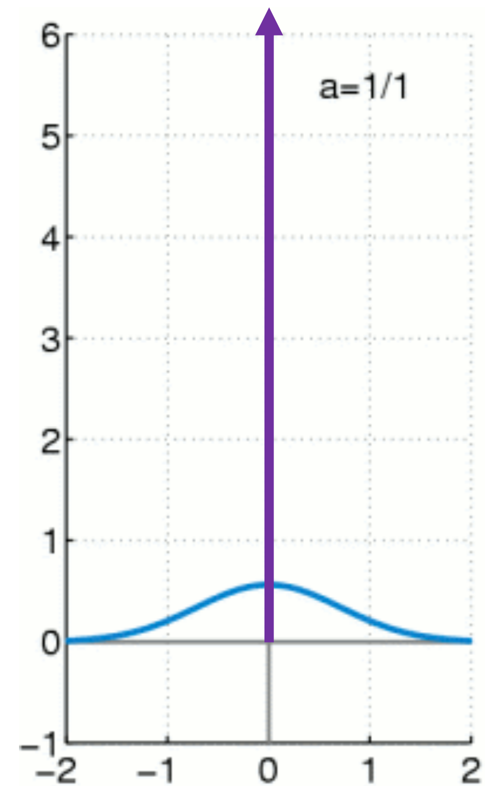
Paul Dirac (8 Aug 1902 – 20 Oct 1984) was an English theoretical physicist.

"I have trouble with Dirac. This balancing on the dizzying path between genius and madness is awful." Einstein

Dirac's delta function (a.k.a unit impulse)

- Dirac's delta function is a function equal to zero everywhere except for zero and whose integral over the entire real line is equal to one. $\int_{-\infty}^{\infty} \delta(\mathrm{t})dt = 1$
- It can be defined as a limit of a pulse whose width goes to zero

$$\delta(\mathrm{t}) = \lim_{a \to 0} \frac{1}{\sqrt{\pi}|a|} e^{-\left(\frac{t}{a}\right)^2}$$

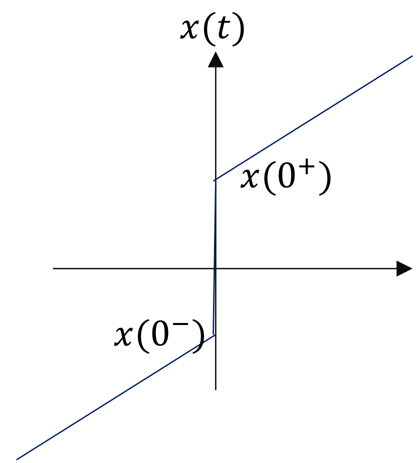


Uses of Dirac's delta function

Differentiation of discontinuous functions

Example: Discontinuity at *t*=0

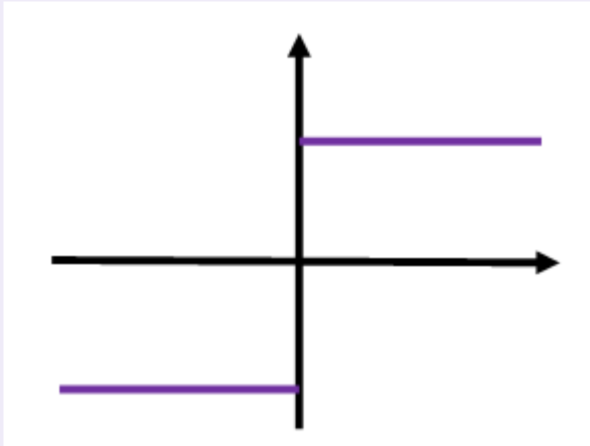
$$\frac{dx(t)}{dt} = (x(0^{+}) - x(0^{-})\ )\delta(\mathrm{t})ç$$



Problem

1. **Let**

$$x(t) = \mathrm{sgn}(t) = \begin{cases} -1 & t < 0 \\ 0 & t = 0 \\ 1 & t > 1 \end{cases}$$



Determine

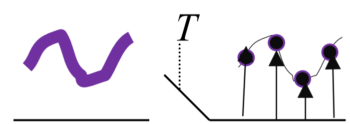
$$y(t) = \frac{dx(t)}{dt}$$

Uses of Dirac's delta function

Sampling

$$\int_{-\infty}^{\infty} x(t)\delta(t)dt = x(0)$$

$$\int_{-\infty}^{\infty} x(t-kT)\delta(t)dt = x(kT)$$



$x(t)$ $\qquad$ $\{x(kT), \quad k \in 0,1,2,\ldots\}$

Problem

Let

$a)\ x(t) = \mathrm{rect}(t)$

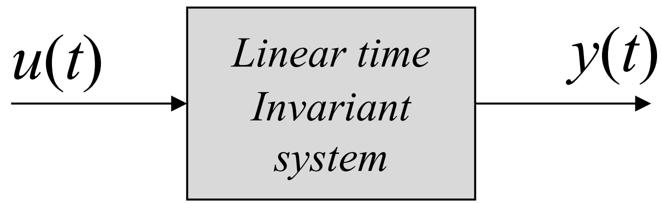
Simplify $x(t)\delta(t)$

$b)\ x(t) = e^{-t^2}$

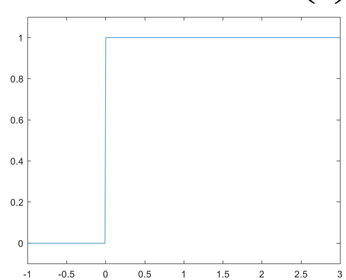
Simplify $x(t)\delta(t-1)$

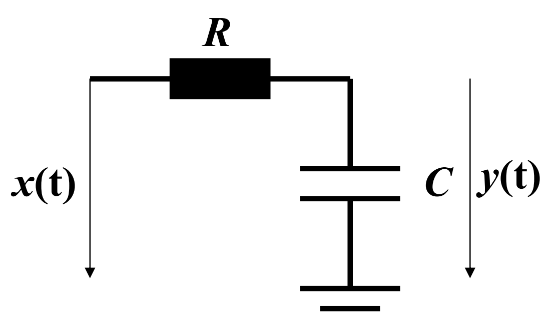
Uses of the unit step function

Linear system modelling: Step response of linear systems

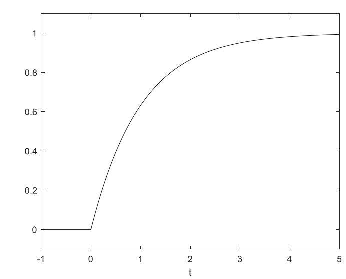


Unit step $u(t)$



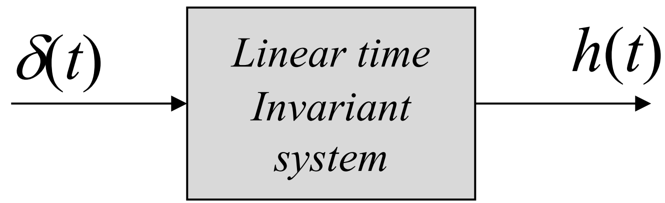


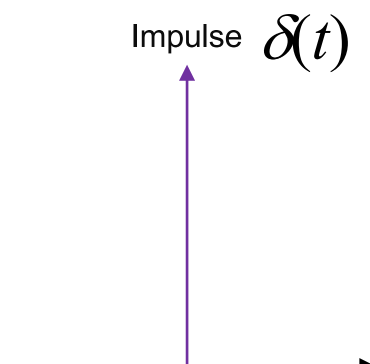
Step response $y(t)$

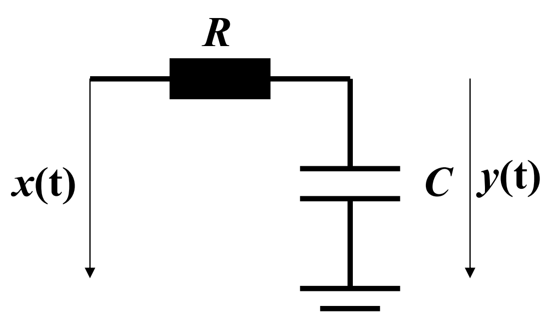


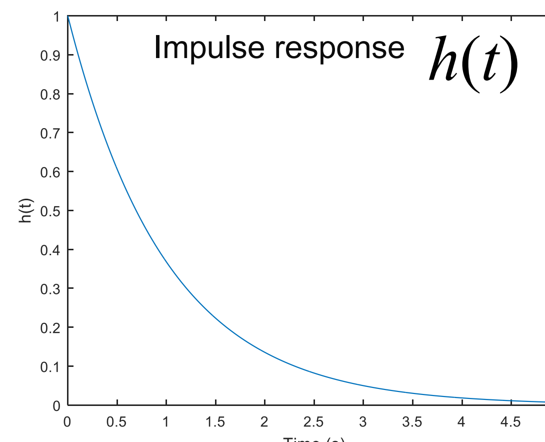
Uses of Dirac's delta function

Linear system modelling: Impulse response of linear systems









Discrete convolution

Discrete convolution

Definition for discrete convolution

$$x(n) \otimes h(n) = \sum_{k=-\infty}^{\infty} h(n-k)x(k)$$

Convolution commutes: $x(n) \otimes h(n) = h(n) \otimes x(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$

But why?

Applications of discrete convolution

Multiplication of polynomials

$$x(z) = x_0 + x_1 z + \cdots x_N z^N$$
$$h(z) = h_0 + h_1 z + \cdots h_N z^M$$

$$y(z) = x(z)h(z) = y_0 + y_1 z + \cdots y_{N+M} z^{N+M}$$

where

$$y_n = \sum_{k=0}^{n} h_{n-k}\, x_k, \quad n = 0,1,2,\ldots,N+M$$

with the assumption that x_k and h_k are zero for indeces that are not given.

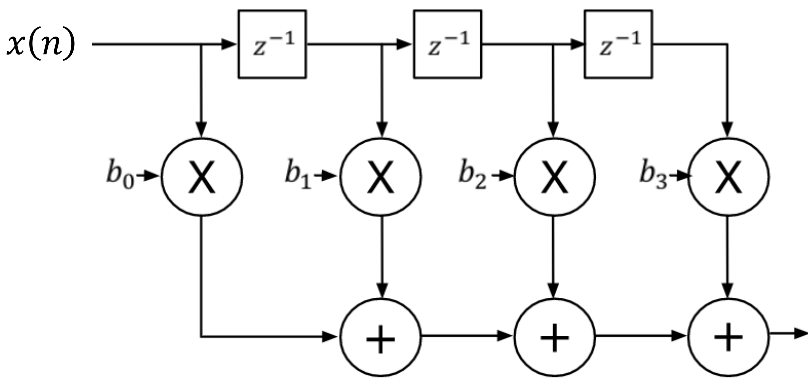
Applications of discrete convolution

Modelling of linear time invariant discrete time systems.

- Digital filters
  - In signal processing, a digital filter is a system that performs mathematical operations on a sampled, discrete-time signal to reduce or enhance certain aspects of that signal.
  - The applications of digital filters include the mitigation of the noise, removal of interfering signals, passing of certain frequency components and rejection of others, shaping of the signal spectrum etc.
- Digital control systems

Discrete convolution: Digital filtering

Finite impulse response (FIR) filter



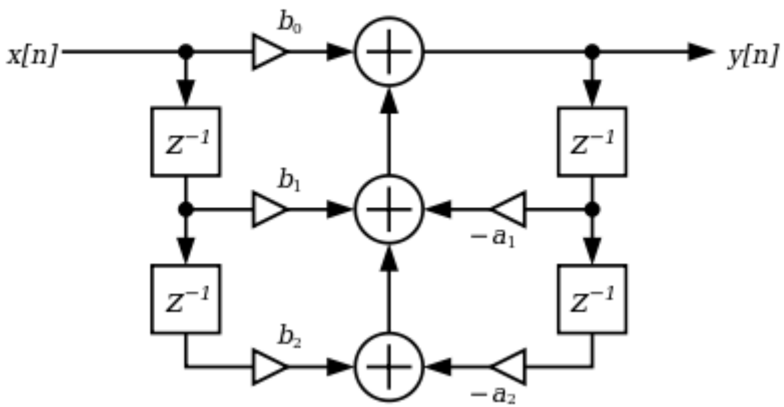
z^{-1} correspond to one sample delay

$y(n) = b_0x(n) + b_1x(n-1) + b_2x(n-2) + b_3x(n-3)$

Impulse $x(n) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$ response => $\{y(n), n = 0,1,2,3\} = \{b_0,\ b_1,\ b_2, b_3\} \triangleq \{h(n)\}$

Discrete convolution: Digital filtering

Infinite Impulse Response Filter (IIR)



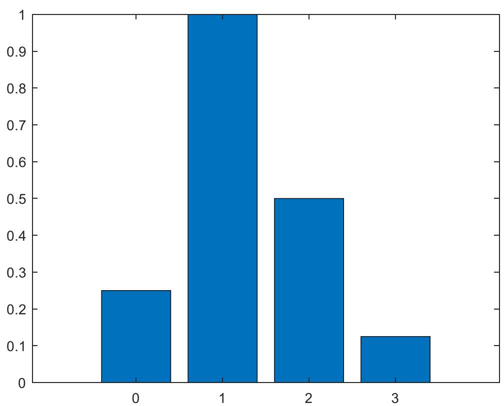
In time domain, the response of a digital filter for arbitrary input signal x(n) is given by discrete convolution between the input and Impulse response:

$$y(n) = \sum_{k=-\infty}^{\infty} h(n-k)x(k)$$

Discrete convolution: FIR filter example

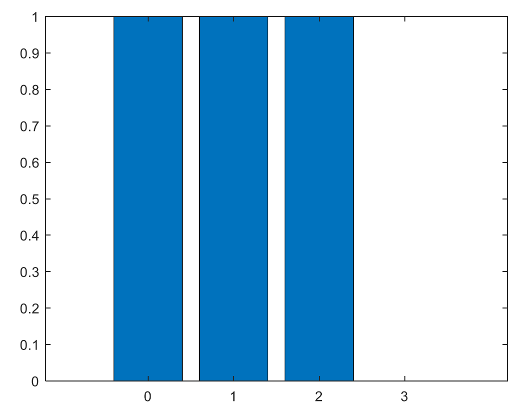
- **Impulse response of FIR filter**

$$\{h(n), n = 0,1,2,3\} = \{\tfrac{1}{4}, 1, \tfrac{1}{2}, \tfrac{1}{8}\}$$



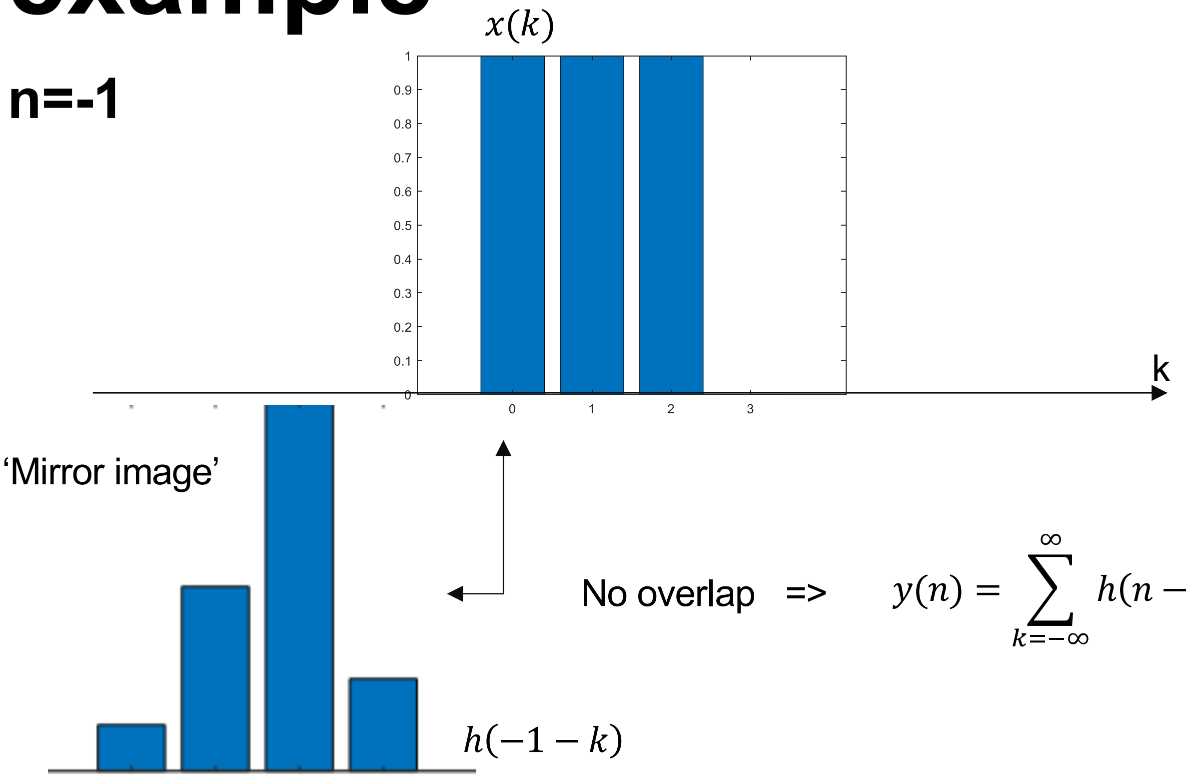
- **Input signal**

$$\{x(n), n = 0,1,2,3\} = \{1,\ 1,\ 1,\ 0\}$$



Discrete convolution: FIR filter example

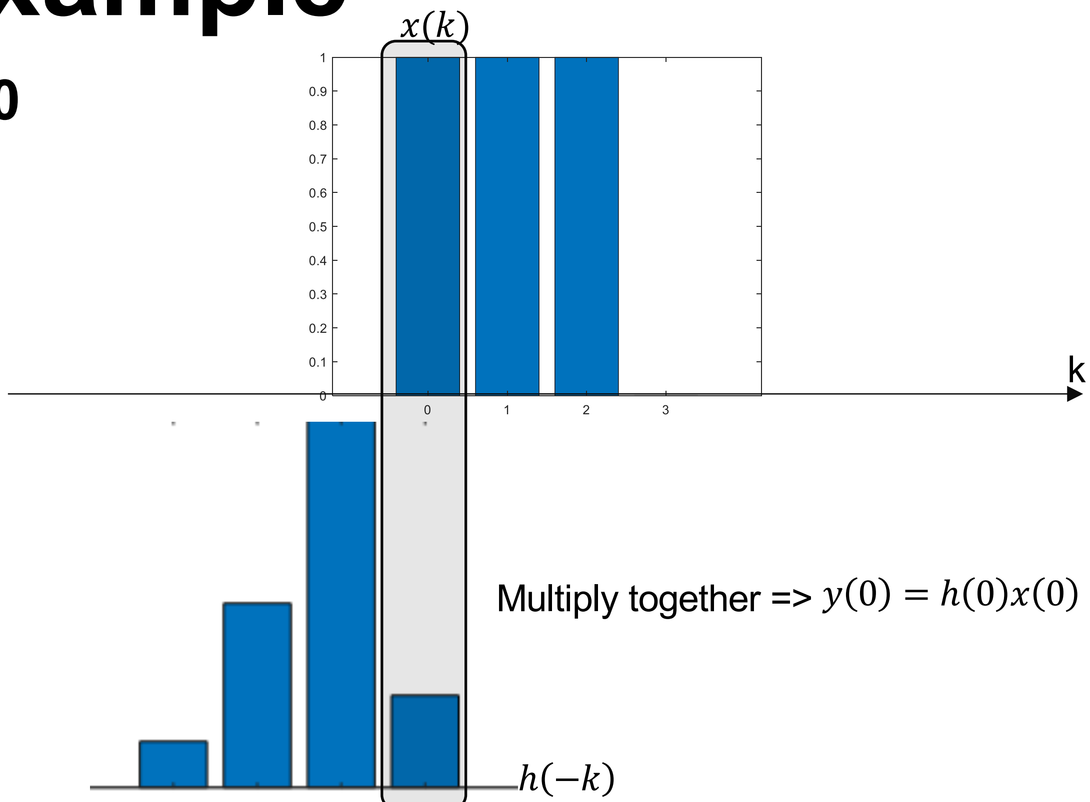
n=-1

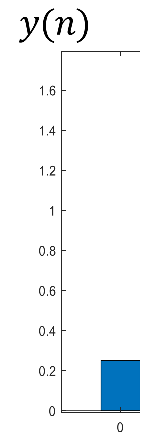


No overlap => $y(n) = \sum_{k=-\infty}^{\infty} h(n-k)x(k) = 0$

Discrete convolution: FIR filter example

n=0





Multiply together => $y(0) = h(0)x(0) = 0.25 * 1$ =0.25

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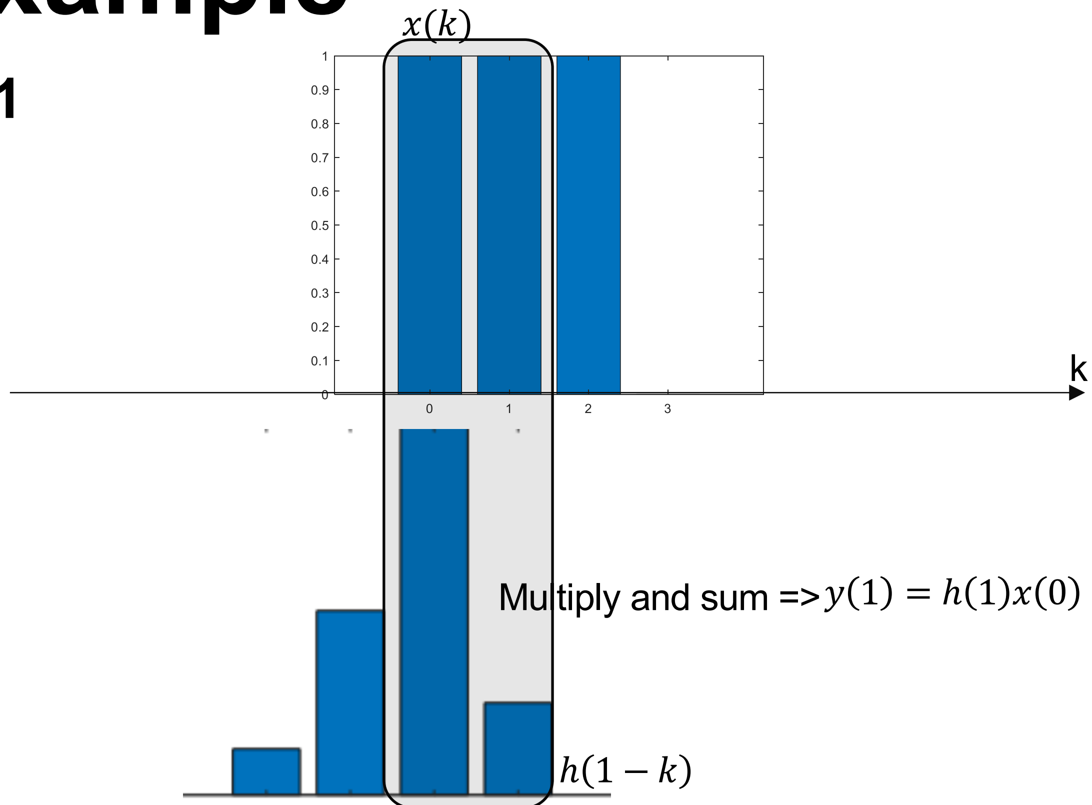
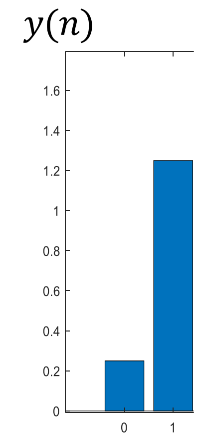
Discrete convolution: FIR filter example

n=1

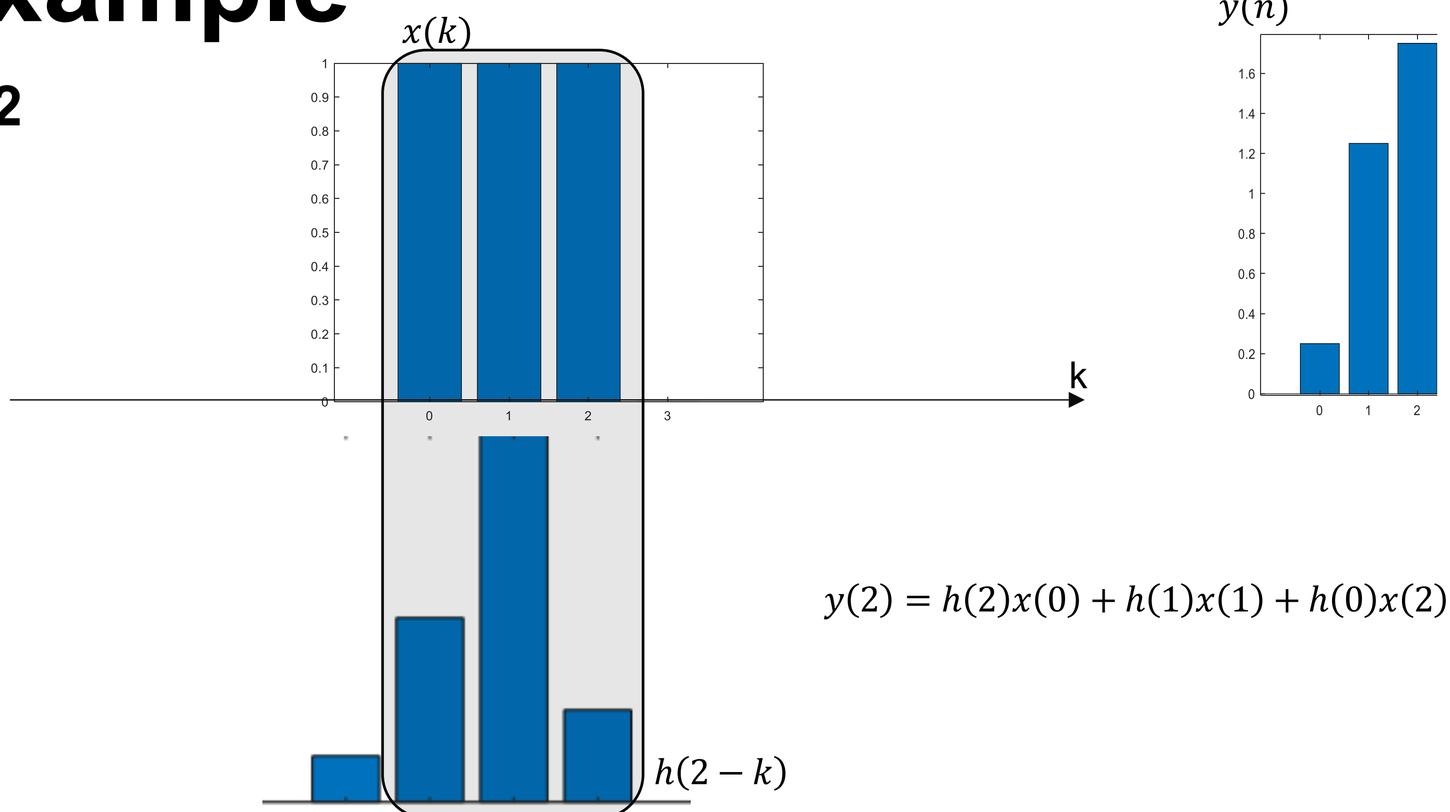




Multiply and sum => $y(1) = h(1)x(0) + h(0)x(1) = 0.25 * 1 + 1 * 1 = 1.25$

Discrete convolution: FIR filter example

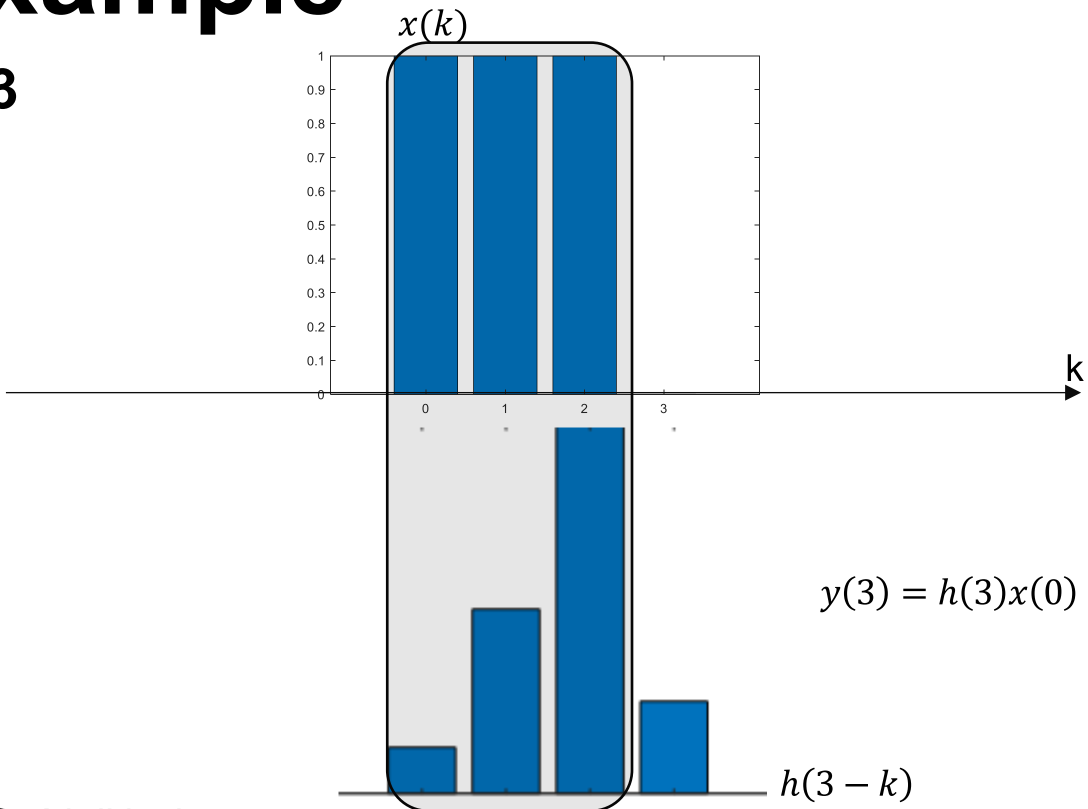
n=2

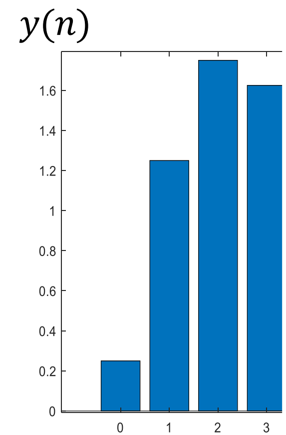


$$y(2) = h(2)x(0) + h(1)x(1) + h(0)x(2) = 1.75$$

Discrete convolution: FIR filter example

n=3

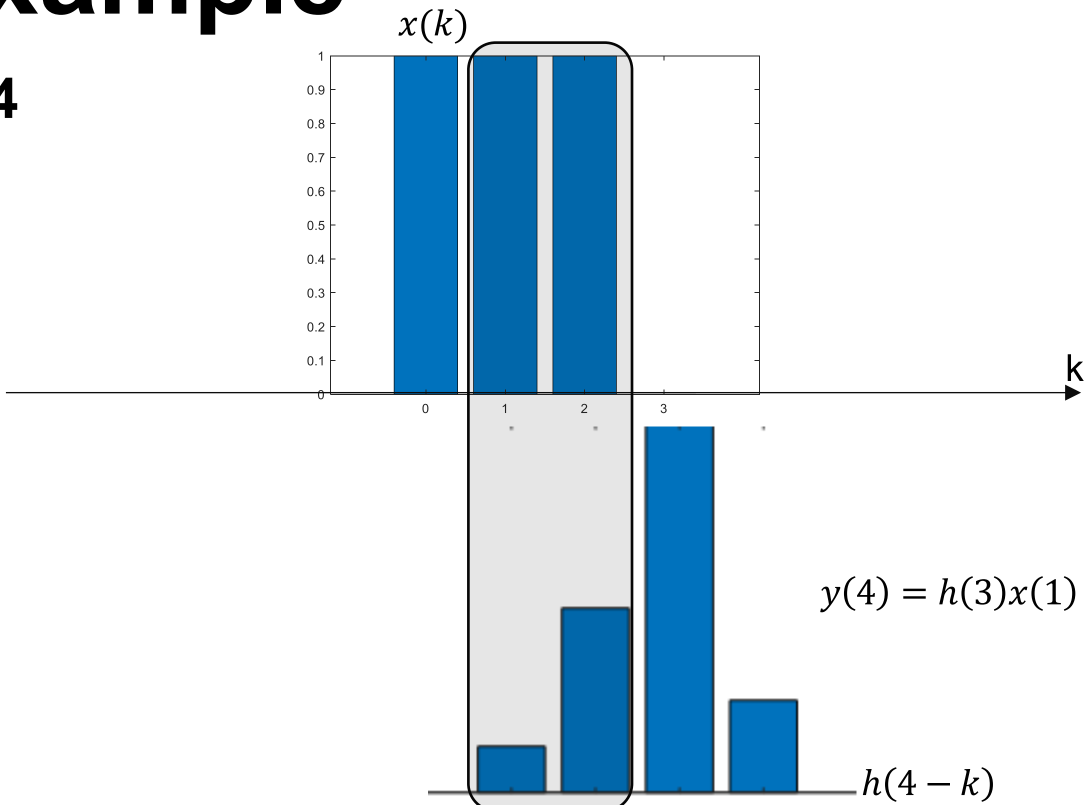




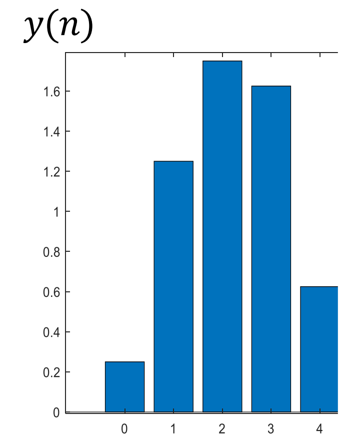
$$y(3) = h(3)x(0) + h(2)x(1) + h(1)x(2) = 1.625$$

Discrete convolution: FIR filter example

n=4



$$y(4) = h(3)x(1) + h(2)x(3) = 0.6250$$



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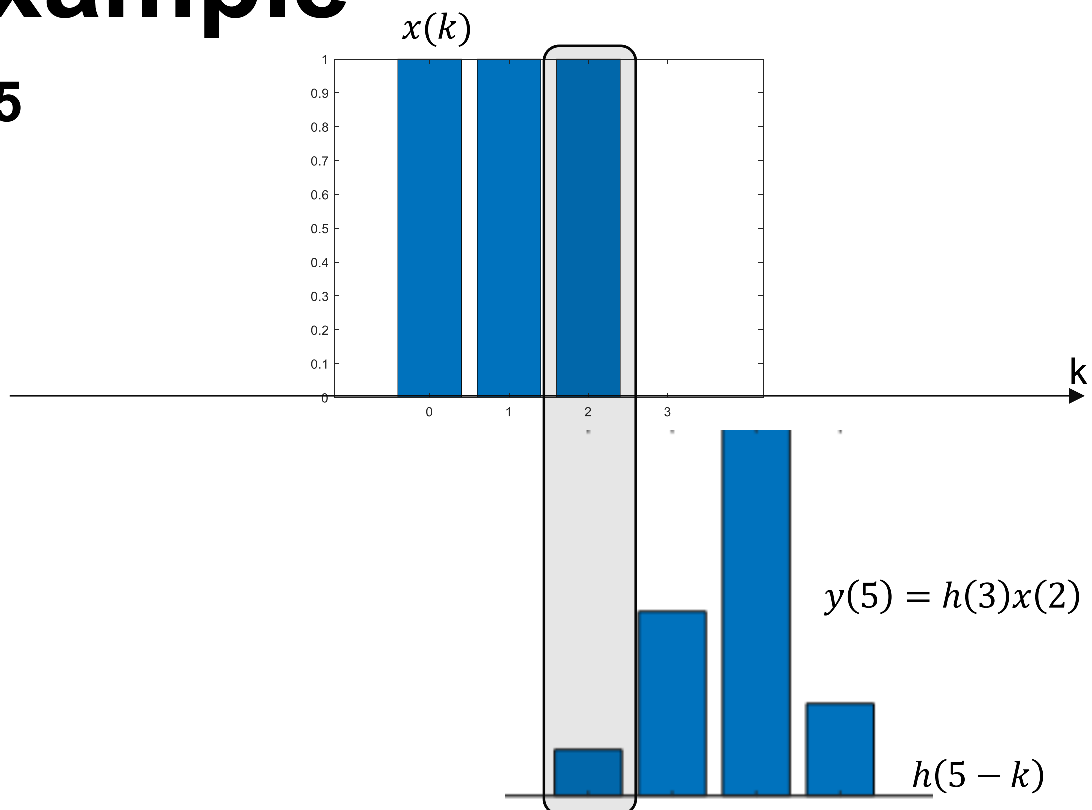
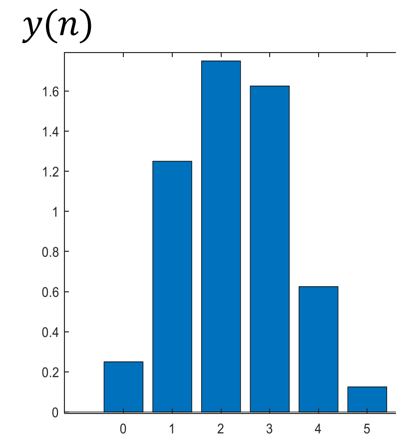
Discrete convolution: FIR filter example

n=5



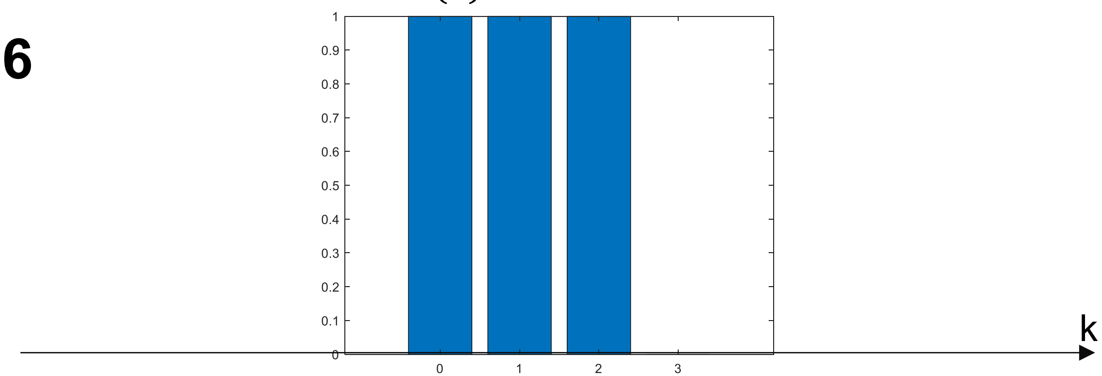
$$y(5) = h(3)x(2) = 0.125$$



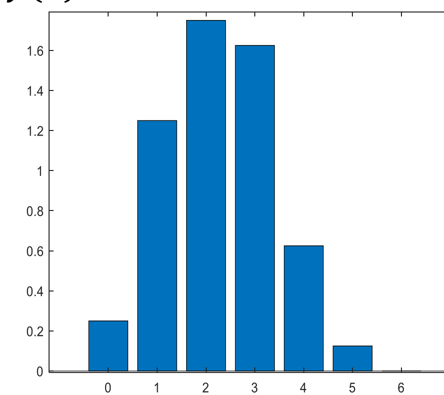
Discrete convolution: FIR filter example

n=6

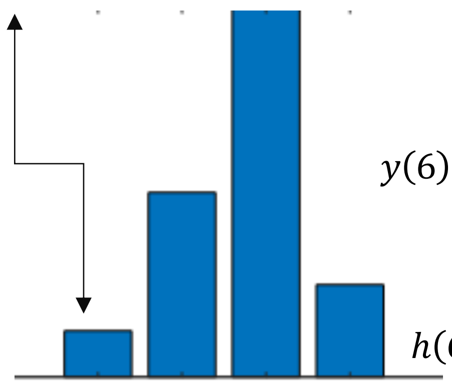
$x(k)$



$y(n)$



No overlap

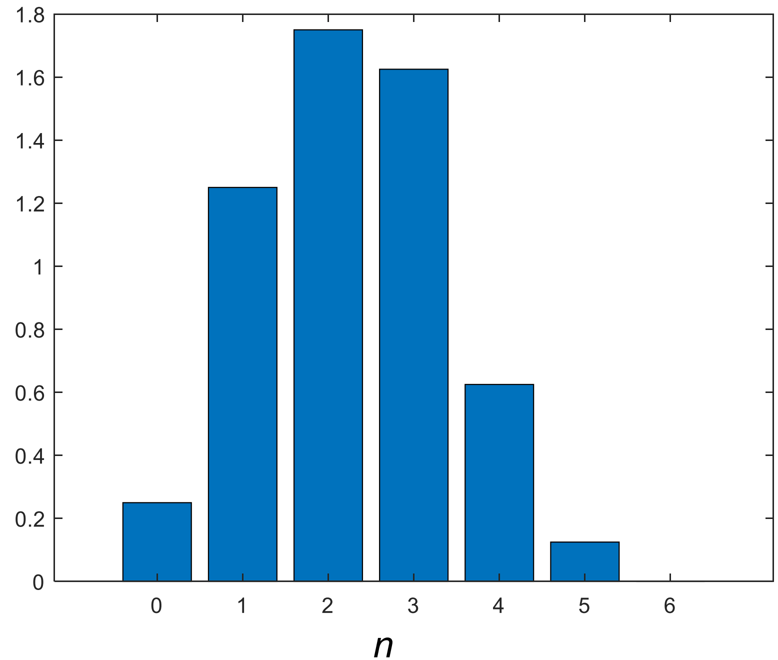


$y(6) = 0$

$h(6-k)$

Discrete convolution: FIR filter example

FIR filter output

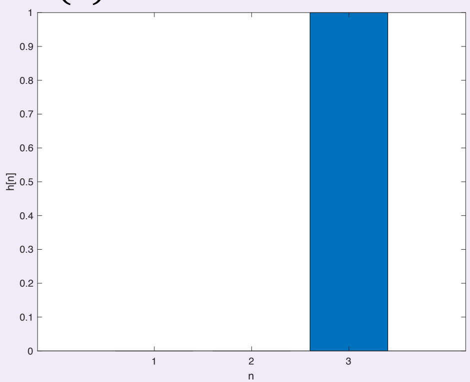


Problem

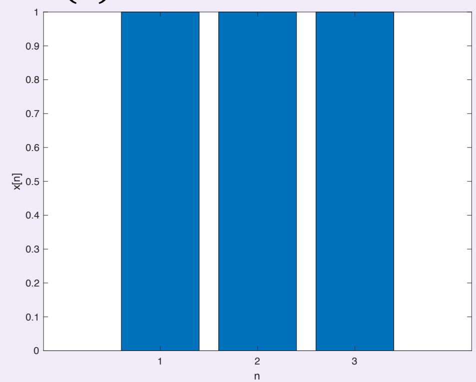
Plot the discrete convolution

$$y(n) = x(n) \otimes h(n) = \sum_{k=-\infty}^{\infty} h(n-k)x(k)$$

$h(n)$



$x(n)$



The convolution integral

The convolution integral

The convolution integral is defined as

$$x(t) \otimes h\,(t) = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)d\tau$$

Convolution commutes: $x(t) \otimes h\,(t) = h\,(t) \otimes x(t) = \int_{-\infty}^{\infty} h(\tau)x(t\tau)d\tau$

But why?

Applications of the convolution integral

Modeling of linear time variant continuous time systems such as electronic circuits or mechanical systems.

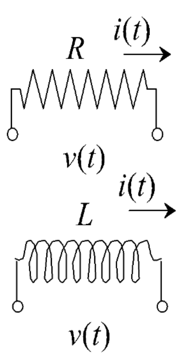
- Analog filtering
  - In signal processing, an analog filter is an electronic circuit that operate on a continuous time signal to reduce or enhance certain aspects of that signal.
  - The applications of analog include those of the digital filters as well as the anti-aliasing filtering before sampling
- Continuous time control systems

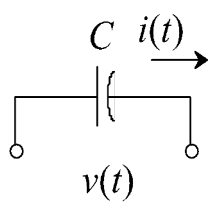
The convolution integral: Linear time invariant systems

Linear time invariant system is described by a differential equation

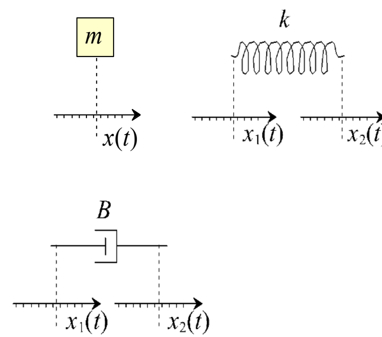
$$\frac{d^n}{dt^n}y(t) = -a_1\frac{d^{n-1}}{dt^{n-1}}y(t) - \cdots - a_n y(t) + b_0\frac{d^m}{dt^m}u(t) + b_1\frac{d^{m-1}}{dt^{m-1}}u(t) + \cdots + b_m u(t)$$

n degree of the system
For any physical system *m<n (strictly proper system)*

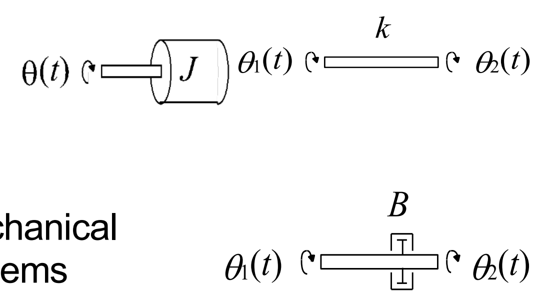




Electric circuits

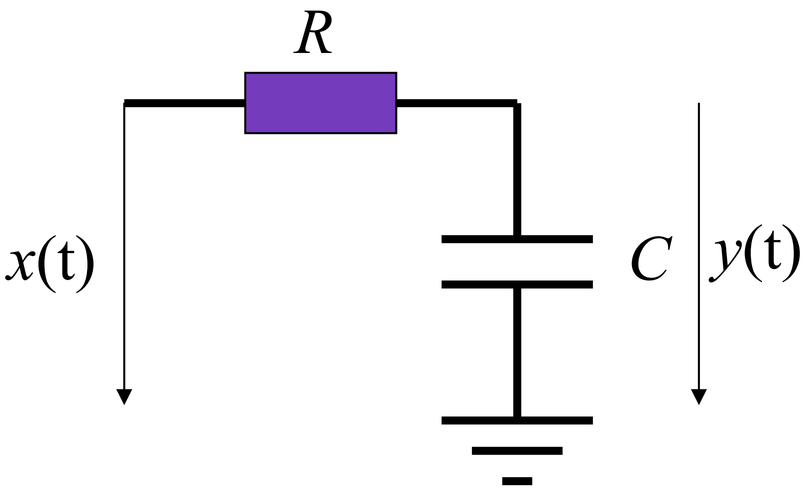


Mechanical systems



The convolution integral: analog filtering

RC-filter for removing high frequency components



$$\frac{d}{dt}i(t) = -\frac{1}{RC}i(t) + \frac{1}{R}x(t)$$

$$\frac{d}{dt}y(t) = \frac{1}{C}i(t)$$

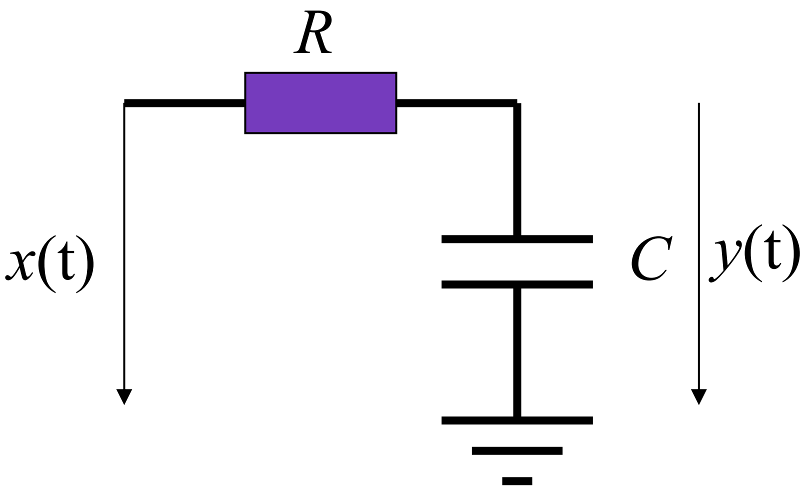
Impulse response:

$$x(t)=\delta(t) \;\Rightarrow\; y(t) = e^{-\frac{1}{RC}t}u(t) \triangleq h(t)$$

Corresponds physically charging the capacitator and then observing it discharge

The convolution integral: analog filtering

RC-filter for removing high frequency components



General input signal

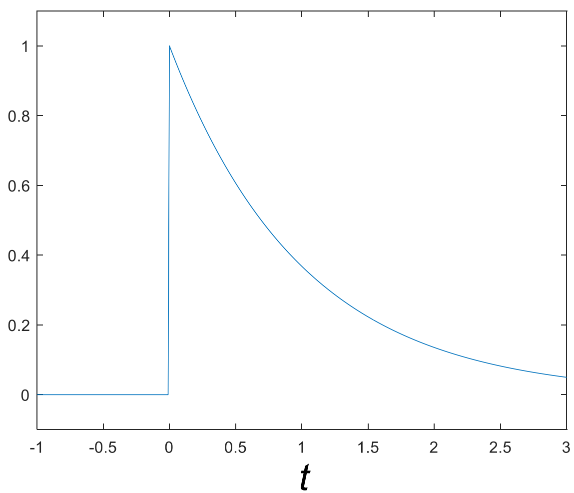
$$x(t) \Rightarrow y(t) = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

The convolution integral: RC Filter step response example

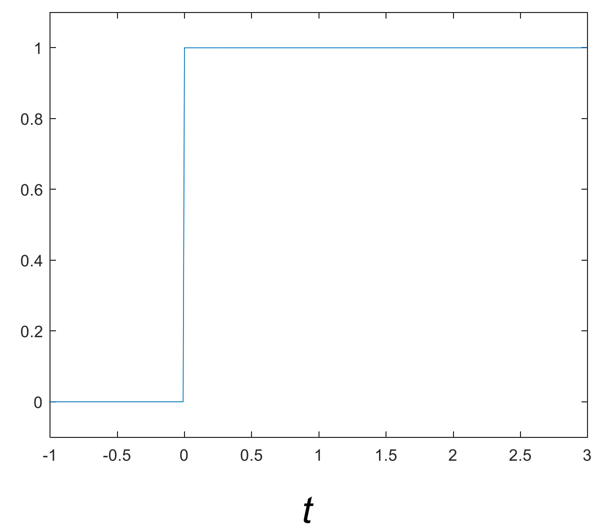
- **Impulse response of FIR filter**

$h(t) = e^{-t}u(t)$



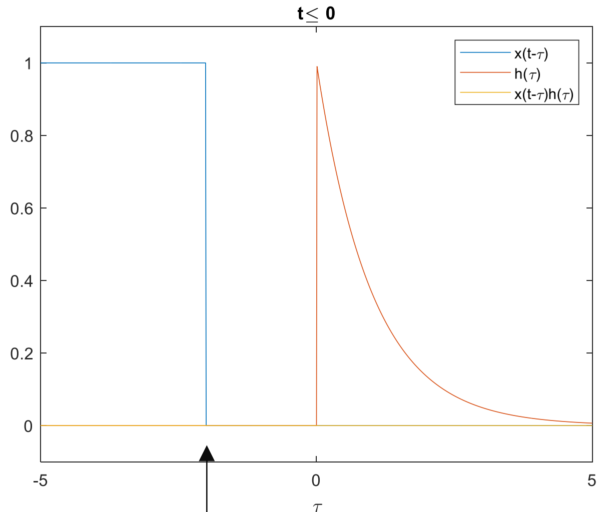
- **Input signal**

$x(t) = u(t)$



The convolution integral: RC Filter step response example

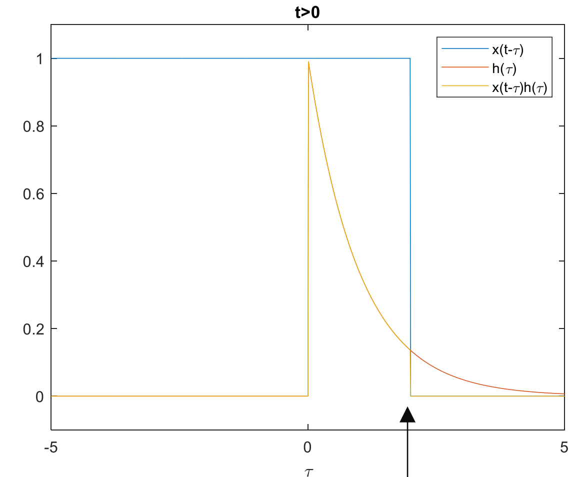
- **Case 1: t ≤ 0**



t

No overlap between $x(t-\tau)$ and $h(\tau)$
=> $y(t)=0$

- **Case 2: t>0**



t

Integrate for the overlap region from 0 to *t*

The convolution integral: RC Filter step response example

Step response for t≤0

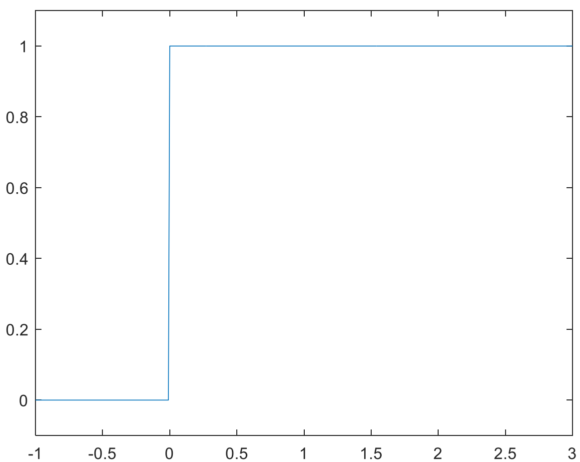
$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = 0$$

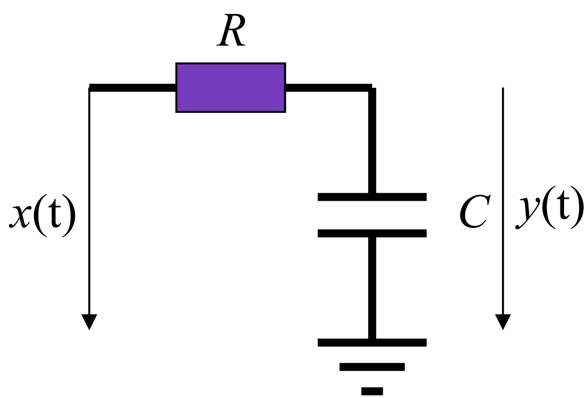
Step response for t>0

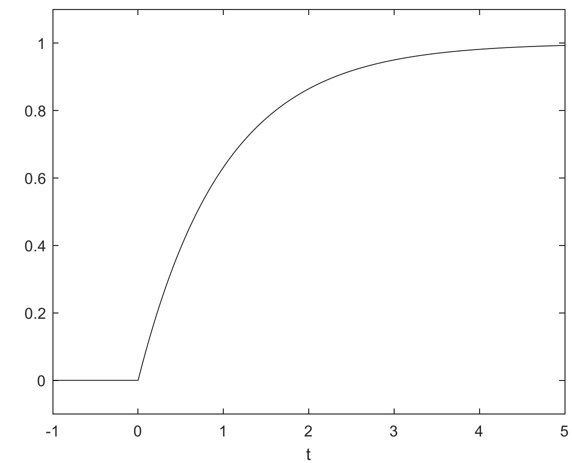
$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = \int_{\mathbf{0}}^{\boldsymbol{t}} e^{-t} \cdot \mathbf{1} d\tau = 1 - e^{-t}$$

The convolution integral: RC Filter step response example

RC=1



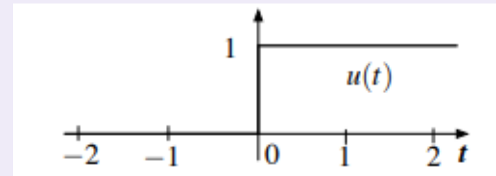




Problem

Calculate the following convolution integral

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$



when $h(t) = x(t) = \mathrm{u}(t)$

Convolution with Dirac's delta function

Convolution integral

$$\int_{-\infty}^{\infty} \delta(\tau)x(t-\tau)d\tau = x(t)$$

$$\int_{-\infty}^{\infty} \delta(\tau - T)x(t-\tau)d\tau = x(t-T)$$

Convolution with Dirac's delta function: Multi-path channel

Channel impulse response

$$h(\tau) = h_0\delta(\tau - \tau_0) + h_1\delta(\tau - \tau_1) + \ldots h_{L-1}\delta(\tau - \tau_{L-1})$$

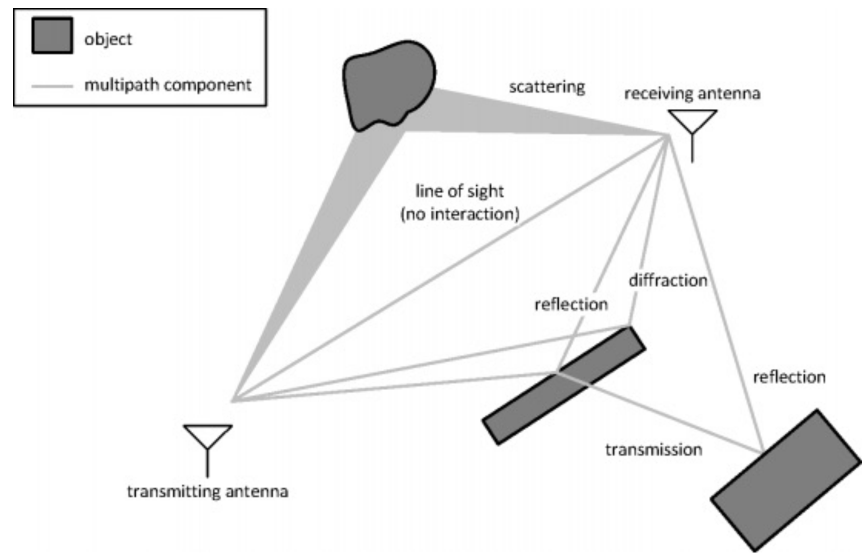
Transmitted signal

$x(\tau)$

Received signal

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t - \tau)d\tau$$

$$= h_0 x(\tau - \tau_0) + h_1 x(\tau - \tau_1) + \ldots h_{L-1} x(\tau - \tau_{L-1})$$



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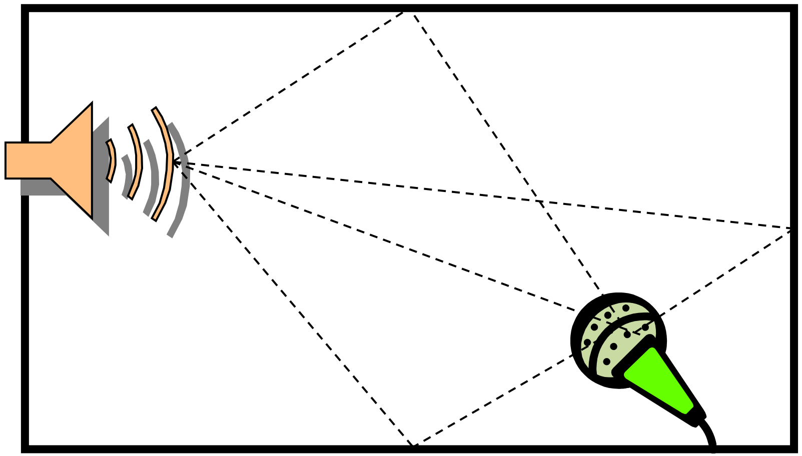
Impulse reponse

Modeling of acoustics in a concert hall



Singing in anechoic studio

$$x(t)$$

http://www.openairlib.net/anechoicdb/content/operatic-voice

Impulse response of a church hall

$$h(t) = \sum_k h_k \delta\left(t - \tau_k\right)$$

http://www.openairlib.net/auralizationdb/content/st-patricks-church-patrington-model

Singing in the church hall

$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$$

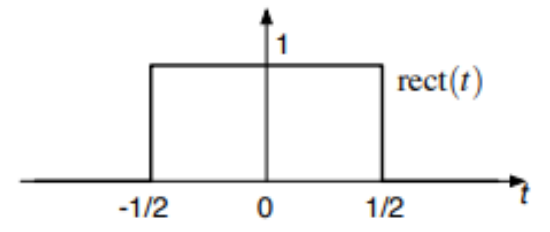
Convolution in chapter 2
+ FFT in chapter 7

Problem

Calculate the following convolution integral

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

when $h(\tau) = \delta(\tau - 1)$ and $x(t) = \text{rect}(t)$



Todays lecture

1. **Special functions**
   - Signum, unit step, Dirac's delta function
2. **Discrete convolution**
3. **The convolution integral**

…applications to linear systems

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