Mathematics for Economists

Mitri Kitti

Aalto University

Vectors and Matrices

Matrices

- $m \times n$ (real) matrix A, we denote $A \in \mathbb{R}^{m \times n}$, is an array

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix},$$

  where $a_{ij} \in \mathbb{R}$, $i = 1, \dots, m$, $j = 1, \dots, m$
  - notation: $A_i = (a_{i1}, \dots, a_{in})$ (ith row), $A^j = (a_{1j}, \dots, a_{mj})$ (jth column)
- Special matrices: square, identity matrix (I), symmetric, diagonal, upper (lower) triangular

Matrix algebra: scalar multiplication

Scalar multiplication: Given an $m \times n$ matrix A and a real number $\alpha \in \mathbb{R}$, the product of A and the number α is a matrix of size $m \times n$ defined as follows:

$$\alpha A = \alpha \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} \alpha a_{11} & \cdots & \alpha a_{1n} \\ \vdots & \ddots & \vdots \\ \alpha a_{m1} & \cdots & \alpha a_{mn} \end{pmatrix}$$

Matrix algebra: addition

Addition. Given two matrices A and B of the same size $m \times n$, their sum is a matrix of size $m \times n$ defined as follows:

$$A + B = \begin{pmatrix} a_{11} + b_{11} & \cdots & a_{1n} + b_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & \cdots & a_{mn} + b_{mn} \end{pmatrix},$$

where

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}, \qquad B = \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{pmatrix}$$

Note: A and B must be of the same size

Matrix algebra: addition

Addition. Matrix addition has the following properties:

- Associativity: $A + (B + C) = (A + B) + C$
- Commutativity: $A + B = B + A$
- Identity: $A + \mathbf{0} = A$, where $\mathbf{0}$ is a matrix of zeros of the same size as A.

Matrix algebra: multiplication

Multiplication. Given two matrices A and B of size $k \times m$ and $m \times n$, respectively, their product AB is a matrix of size $k \times n$ obtained as follows:

- For every $i = 1, \ldots, k$ and $j = 1, \ldots, n$, the (i, j)th entry of AB is the product between the ith row of A and jth column of B:

$$\begin{pmatrix} a_{i1} & a_{i2} & \cdots & a_{im} \end{pmatrix} \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{mj} \end{pmatrix} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{im} b_{mj}.$$

Example:

$$\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} \begin{pmatrix} X & Y \\ W & Z \end{pmatrix} = \begin{pmatrix} aX + bW & aY + bZ \\ cX + dW & cY + dZ \\ eX + fW & eY + fZ \end{pmatrix}$$

Matrix algebra: multiplication

Multiplication. Matrix multiplication satisfies:

- Associativity: $A(BC) = (AB)C$.

- However, multiplication does not satisfy commutativity. If AB is well-defined, it could happen that
  - BA is not well-defined, or
  - $AB \neq BA$

Example:

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix}$$

Matrix algebra: multiplication

- What matrices do to vectors? Twist and scale
- Examples

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

try what happens to vector $\boldsymbol{x} = (1, 1)$ with these two, i.e., what are $A\boldsymbol{x}$ and $B\boldsymbol{x}$

Example: rotating an image

- Matrix $A = \begin{pmatrix} 0.5 & -0.87 \\ 0.87 & 0.5 \end{pmatrix}$
- A(Queen Elizabeth II of the United Kingdom by Andy Warhol 1985)

Example: rotating an image

- Matrix $A = \begin{pmatrix} 0.5 & -0.87 \\ 0.87 & 0.5 \end{pmatrix}$
- A(Queen Elizabeth II of the United Kingdom by Andy Warhol 1985)

Example: nonlinear image filtering

- vectors = pixels at (x, y) coordinates
- map each pixel to a new position, but depending on the pixel use a different mapping ("$A(\boldsymbol{x})\boldsymbol{x}$" becomes nonlinear)

$$A(x,y) = \begin{bmatrix} \cos\left(\theta n \sqrt{(x^2+y^2)/k}\right) & -\sin\left(\theta n \sqrt{(x^2+y^2)/k}\right) \\ \sin\left(\theta n \sqrt{(x^2+y^2)/k}\right) & \cos\left(\theta n \sqrt{(x^2+y^2)/k}\right) \end{bmatrix}$$

Matrix algebra: identity matrix

The *identity* matrix is an $n \times n$ matrix of the following form:

$$I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

- For any A of size $m \times n$, we have $AI = A$
- For any B of size $n \times k$, we have $IB = B$
- For any C of size $n \times n$, we have $IA = AI = A$

Matrix algebra: transpose

- **Transpose.** Given a $k \times n$ matrix A, the transpose of A is the $n \times k$ matrix A^T obtained by interchanging the rows and columns of A. That is, the (i, j)th entry of A^T is the (j, i)th entry of A.
- Example:

$$\begin{pmatrix} 3 & 4 & 1 \\ 0 & 1 & 0 \end{pmatrix}^T = \begin{pmatrix} 3 & 0 \\ 4 & 1 \\ 1 & 0 \end{pmatrix}$$

Matrix algebra: transpose

Transpose. The transpose satisfies the following properties:

- $(A+B)^T = A^T + B^T$
- $(A^T)^T = A$
- $(\alpha A)^T = \alpha A^T$, where $\alpha \in \mathbb{R}$
- $(AB)^T = B^T A^T$
- $\mathrm{rank} A = \mathrm{rank} A^T$.

Example: portfolio variance

Covariance matrix of annualized daily returns of Faang-stocks (AAPL,AMZN,FB,GOOGL,NFLX)

$$V = \begin{pmatrix} 0.095 & 0.065 & 0.097 & 0.060 & 0.064 \\ 0.068 & 0.108 & 0.148 & 0.063 & 0.088 \\ 0.097 & 0.148 & 0.300 & 0.124 & 0.206 \\ 0.060 & 0.063 & 0.124 & 0.082 & 0.066 \\ 0.064 & 0.088 & 0.206 & 0.066 & 0.195 \end{pmatrix}$$

Assume investing one euro to a portfolio; weight vector $\boldsymbol{w} = (w_1, w_2, w_3, w_4, w_5)$, $\sum_i w_i = 1$

Variance of the portfolio is $\boldsymbol{w}^T V \boldsymbol{w}$

Systems of linear equations in matrix form

Recall that a linear system with m equations and n unknowns is:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad\qquad \vdots \qquad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m, \end{aligned}$$

where a_{ij} and b_i are given parameters, and x_j are the unknown variables.

This system can also be written in matrix form:

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

$$A\mathbf{x} = \mathbf{b},$$

where the m by n matrix A is called the **coefficient matrix**, $\mathbf{x}$ is an $n \times 1$ vector of unknowns and $\mathbf{b}$ is an $m \times 1$ vector of parameters

Systems of linear equations in matrix form

We can represent the system of equations in an even more compact form by defining the **augmented coefficient matrix** $\hat{A}$:

$$\hat{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{pmatrix}.$$

The augmented coefficient matrix has dimension m by $(n+1)$

Exercise (from Lecture 1)

- System of 4 linear equations in 4 unknowns:

$$\begin{aligned} 2x_1 - x_2 &= 0 \\ -x_1 + 2x_2 - x_3 &= 0 \\ -x_2 + 2x_3 - x_4 &= 0 \\ -x_3 + 2x_4 &= 5 \end{aligned}$$

- The augmented matrix corresponding to the above system:

$$\left[\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & 5 \end{array}\right]$$

Matrix inversion

From now on, we will focus on $n \times n$ matrices, i.e. *square* matrices

- For a given matrix A, we say that B is an **inverse** for A if $AB = BA = I$.
  - Not every matrix has an inverse
  - If A has an inverse, then the inverse is unique
  - The inverse of A is usually denoted as A^{-1}

Matrix inversion

For any two invertible matrices A and B having the same size, the following properties hold:

- $(AB)^{-1} = B^{-1}A^{-1}$
- $(A^T)^{-1} = (A^{-1})^T$
- $(A^{-1})^{-1} = A$

Matrix algebra

- Consider a generic 2×2 matrix:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

- The inverse is

$$A^{-1} = \begin{pmatrix} \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

- We have that A is invertible (and, equivalently, nonsingular) if and only if $ad - bc \neq 0$

- The scalar $ad - bc$ is called the **determinant** of A

Determinant

The **determinant** of a square matrix A, denoted as $\det A$ or $|A|$, is defined recursively as follows:

1. For 1×1 matrices, $\det A = a_{11}$
2. For $n \times n$ matrices,

$$\det A = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{-1,-j}$$

   where $A_{-1,-j}$ is the $(n-1) \times (n-1)$ matrix obtained by deleting the 1st row and jth column of A.

Terminology:

- $\det A_{-1,-j}$ is called a **minor** of A
- $(-1)^{1+j} \det A_{-1,-j}$ is called a **cofactor** of A

Determinant

- Example: Consider the 3×3 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

- The minors of A are:

$$\det A_{-1,-1} = \det \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}$$

$$\det A_{-1,-2} = \det \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix}$$

$$\det A_{-1,-3} = \det \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

Determinant

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

- The determinant of A is

$$\det A = a_{11} \det A_{-1,-1} - a_{12} \det A_{-1,-2} + a_{13} \det A_{-1,-3}.$$

Determinant: a tip

Checkboard pattern for the signs of the cofactors

$$\begin{pmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{pmatrix}$$

Example

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

$$A_{23} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 7 & 8 \end{pmatrix} \qquad C_{23} = (-1)^{2+3} \det \begin{pmatrix} 1 & 2 \\ 7 & 8 \end{pmatrix} = (-1)(-6) = 6$$

Rules of determinant

For any two square matrices A and B of the same size, the determinant satisfies the following properties:

- $\det A^T = \det A$
- $\det(AB) = (\det A)(\det B)$
- $\det A^{-1} = \frac{1}{\det A}$
- In general, $\det(A + B) \neq \det A + \det B$

Nonsingular matrices

Proposition

For a square matrix A, the following are equivalent:

1. *A is nonsingular*
2. *A is invertible*
3. $\det A \neq 0$.

Linear independence of a set of vectors

Proposition

Vectors $\boldsymbol{u}_1, \dots, \boldsymbol{u}_m$ *in* $\mathbb{R}^n$ *are linearly dependent if and only if the linear system*

$$A \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix} = 0$$

has a **nonzero** *solution* $(a_1, \dots, a_m)$*, where* A *is the* $n \times m$ *matrix whose columns are the vectors* $\boldsymbol{u}_1, \dots, \boldsymbol{u}_m$*.*

Determinant and linear independence

Proposition

A set of n vectors $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n$ *in* $\mathbb{R}^n$ *is linearly* **independent** *if and only if*

$$\det \begin{pmatrix} \boldsymbol{u}_1 & \ldots & \boldsymbol{u}_n \end{pmatrix} \neq 0$$

Proposition

If $k > n$*, any set of k vectors in* $\mathbb{R}^n$ *is linearly* **dependent**.

Exercise

Determine whether or not the following three vectors are linearly independent:

$$\boldsymbol{u}_1 = (1, 2, 3)$$
$$\boldsymbol{u}_2 = (0, 0, 0)$$
$$\boldsymbol{u}_3 = (1, 5, 6).$$

Cramer's rule

We can use the determinant to solve systems of linear equations via the so-called **Cramer's rule**.

For a nonsingular A, the unique solution to $A\boldsymbol{x} = \boldsymbol{b}$ is obtained as

$$x_i = \frac{\det B_i}{\det A} \quad \text{for any } i = 1, \dots, n,$$

where B_i is the matrix A with the ith column replaced by $\boldsymbol{b}$.

Cramer's rule: exercise

Exercise. Remember the IS-LM model from Lecture 1:

$$sY + ar = I^0 + G \tag{IS}$$

$$mY - hr = M_s - M^0 \tag{LM}$$

where Y and r are unknown variables and all other parameters are positive constants. Solve this system using Cramer's rule.

Cramer's rule: exercise

Multiplier matrix (let us denote it by A) is

$$\begin{pmatrix} s & a \\ m & -h \end{pmatrix}.$$

Determinant $\det(A) = s \cdot (-h) - a \cdot m = -sh - am$ (assume this is non zero) Matrices B_1 and B_2 are

$$B_1 = \begin{pmatrix} I^0 + G & a \\ M_s - M^0 & -h \end{pmatrix}.$$

and

$$B_2 = \begin{pmatrix} s & I^0 + G \\ m & M_s - M^0 \end{pmatrix}.$$

Cramer's rule: exercise

$\det(B_1) = -h(I^0 + G) - a(M_s - M^0)$ and
$\det(B_2) = s(M_s - M^0) - m(I^0 + G)$

Solution
$Y = \det(B_1)/\det(A) = [-h(I^0 + G) - a(M_s - M^0)]/(-sh - am)$ and
$r = \det(B_1)/\det(A) = [s(M_s - M^0) - m(I^0 + G)]/(-sh - am)$