ELEC-E8101 Digital and Optimal Control

Solutions 1

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1. a)

Let's do the transform of the sequence based on the definition of the Z-transformation. The definition is

$$F(z) = \sum_{k=0}^{\infty} f(k) z^{-k} \,.$$

Hence the transform of sequence

$$y(k) = 1 \,, k = 0,1,2,3,..$$

is

$$Y(z) = \sum_{k=0}^{\infty} y(k) z^{-k} = 1 + z^{-1} + z^{-2} + z^{-3} + \ldots$$

If the equality for the geometric sums

$$\sum_{k=0}^{\infty} q^k = \frac{1}{1-q}, \text{ if } |q| < 1$$

is applied, we get

$$Y(z) = \sum_{k=0}^{\infty} y(k) z^{-k} = 1 + z^{-1} + z^{-2} + z^{-3} + \ldots$$

$$= \frac{1}{1 - \frac{1}{z}} = \frac{z}{z-1}$$

b)

In a similar way as in part *a)* we set

$$Y(z) = \sum_{k=0}^{\infty} e^{-ak} z^{-k} = 1 + e^{-a} z^{-1} + e^{-2a} z^{-2} + e^{-3a} z^{-3} + \ldots$$

$$= 1 + (e^{a} z)^{-1} + (e^{a} z)^{-2} + (e^{a} z)^{-3} + \ldots = \frac{1}{1 - e^{-a} z^{-1}} = \frac{z}{z - e^{-a}}$$

You can verify both results by using the z-transform tables.

2. a) $F(z) = Z\{f(k)\} = \sum_{k=0}^{\infty} a^k z^{-k} = 1 + az^{-1} + a^2 z^{-2} + ... = \frac{1}{1-az^{-1}} = \frac{z}{z-a}$ The geometric series converges for $|az^{-1}| < 1$. However, here we do not calculate the expression for any special value of *z*. Therefore, the convergence condition is not a matter of concern. Compare to Laplace transformation and its convergence region in the continuous time case.

b) $Z\left[e^{-kh/T}\right] = \frac{z}{z - e^{-h/T}}$ holds because $e^{-\frac{kh}{T}} = \left(e^{-\frac{h}{T}}\right)^k$. But $e^{-h/T}$ is always >=0. So this expression is more restrictive than that in part a).

3.

Define the value of *y*(*kh*), when $k \to \infty$, using the final-value theorem.

Final-value theorem: If $\lim_{k\to\infty} y(kh)$ exists, then it holds

$$\lim_{k\to\infty} y(kh) = \lim_{z\to 1}(1 - z^{-1})Y(z),$$

A sufficient (but not necessary) condition for the existence of $\lim_{k\to\infty} y(kh)$ is that $(1-z^{-1})Y(z)$ has no poles on or outside the unit circle.

$$Y(z) = \frac{0{,}792z^2}{(z-1)(z^2 - 0{,}416z + 0{,}208)}$$

Let's check whether $(1 - z^{-1})Y(z)$ has poles on or outside the unit circle.

$$(1-z^{-1})Y(z) = (1-z^{-1})\frac{0{,}792z^2}{(z-1)(z^2 - 0{,}416z + 0{,}208)}$$

$$= \frac{0{,}792z}{(z^2 - 0{,}416z + 0{,}208)}$$

which has the poles :

$$z^2 - 0{,}416z + 0{,}208 = 0$$

$$z = 0{,}208 \pm 0{,}406i$$

The absolute values of the complex poles (see also MATLAB-command *roots*) are about 0,46

⇨ they are inside the unit circle, and the final-value theorem can be used.

$$\lim_{k\to\infty} y(kh) = \lim_{z\to 1}\left(1-z^{-1}\right)Y(z) = \lim_{z\to 1}\frac{0,792z}{\left(z^2-0,416z+0,208\right)}$$

$$=\frac{0,792}{1-0,416+0,208}=\frac{0,792}{0,792}=1$$

For verification in Matlab the signal *y* can be constructed by letting a unit step function $\frac{z}{z-1}$ go through the system with a pulse transfer function $H(z)=\frac{0.792z}{z^2-0.416z+0.208}$. The following commands can be used.

H=0.792*tf([1 0],[1 -0.416 0.208],1); % Note the last “1” which tells Matlab that we consider a discrete-time system with discretization interval 1. If left out, Matlab would consider *H* as a continuous time transfer function.

step(H) % Plots the response and shows that it indeed approaches 1.

Note: for defining *H* we could also write

z=tf('z');

H=0.792*z/(z^2-0.416*z+0.208)

4.

Take the inverse-transform of the following equation

$$Y(z)=\frac{\left(1-e^{-ah}\right)z}{(z-1)\left(z-e^{-ah}\right)}$$

If *Z*-transformation tables are studied, it is noted that in every transform *z* is in the numerator. Therefore, for convenience, we first divide the previous equation with *z*,

$$\frac{Y(z)}{z}=\frac{\left(1-e^{-ah}\right)}{(z-1)\left(z-e^{-ah}\right)}=\frac{A}{z-1}+\frac{B}{z-e^{-ah}}$$

Let’s solve A and B with Heaviside’s method. (The partial fraction method could as well be used)

$$A = \lim_{z-1}(z-1)\frac{\left(1-e^{-ah}\right)}{(z-1)\left(z-e^{-ah}\right)} = 1$$

$$B = \lim_{z-e^{-ah}}\left(z-e^{-ah}\right)\frac{\left(1-e^{-ah}\right)}{(z-1)\left(z-e^{-ah}\right)} = -1$$

Then

$$\frac{Y(z)}{z} = \frac{1}{z-1} - \frac{1}{z-e^{-ah}}$$

$$\Rightarrow Y(z) = \frac{z}{z-1} - \frac{z}{z-e^{-ah}}$$

and this can be inverse-transformed with the transformation tables.

$$y(kh) = 1 - e^{-akh}$$

***5.**

Prove the following Z-transformation

$$Z\left\{\frac{1}{2}(kh)^2\right\} = \frac{h^2 z(z+1)}{2(z-1)^3}$$

Let's do what the hint says and transform Z{kh}.

$$y(kh) = kh,\ k = 0,1,2,3,...$$

$$\Rightarrow Y(z) = \sum_{k=0}^{\infty} khz^{-k} = 0 + hz^{-1} + 2hz^{-2} + 3hz^{-3} + ...$$

This is not a geometric series. But let's multiply with (z - 1)

$$\Rightarrow \quad (z-1)Y(z) = h\left[1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + ... - z^{-1} - 2z^{-2} - 3z^{-3} - ...\right]$$
$$= h\left[1 + z^{-1} + z^{-2} + z^{-3} + ...\right]$$

Now it can be seen that this is a geometric sum with ratio of z^{-1} between the consecutive terms. Hence we get

$$(z-1)Y(z) = \frac{h}{1-z^{-1}} = \frac{hz}{z-1}$$

$$\Rightarrow Y(z) = \frac{hz}{(z-1)^2}$$

Let's use the same trick into the actual problem and see what happens.

$$Z\left\{\frac{1}{2}(kh)^2\right\}=\frac{1}{2}h^2\left[0+z^{-1}+4z^{-2}+9z^{-3}+16z^{-4}+...\right]=Y(z)$$

$$(z-1)Y(z)=\frac{1}{2}h^2\left[1+4z^{-1}+9z^{-2}+16z^{-3}+...-z^{-1}-4z^{-2}-9z^{-3}-16z^{-4}\right]$$

$$\underset{arrangements}{=}\frac{1}{2}h^2\left[1+3z^{-1}+5z^{-2}+7z^{-3}+...\right]\underset{arrangements}{=}\frac{1}{2}h^2\left[\left(1+2z^{-1}+3z^{-2}+4z^{-3}+...\right)\right.$$

$$\left.+\underbrace{\left(z^{-1}+2z^{-2}+3z^{-3}+...\right)}_{Z\{kh\}}\right]=\frac{1}{2}h^2\left[\frac{z}{z}\left(1+2z^{-1}+3z^{-2}+4z^{-3}+...\right)+\frac{z}{(z-1)^2}\right]$$

$$=\frac{1}{2}h^2\left[z\left(z^{-1}+2z^{-2}+2z^{-3}+...\right)+\frac{z}{(z-1)^2}\right]=\frac{1}{2}h^2\left[z\frac{z}{(z-1)^2}+\frac{z}{(z-1)^2}\right]=\frac{h^2z(z+1)}{2(z-1)^2}$$

By dividing by (z - 1) the final result can be obtained:

$$Y(z)=\frac{h^2z(z+1)}{2(z-1)^3}$$

6.

Define $y(k)$ of the following difference equation by using the Z-transformation

$$y(k+2)-1{,}5y(k+1)+0{,}5y(k)=u(k+1)$$

$u(k)$ is the unit step at $k = 0$, $y(0)=0{,}5$ and $y(-1)=1$

Let us take the Z-transformation of the above equation:

$$z^2Y(z) - z^2y(0) - zy(1) - 1.5[zY(z) - zy(0)] + 0.5Y(z) = zU(z) - zu(0).$$

The initial value $y(1)$ of time domain is yet unknown, but it can be solved with the difference equation:

$$y(1)=1.5y(0)-0.5y(-1)+u(0)=0.75-0.5+1=1.25.$$

By substituting the initial values to the z-transformed equation we get:

$$\Rightarrow [z^2 - 1.5z + 0.5]Y(z) = 0.5z^2 + 1.25z - 0.75z + z\frac{z}{z-1} - z$$

$$\Rightarrow [(z - 1)(z - 0.5)]Y(z) = 0.5z^2 + 0.5z + \frac{z}{z-1}.$$

Let us solve $Y(z)$:

$$Y(z) = \frac{0.5z^2 + 0.5z + \dfrac{z}{z-1}}{(z-1)(z-0.5)} = \frac{(z-1)(0.5z^2 + 0.5z) + z}{(z-1)^2 (z-0.5)} =$$

$$= \frac{0.5z^3 + 0.5z^2 - 0.5z^2 - 0.5z + z}{(z-1)^2 (z-0.5)} = \frac{0.5z(z^2 + 1)}{(z-1)^2 (z-0.5)}$$

and by the partial fractions this is:

$$\frac{0.5z(z^2+1)}{(z-1)^2 (z-0.5)} = z\left[\frac{A}{z-0.5} + \frac{B}{(z-1)^2} + \frac{C}{z-1}\right]$$

$\Rightarrow A = 2.5$, $B = 2$ ja $C = -2$.

$$\Rightarrow Y(z) = \frac{2.5z}{z-0.5} + \frac{2z}{(z-1)^2} - \frac{2z}{z-1}.$$

Inverse-transformation:

$y(k) = 2.5 \cdot 0.5^k + 2k - 2 = 2(k - 1) + 2.5 \cdot 0.5^k$.