KOORDINAATISTOT

Karteesinen koordinaatisto

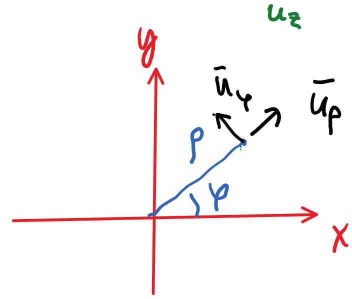
x, y, z

$\bar{u}_x, \bar{u}_y, \bar{u}_z$

Sylinterikoordinaatisto

ρ, φ, z

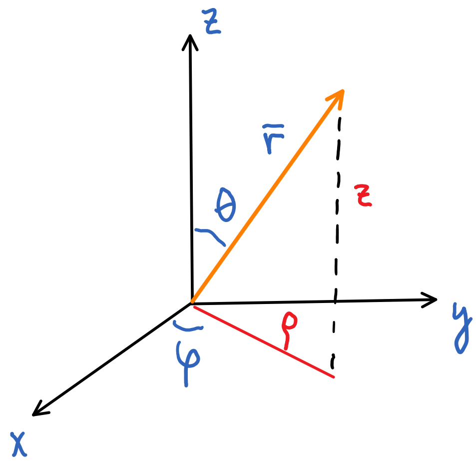
$\bar{u}_\rho$
$\bar{u}_\varphi$
$\bar{u}_z$



Pallokoordinaatisto

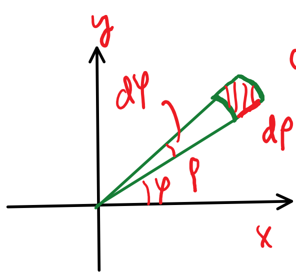
r, θ, φ

$\bar{u}_r$
$\bar{u}_\theta$
$\bar{u}_\varphi$

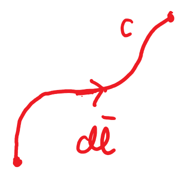


$r = \sqrt{\rho^2 + z^2}$

$\cos\theta = \frac{z}{r}$



$dS = \rho \, d\rho \, d\varphi$



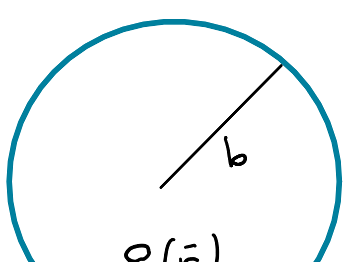
$\int_C d\bar{l}$

$d\bar{l} = \bar{u}_x \, dx + \bar{u}_y \, dy + \bar{u}_z \, dz$

$d\bar{l} = \bar{u}_\rho \, d\rho + \rho \, d\varphi \, \bar{u}_\varphi + \bar{u}_z \, dz$

$d\bar{l} = \bar{u}_r \, dr + r \, d\theta \, \bar{u}_\theta + r \sin\theta \, d\varphi \, \bar{u}_\varphi$

EPÄHOMOGEENISEN
PALLON
MASSA

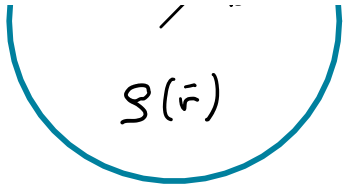


$\rho(r) = \rho_0 \left(1 - \frac{r}{b}\right)$

$M = \int \rho(r) \, dV$

$0 \dots \pi$

$r^2 \sin\theta \, dr \, d\theta \, d\varphi$



$$M = \int \rho(\bar r)\, dV$$

$dV = r^2 \sin\theta \; dr \; d\theta \; d\varphi$, with r: $0 \dots b$ and φ: $0 \dots 2\pi$

$$M = \int_{r=0}^{b} \int_{\theta=0}^{\pi} \int_{\varphi=0}^{2\pi} \rho_0 \left(1 - \frac{r}{b}\right) r^2 \sin\theta \; dr \; d\theta \; d\varphi$$

$$\int_0^{2\pi} d\varphi = 2\pi$$

$$\int_0^{\pi} \sin\theta \; d\theta = \Big/_0^{\pi} -\cos\theta = 1 - (-1) = 2$$

$$M = 4\pi \rho_0 \int_0^b \left(r^2 - \frac{r^3}{b}\right) dr$$

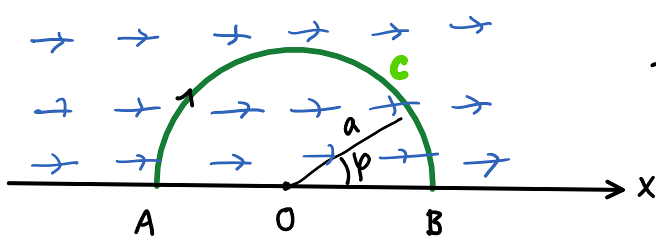
$$= 4\pi \rho_0 \Big/_0^b \left(\frac{1}{3} r^3 - \frac{1}{4}\frac{r^4}{b}\right) = 4\pi \rho_0 \left(\frac{1}{3} b^3 - \frac{1}{4} b^3\right) = 4\pi b^3 \frac{\rho_0}{12}$$

$$= \frac{1}{4} \cdot \rho_0 \cdot V$$

Viivaintegraalit

Esimerkki:

$$\bar E = E_0 \bar u_x$$



$$\int_C \bar E \cdot d\bar l$$

$d\bar l = \bar u_\varphi \, a \, d\varphi$, $\bar E = \underline{E_0 \bar u_x}$

$$\bar u_x \cdot \bar u_\varphi = -\sin\varphi$$

$$\bar u_\varphi = \bar u_y \cos\varphi - \bar u_x \sin\varphi$$

$$\int_C \bar E \cdot d\bar l = a E_0 \underbrace{\int_\pi^0 -\sin\varphi \, d\varphi}_{2} = 2 a E_0$$

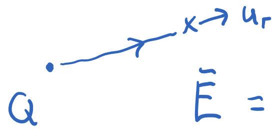
Varausrenkaan sähkökenttä symmetria-akselilla

Origonsiirto

0D · Q

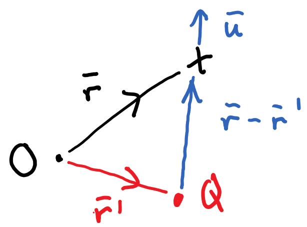
1D q

Origonsiirto



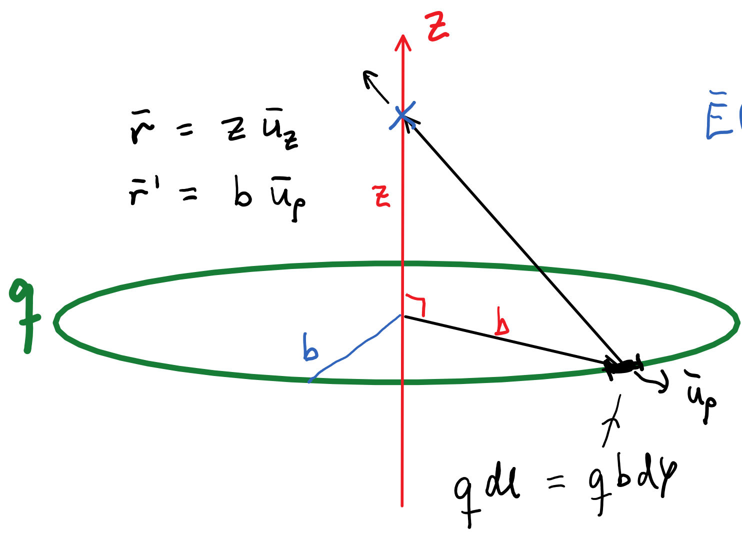
$$\bar{E} = \frac{Q\,\bar{u}_r}{4\pi\varepsilon_0 r^2}$$

0D $\cdot$ Q
1D q
2D ρ_s
3D ρ



$$\bar{u} = \frac{\bar{r}-\bar{r}'}{|\bar{r}-\bar{r}'|}$$

$$\bar{E}(\bar{r}) = \frac{Q\,(\bar{r}-\bar{r}')}{4\pi\varepsilon_0 |\bar{r}-\bar{r}'|^3}$$



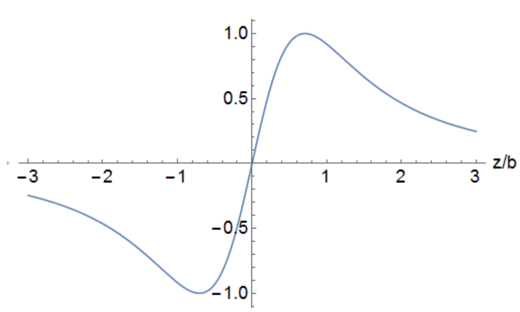
$$\bar{E}(z) = \int_0^{2\pi} \frac{q\,b\,d\varphi\,(z\bar{u}_z - b\bar{u}_\rho)}{4\pi\varepsilon_0 \underbrace{|z\bar{u}_z - b\bar{u}_\rho|^3}_{(z^2+b^2)^{3/2}}}$$

$$= \frac{q\,b}{4\pi\varepsilon_0 (z^2+b^2)^{3/2}} \int_0^{2\pi} (z\,\bar{u}_z - b\,\bar{u}_\rho)\,d\varphi$$

$$= \frac{q\,b\,z\,\bar{u}_z}{4\pi\varepsilon_0 (z^2+b^2)^{3/2}} \int_0^{2\pi} d\varphi$$

$$\int_0^{2\pi} \bar{u}_\rho\,d\varphi = 0$$

$$\bar{E}(z) = \frac{\overbrace{q\,2\pi b}^{Q}\,z}{4\pi\varepsilon_0 (z^2+b^2)^{3/2}}\,\bar{u}_z$$

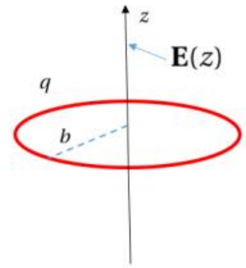


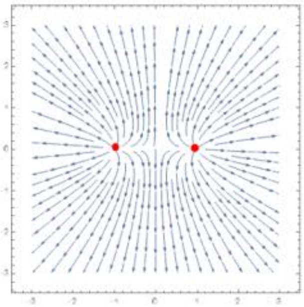
$$\bar{E}(0) = 0$$

$$z \gg b \quad \Rightarrow \quad (z^2 + \cancel{b^2})^{3/2} = z^3$$

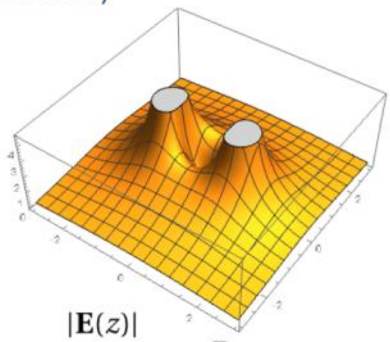
$$z \gg b: \quad \frac{Q}{4\pi\varepsilon_0 z^2}\,\bar{u}_z$$

Varausrenkaan (vakioviivavaraus) kenttä kaikkialla (elliptiset integraalit!)

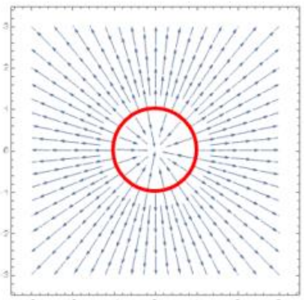




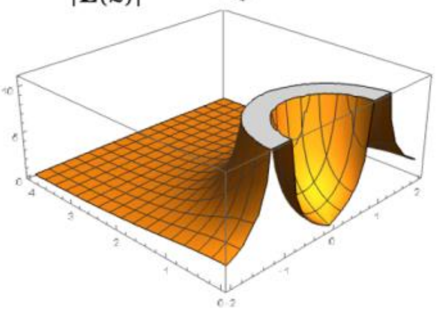
zx-taso



$|\mathbf{E}(z)|$



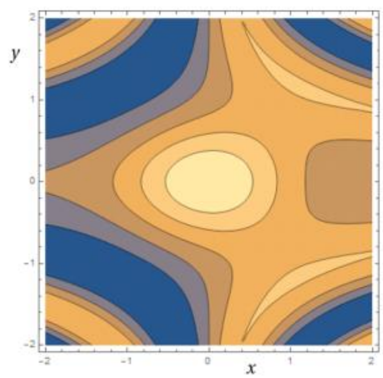
xy-taso

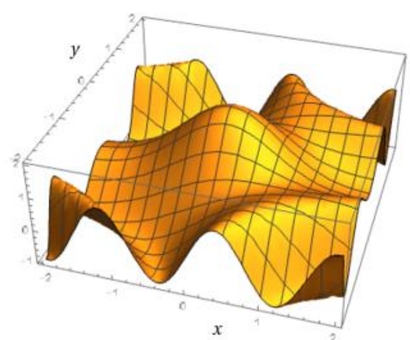


Oppikirja (staattinen): luku 3.4.5, sivut 58-59

Kahden muuttujan skalaarikenttä

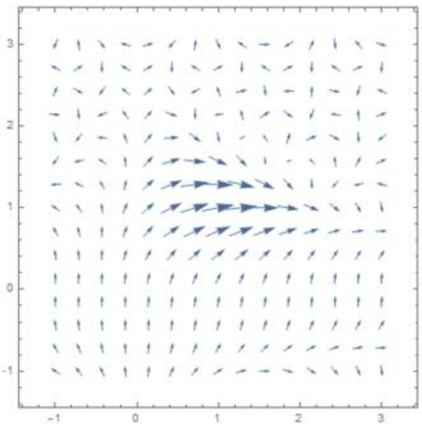
$$F(x, y) = \sin(xy^2) + 2\mathrm{e}^{-(x^2+2y^2)}$$

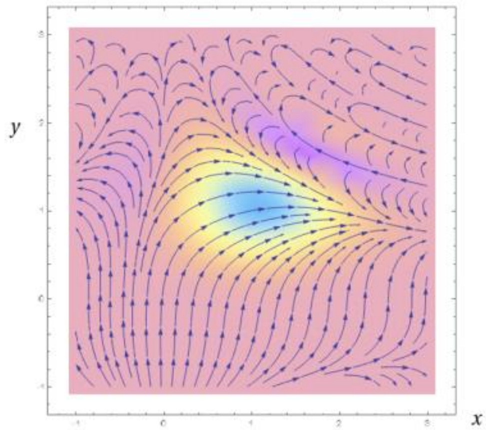




Kahden muuttujan vektorikenttä

$$\mathbf{G}(x,y) = \mathbf{u}_x\left[\sin(xy^2) + 2e^{-(x-1)^2-2(y-1)^2}\right] + \mathbf{u}_y\cos(xy^2)$$





NABLA

$$\nabla = \bar{u}_x \frac{\partial}{\partial x} + \bar{u}_y \frac{\partial}{\partial y} + \bar{u}_z \frac{\partial}{\partial z}$$

Gradientti

$$F(\bar{r}) = F(x,y,z)$$

$$\nabla F = \bar{u}_x \frac{\partial F}{\partial x} + \bar{u}_z \frac{\partial F}{\partial y} + \bar{u}_z \frac{\partial F}{\partial z}$$

$$r = \sqrt{x^2+y^2+z^2} = (x^2+y^2+z^2)^{1/2}$$

$$\frac{\partial r}{\partial x} = \frac{1}{2}(\quad)^{-1/2} \cdot 2x = \frac{x}{\sqrt{x^2+y^2+z^2}}$$

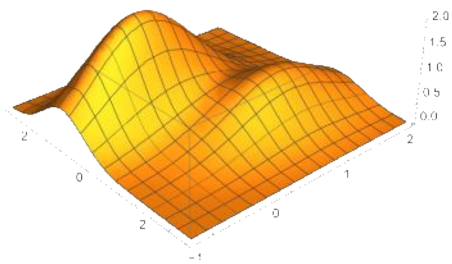
$$F(\bar{r}) = r \quad \Rightarrow \quad \nabla F = \bar{u}_x \frac{x}{r} + \bar{u}_y \frac{y}{r} + \bar{u}_z \frac{z}{r}$$

$$= \frac{x\bar{u}_x + y\bar{u}_y + z\bar{u}_z}{r} = \frac{\bar{r}}{r} = \bar{u}_r$$

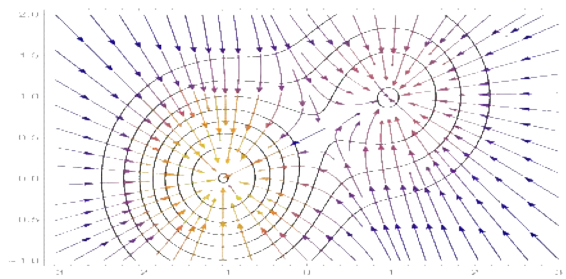
$$\nabla = \bar{u}_r \frac{\partial}{\partial r} + \bar{u}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \bar{u}_\varphi \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}$$ (PALLOKOORDINAATISTOSSA)

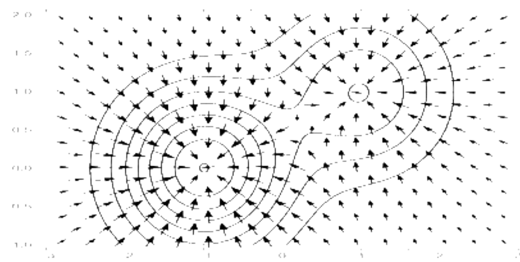
$$\nabla r = \bar{u}_r \frac{\partial r}{\partial r} = \bar{u}_r$$

Esimerkki: $F(x,y) = e^{-(x-1)^2-(y-1)^2} + 2e^{-(x+1)^2-y^2}$



Gradientti on kohtisuorassa tasa-arvoviivoja vastaan



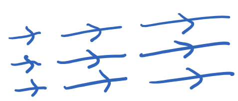


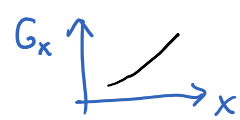
Divergenssi

$$\nabla \cdot \bar{G}(\bar{r}) = \left(\bar{u}_x \frac{\partial}{\partial x} + \bar{u}_y \frac{\partial}{\partial y} + \bar{u}_z \frac{\partial}{\partial z}\right) \cdot \left(\bar{u}_x G_x + \bar{u}_y G_y + \bar{u}_z G_z\right)$$

$$= \frac{\partial G_x}{\partial x} + \frac{\partial G_y}{\partial y} + \frac{\partial G_z}{\partial z}$$

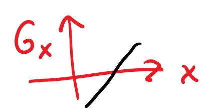
$\bar{G} = \bar{u}_x G_x(x)$

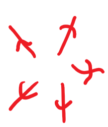


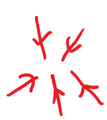


$\frac{\partial G_x}{\partial x} > 0$

$\nabla \cdot \bar{G} > 0$





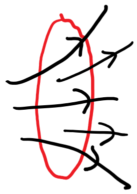


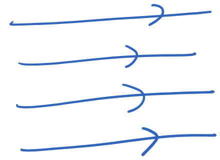
$\nabla \cdot \bar{G} < 0$

$\bar{G}(\bar{r}) = \bar{r} = \bar{u}_x x + \bar{u}_y y + \bar{u}_z z$ $\qquad \nabla \cdot \bar{G} = 1 + 1 + 1 = 3$

$\bar{G}(\bar{r}) = \bar{u}_r r$

$$\nabla \cdot \bar{G} = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 G_r\right) = \frac{1}{r^2} \frac{\partial}{\partial r} r^3 = \frac{1}{r^2} 3r^2 = 3$$





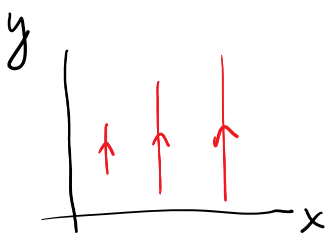
$3\bar u_x$

Roottori

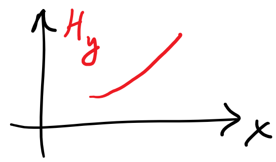
$$\nabla\times\bar H(\bar r) = \left(\bar u_x\frac{\partial}{\partial x}+\bar u_y\frac{\partial}{\partial y}+\bar u_z\frac{\partial}{\partial z}\right)\times\left(\bar u_x H_x+\bar u_y H_y+\bar u_z H_z\right)$$

$$= \begin{vmatrix} \bar u_x & \bar u_y & \bar u_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ H_x & H_y & H_z \end{vmatrix} = \bar u_x\left(\frac{\partial H_z}{\partial y}-\frac{\partial H_y}{\partial z}\right)+\bar u_y\left(\frac{\partial H_x}{\partial z}-\frac{\partial H_z}{\partial x}\right)$$

$$+\bar u_z\left(\underset{+}{\frac{\partial H_y}{\partial x}}-\frac{\partial H_x}{\partial y}\right)$$



$$\bar H = \bar u_y H_y(x)$$



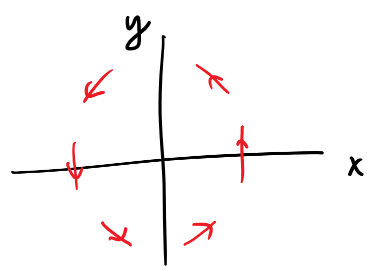
$$\frac{\partial H_y}{\partial x} > 0$$

$$\nabla\times\bar H = \bar u_z \underset{>0}{\frac{\partial H_y}{\partial x}}$$

$$\bar G(\bar r) = \bar u_r r = \bar u_x \underset{G_x}{x} + \bar u_y \overset{G_y}{y} + \bar u_z \overset{G_z}{z}$$

$$\nabla\times\bar G = \begin{vmatrix} \bar u_x & \bar u_y & \bar u_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix} = \bar u_x\left(\frac{\partial z}{\partial y}-\frac{\partial y}{\partial z}\right)+\dots = 0$$

$$\bar H = \overset{\downarrow}{\rho}\,\bar u_\varphi = H_\varphi \bar u_\varphi$$

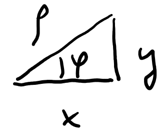


$$\nabla\times\bar H = \frac{1}{\rho}\begin{vmatrix} \bar u_\rho & \rho\bar u_\varphi & \bar u_z \\ \partial/\partial\rho & \partial/\partial\varphi & \partial/\partial z \\ \underset{0}{\cancel{H_\rho}} & \underset{\rho}{\rho\cdot H_\varphi} & \underset{0}{\cancel{H_z}} \end{vmatrix}$$

$$= \frac{1}{\ } \left(\bar u_\rho \left(-\frac{\partial}{\ }\rho^2\right) + \rho\bar u_\varphi (0-0) + \bar u_z\left(\frac{\partial}{\partial\rho}\rho^2 - 0\right)\right) = 2\bar u_z$$

$$= \frac{1}{\rho}\left(\bar{u}_\rho\left(-\underbrace{\frac{\partial}{\partial z}\rho^2}_{0}\right) + \rho\,\bar{u}_\varphi\,(0-0) + \bar{u}_z\left(\underbrace{\frac{\partial}{\partial \rho}\rho^2}_{2\rho} - 0\right)\right) = 2\bar{u}_z$$

$$\bar{H} = \rho\,\bar{u}_\varphi = \underbrace{\sqrt{x^2+y^2}}_{\rho}\left(\bar{u}_y \underbrace{\cos\varphi}_{+x/\rho} - \bar{u}_x \underbrace{\sin\varphi}_{y/\rho}\right)$$



$$\cos\varphi = \frac{x}{\rho}$$

$$= -\bar{u}_x\,y + \bar{u}_y\,x$$

$$\nabla\times\bar{H} = \begin{vmatrix} \bar{u}_x & \bar{u}_y & \bar{u}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & x & 0 \end{vmatrix} = \bar{u}_z\left(\frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y}\right) = \bar{u}_z\left(1-(-1)\right)$$

$$= 2\bar{u}_z$$