Mathematics for Economists

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Linear equations

Linear combinations revisited

- For a given set of m vectors $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m$ in $\mathbb{R}^n$, the corresponding set of all linear combinations is

$$\mathcal{L}[\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m] := \{a_1\boldsymbol{u}_1 + \cdots + a_m\boldsymbol{u}_m : a_1, \ldots, a_m \in \mathbb{R}\}$$

- For a given subset $V \subseteq \mathbb{R}^n$, if there exist m vectors $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m$ in $\mathbb{R}^n$ such that

$$V = \mathcal{L}[\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m],$$

then we say the V is **spanned** by $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m$.

Span

- Let $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m$ be a set of m vectors in $\mathbb{R}^n$. Form the $n \times m$ matrix A:

$$A = \begin{pmatrix} \boldsymbol{u}_1 & \ldots & \boldsymbol{u}_m \end{pmatrix}.$$

- Then we say that a vector $\boldsymbol{b} \in \mathbb{R}^n$ is contained in the space spanned by $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m$ if and only if the system $A\boldsymbol{c} = \boldsymbol{b}$ has at least one solution $\boldsymbol{c}$.

- In addition, a set of vectors that spans $\mathbb{R}^n$ must contain at least n vectors. [Note: not every set of n vectors spans $\mathbb{R}^n$!]

Linear independence

- Different sets of vectors can span the same space. For example, each of the following sets of vectors spans $\mathbb{R}^2$:

$$(a) \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$(b) \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$(c) \quad \begin{pmatrix} 2 \\ -2 \end{pmatrix}, \quad \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

Basis and dimension

- Let $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m$ be a set of m vectors in $\mathbb{R}^n$. Let $U \subseteq \mathbb{R}^n$. We say that $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m$ forms a **basis** of U if:
  1. U is spanned by $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m$;
  2. $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m$ are linearly independent.
- Loosely put, a basis of U is a "minimal" set of vectors that spans U
  - the dimension of U is m.
- The same set can have different bases
- Every basis of $\mathbb{R}^n$ contains exactly n vectors

Linear independence

Proposition

Let $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n$ *be a set of* n *vectors in* $\mathbb{R}^n$*. Let* A *be the* $n \times n$ *matrix*

$$A = \begin{pmatrix} \boldsymbol{u}_1 & \ldots & \boldsymbol{u}_n \end{pmatrix}.$$

Then the following are equivalent:

1. $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n$ *are linearly independent*
2. $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n$ *span* $\mathbb{R}^n$
3. $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n$ *form a basis of* $\mathbb{R}^n$
4. *the determinant of* A *is nonzero.*

Linear independence

The **canonical basis** of $\mathbb{R}^n$ consists of vectors $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$, where

$$\begin{aligned}
\boldsymbol{e}_1 &= (1, 0, 0, \ldots, 0, 0) \\
\boldsymbol{e}_2 &= (0, 1, 0, \ldots, 0, 0) \\
\ldots &\quad \ldots \quad \ldots \\
\boldsymbol{e}_n &= (0, 0, 0, \ldots, 0, 1)
\end{aligned}$$

Exercise

Write $\boldsymbol{v} = (2, -5, 3)$ as a linear combination of the following three vectors

$$\begin{aligned}
\boldsymbol{u}_1 &= (1, -3, 2) \\
\boldsymbol{u}_2 &= (2, -4, -1) \\
\boldsymbol{u}_3 &= (1, -5, 7).
\end{aligned}$$

Exercise

Show that the following three vectors form a basis of $\mathbb{R}^3$:

$$\begin{aligned}
\boldsymbol{u}_1 &= (1, 1, 1) \\
\boldsymbol{u}_2 &= (1, 2, 3) \\
\boldsymbol{u}_3 &= (1, 5, 8).
\end{aligned}$$

Rank

- $\text{rank}(A)$ = dimension of the set $\mathcal{L}[A^1, \ldots, A^n]$, where A^i, $i = 1, \ldots, m$, are to columns vectors of A
- column rank = dimension of $\text{span}(A^1, \ldots, A^n)$
- row rank = dimension of $\text{span}(A_1, \ldots, A_m)$
- rank = column rank = row rank
- rank is at most $\min(m, n)$, if $\text{rank}(A) = \min(m, n)$ we say that A has full rank
- if $\text{rank}(A) = m$ it has full row rank, if $\text{rank}(A) = n$ it has full column rank

Row-echelon forms

A bit of terminology:

- A matrix is in **row echelon form** if each row begins with more zeros than the row above it
- The first non-zero entry in each row of a matrix in row echelon form is called a **pivot**

Examples:

$$B = \begin{bmatrix} \mathbf{4} & 6 & 0 & 0 & 1 \\ 0 & \mathbf{3} & 0 & 5 & 2 \\ 0 & 0 & \mathbf{1} & 4 & 4 \\ 0 & 0 & 0 & \mathbf{6} & 5 \end{bmatrix}$$

$$C = \begin{bmatrix} \mathbf{3} & 6 & 7 & 0 & 1 \\ 0 & 0 & \mathbf{4} & 5 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Row-echelon forms

A bit of terminology:

- A matrix is in **reduced row echelon form** if it is in row echelon form with each pivot equal to one and each column that contains a pivot has no other nonzero entries

Examples:

$$D = \begin{bmatrix} \mathbf{1} & 0 & 0 & 0 & 1 \\ 0 & \mathbf{1} & 0 & 0 & 2 \\ 0 & 0 & \mathbf{1} & 0 & 3 \\ 0 & 0 & 0 & \mathbf{1} & 4 \end{bmatrix}$$

$$E = \begin{bmatrix} \mathbf{1} & 3 & 0 & 0 & 1 \\ 0 & 0 & \mathbf{1} & 5 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

With the Gauss-Jordan elimination method, we transform the augmented coefficient matrix in reduced row echelon form

Finding the rank

A bit of terminology:

- The **rank** of a matrix is the number of nonzero rows in its row echelon form (or, equivalently, in its reduced row echelon form)

Examples:

$$B = \begin{bmatrix} \mathbf{4} & 6 & 0 & 0 & 1 \\ 0 & \mathbf{3} & 0 & 5 & 2 \\ 0 & 0 & \mathbf{1} & 4 & 4 \\ 0 & 0 & 0 & \mathbf{6} & 5 \end{bmatrix}$$

$$C = \begin{bmatrix} \mathbf{3} & 6 & 7 & 0 & 1 \\ 0 & 0 & \mathbf{4} & 5 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We have that $\mathrm{rank}(B) = 4$ and $\mathrm{rank}(C) = 2$

Finding the rank

- Turn matrix into its row echelon form by using the three elementary row operations (that constitute the Gauss-Jordan elimination method):
  1. multiplying both sides of an equation by a nonzero real number;
  2. adding a multiple of one equation to another;
  3. interchanging (swapping) two equations.

- Write a system in matrix form, perform elementary operations until row-echelon form is obtained
  - if k is the number of zero rows and n is the number of rows, rank of A is $n - k$

Properties of the rank of a matrix

A few properties:

- Let A be an $m \times n$ matrix. Then we have:
  - $0 \leq \mathrm{rank}(A) \leq \min\{m, n\}$
  - $\mathrm{rank}(A) = 0$ if and only if A is a zero matrix, i.e. a matrix all of whose entries are zero
- Let A be a coefficient matrix and $\hat{A}$ be an augmented coefficient matrix for a system of linear equations. Then we have:
  - $\mathrm{rank}(A) \leq \mathrm{rank}(\hat{A})$

Existence of solutions for systems of linear equations

Proposition (Existence of solutions)

A system of linear equations with coefficient matrix A and augmented coefficient matrix $\hat{A}$ has a solution if and only if

$$\mathrm{rank}(A) = \mathrm{rank}(\hat{A}).$$

NOTE: The solution is not necessarily unique.

How many solutions can a system of linear equations have?

Proposition (Number of solutions)

In a system of linear equations, exactly one of the following is true:

- *The system has* **no** *solution;*
- *The system has exactly* **one** *solution;*
- *The system has* **infinitely** *many solutions.*

For a given system of linear equations, the proposition does not tell us which of the three cases holds. We can say something more about this depending on the number of equations (m) and unknowns (n). We will consider three cases:

1. $m < n$, i.e. more unknowns than equations (underdetermined systems)
2. $m > n$, i.e. more equations than unknowns (overdetermined systems)
3. $m = n$, i.e. as many equations as unknowns

1) Solutions when $m < n$

- In general, the system has 0 or infinitely many solutions:
  - A system with no solution:
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 2 \end{array}\right]$$
  - A system with infinitely many solutions:
$$\left[\begin{array}{ccc|c} 1 & 0 & 4 & 1 \\ 0 & 1 & 2 & 1 \end{array}\right]$$

    The solution(s) can be written as $x_1 = 1 - 4x_3$ and $x_2 = 1 - 2x_3$, where x_3 is a *free variable*. That is, the free variable can take on any value in $\mathbb{R}$. Once one chooses a value for the free variable, the value of the other variables (called *basic variables*) is uniquely determined

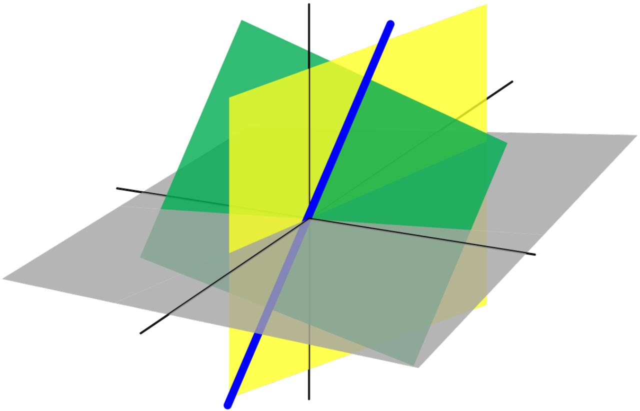
1) Solutions when $m < n$

- If $\boldsymbol{b} = \mathbf{0}$, the system has infinitely many solutions:
  - Example:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

- If $rank(A) = m$, then the system has infinitely many solutions

Reminder: Solutions as Intersections of hyperplanes



2) Solutions when $m > n$

- In general, the system has 0, 1, or infinitely many solutions:
  - A system with no solution:

$$\left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right]$$

  - A system with exactly one solution:

$$\left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 2 & 0 & 8 \end{array}\right]$$

  - A system with infinitely many solutions:

$$\left[\begin{array}{cc|c} 1 & 1 & 4 \\ 2 & 2 & 8 \\ 3 & 3 & 12 \end{array}\right]$$

2) Solutions when $m > n$

- If $\boldsymbol{b} = \mathbf{0}$, the system has either 1 or infinitely many solutions:
  - A system with exactly one solution:

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{array}\right]$$

  - A system with infinitely many solutions:

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 2 & -2 & 0 \\ -1 & 1 & 0 \end{array}\right]$$

- If $\mathrm{rank}(A) = n$, the system has either 0 or 1 solution

3) Solutions when $m = n$

- In general, the system has 0, 1, or infinitely many solutions:
  - A system with no solution:
    $$\left[\begin{array}{cc|c} 1 & 1 & 4 \\ 2 & 2 & 0 \end{array}\right]$$
  - A system with exactly one solution:
    $$\left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 0 \end{array}\right]$$
  - A system with infinitely many solutions:
    $$\left[\begin{array}{cc|c} 1 & 1 & 4 \\ 2 & 2 & 8 \end{array}\right]$$

3) Solutions when $m = n$

- If $\boldsymbol{b} = \mathbf{0}$, the system has 1 or infinitely many solutions:
  - A system with exactly one solution:

$$\left[\begin{array}{cc|c} 1 & 4 & 0 \\ 5 & 7 & 0 \end{array}\right]$$

  - A system with infinitely many solutions:

$$\left[\begin{array}{cc|c} 1 & 4 & 0 \\ 2 & 8 & 0 \end{array}\right]$$

- If $\mathrm{rank}(A) = m = n$, the system has exactly 1 solution

Exercise 1

Compute the rank of the following matrix:

$$\begin{bmatrix} 1 & 6 & -7 & 3 \\ 1 & 9 & -6 & 4 \\ 1 & 3 & -8 & 4 \end{bmatrix}$$

Exercise 1

Multiply row 1 with (-1) and add to other two rows:

$$\begin{bmatrix} 1 & 6 & -7 & 3 \\ 0 & 3 & 1 & 1 \\ 0 & -3 & -1 & 1 \end{bmatrix}$$

Add up last two rows:

$$\begin{bmatrix} 1 & 6 & -7 & 3 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Rank is three!

Exercise 2

Find all the values of the parameter a such that the following system of linear equations has:

- no solution
- exactly one solution
- infinitely many solutions.

$$\begin{aligned} 6x + y &= 7 \\ 3x + y &= 4 \\ -6x - 2y &= a. \end{aligned}$$

Exercise (from Lecture 1)

- Use the Gauss-Jordan elimination method to solve the following system of 4 linear equations in 4 unknowns:

$$\begin{aligned} 2x_1 - x_2 &= 0 \\ -x_1 + 2x_2 - x_3 &= 0 \\ -x_2 + 2x_3 - x_4 &= 0 \\ -x_3 + 2x_4 &= 5 \end{aligned}$$

- Rewrite the system in matrix form:

$$\left[\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & 5 \end{array}\right]$$

Exercise (from Lecture 1)

- Apply the Gauss-Jordan elimination method directly to the matrix. At the first step, multiply the first row by $\frac{1}{2}$ and add it to the second row:

$$\left[\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 0 \\ -1+\frac{1}{2}\times 2 & 2+\frac{1}{2}\times(-1) & -1+\frac{1}{2}\times 0 & 0+\frac{1}{2}\times 0 & 0+\frac{1}{2}\times 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & 5 \end{array}\right]$$

that is

$$\left[\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & 5 \end{array}\right]$$

Exercise (from Lecture 1)

Multiply row 2 with $2/3$ and add to third row:

$$\left[\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 0 \\ 0 & 3/2 & -1 & 0 & 0 \\ 0 & -1+1 & 2-2/3 & -1 & 0 \\ 0 & 0 & -1 & 2 & 5 \end{array}\right]$$

that is

$$\left[\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 0 \\ 0 & 3/2 & -1 & 0 & 0 \\ 0 & 0 & 4/3 & -1 & 0 \\ 0 & 0 & -1 & 2 & 5 \end{array}\right]$$

Exercise (from Lecture 1)

Multiply row 3 with $3/4$ and add to the last row:

$$\left[\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 0 \\ 0 & 3/2 & -1 & 0 & 0 \\ 0 & 0 & 4/3 & -1 & 0 \\ 0 & 0 & -1+1 & 2-3/4 & 5 \end{array}\right]$$

that is

$$\left[\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 & 0 \\ 0 & 0 & 4/3 & -1 & 0 \\ 0 & 0 & 0 & 5/4 & 5 \end{array}\right]$$

We are now done with the "forward sweep"

Exercise (from Lecture 1)

Multiply the last row with $4/5$:

$$\left[\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 & 0 \\ 0 & 0 & 4/3 & -1 & 0 \\ 0 & 0 & 0 & 1 & 20/5 \end{array}\right]$$

and then and to the second last

$$\left[\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 & 0 \\ 0 & 0 & 4/3 & 0 & 4 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

Exercise (from Lecture 1)

Multiply the second last row with 3/4:

$$\left[\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 12/4 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

and then and to the second

$$\left[\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

Exercise (from Lecture 1)

Multiply the second last row with $2/3$:

$$\left[\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

and then and to the first row

$$\left[\begin{array}{cccc|c} 2 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

Exercise (from Lecture 1)

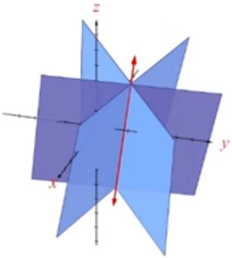
- At the end of the Gauss-Jordan elimination algorithm, we obtain the following matrix

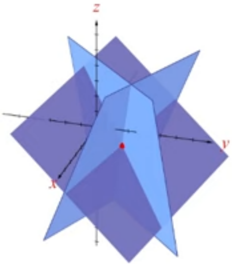
$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right],$$

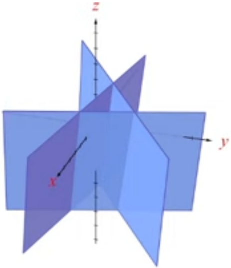
which gives us the solution

$$\begin{aligned} x_1 &= 1 \\ x_2 &= 2 \\ x_3 &= 3 \\ x_4 &= 4. \end{aligned}$$

A video of a 3d inconsistent systems







$$\left[\begin{array}{ccc|c} 4 & -4 & -1 & -10 \\ 4 & 12 & -5 & 14 \\ 16 & 4 & -9 & -10 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 4 & -4 & -1 & -10 \\ 4 & 12 & -5 & 14 \\ 4 & 5 & 6 & -15 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 4 & -4 & -1 & -6 \\ 12 & 4 & -7 & 22 \\ 4 & 12 & -5 & 2 \end{array}\right]$$

Matrix inversion by Gauss-Jordan elimination

- Idea, let e_i denote ith coordinate vector
  - finding solutions $\mathbf{x}_i$ to equations $A\mathbf{x} = \mathbf{e}_i$, $i = 1, \ldots, n$
  - $\Rightarrow A^{-1} = (\mathbf{x}_1 \ \mathbf{x}_2 \ \cdots \ \mathbf{x}_n)$
  - Note 1: in the identity matrix I the columns are $\mathbf{e}_1, \cdots, \mathbf{e}_n$, i.e., $I = (\mathbf{e}_1 \ldots \mathbf{e}_n)$
  - Note 2: by the definition of inverse $AA^{-1} = I$ which means that $A\mathbf{x}_i = \mathbf{e}_i$
- Gauss-Jordan for inversion
  - form an augmented matrix $\hat{A} = [A|I]$
  - use elementary row operations to turn $\hat{A}$ into for $[I|B]$ now B is the inverse (assuming A has full rank)

Matrix inversion: an example

$$A = \begin{pmatrix} 1 & 5 & -1 \\ 2 & 2 & -2 \\ -1 & 4 & 3 \end{pmatrix}$$

Augmented matrix: $\left(\begin{array}{ccc|ccc} 1 & 5 & -1 & 1 & 0 & 0 \\ 2 & 2 & -2 & 0 & 1 & 0 \\ -1 & 4 & 3 & 0 & 0 & 1 \end{array}\right)$

Add $-12R_1$ to R_2, and add R_1 to R_3: $\left(\begin{array}{ccc|ccc} 1 & 5 & -1 & 1 & 0 & 0 \\ 0 & -8 & 0 & -2 & 1 & 0 \\ 0 & 9 & 2 & 1 & 0 & 1 \end{array}\right)$

Matrix inversion: an example

Divide R_2 by -8, next add $-9R_2$ to R_3: $\left(\begin{array}{ccc|ccc} 1 & 5 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0.25 & -0.125 & 0 \\ 0 & 0 & 2 & -1.25 & 1.125 & 1 \end{array}\right)$

Divide R_3 by 2: $\left(\begin{array}{ccc|ccc} 1 & 5 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0.25 & -0.125 & 0 \\ 0 & 0 & 1 & -0.625 & 0.5625 & 0.5 \end{array}\right)$

Add R_3 to R_1 and $-5R_2$ to R_1: $\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -0.875 & 1.1875 & 0.5 \\ 0 & 1 & 0 & 0.25 & -0.125 & 0 \\ 0 & 0 & 1 & -0.625 & 0.5625 & 0.5 \end{array}\right)$