ELEC-A7200

Signals and Systems

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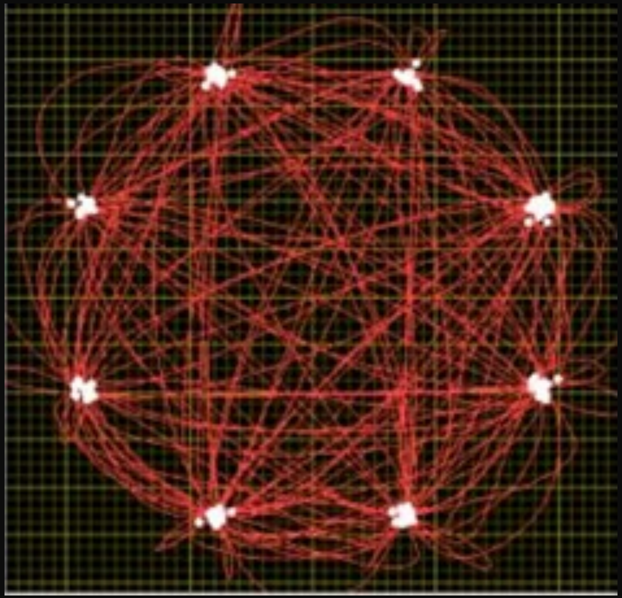
Engineering



Lecture 3
Signal space

Vector space & Signal space

Vector space

Vector (or discrete time pulse)

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{C}^n \ , x_k = a_k + jb_k$$

Inner product of two vectors *x* and *y*

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle \triangleq \boldsymbol{y}^H \boldsymbol{x} = \sum_{k=1}^{n} x_k \ y_k^*$$

Length of vector *x* (norm of a vector)

$$\|\boldsymbol{x}\| = \sqrt{\langle \boldsymbol{x}, \boldsymbol{x} \rangle} = \sqrt{\sum_{k=1}^{n} |x_k|^2}$$

Orthonormal basis $\{ \phi_k, k = 1,2,\ldots,n\}$:

$$\langle \phi_{k}, \phi_{L} \rangle = \begin{cases} 1, & k = l \\ 0, & k \neq l \end{cases}$$

$$\boldsymbol{x} = \sum_{k=1}^{n} \langle \boldsymbol{x}, \ \phi_k \rangle \ \phi_k$$

Signal space

Pulse (Energy signal)

$$x(t) \in \mathbb{C}\ , t_0 \leq t \leq t_1, \ x(t) = a(t) + jb(t)$$

Inner product of two pulses $x(t)$ *and* $y(t)$

$$\langle x(t), y(t) \rangle \triangleq \int_{t_0}^{t_1} x(t) y^*(t) dt$$

Signal energy

$$E = \|x(t)\|^2 = \langle x(t), x(t) \rangle = \int_{t_0}^{t_1} |x(t)|^2 \, dt$$

Periodic signal (Power signal)

$$x(t) \in \mathbb{C}\ , x(t + T_0) = x(t), \ x(t) = a(t) + jb(t)$$

Inner product of two pulses $x(t)$ *and* $y(t)$

$$\langle x(t), y(t) \rangle \triangleq \int_{T_0} x(t) y^*(t) dt$$

Signal power

$$P = \frac{1}{T_0} \|x(t)\|^2 = \frac{1}{T_0} \langle x(t), x(t) \rangle = \frac{1}{T_0} \int_{t_0}^{t_1} |x(t)|^2 \, dt$$

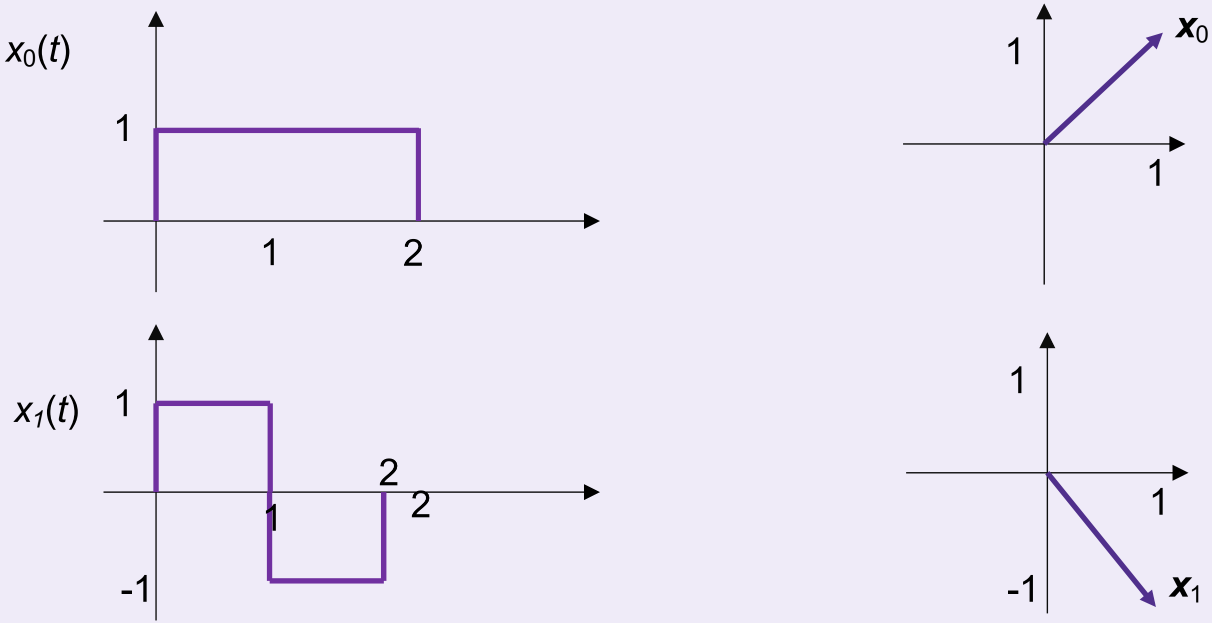
Orthonormal basis $\{\phi_k(t), k = 1,2,\ldots,m\}$:

$$\langle \phi_k(t), \phi_l(t) \rangle = \begin{cases} 1, & k = l \\ 0, & k \neq l \end{cases}$$

$$x(t) = \sum_{k=1}^{n} \langle x(t), \phi_k(t) \rangle \, \phi_k(t)$$

Problem

Determine the norm and the inner product for signals and vectors



Representing vector with orthonormal basis

Orhonormal basis

$$\phi_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix},\ \phi_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$$

Vector

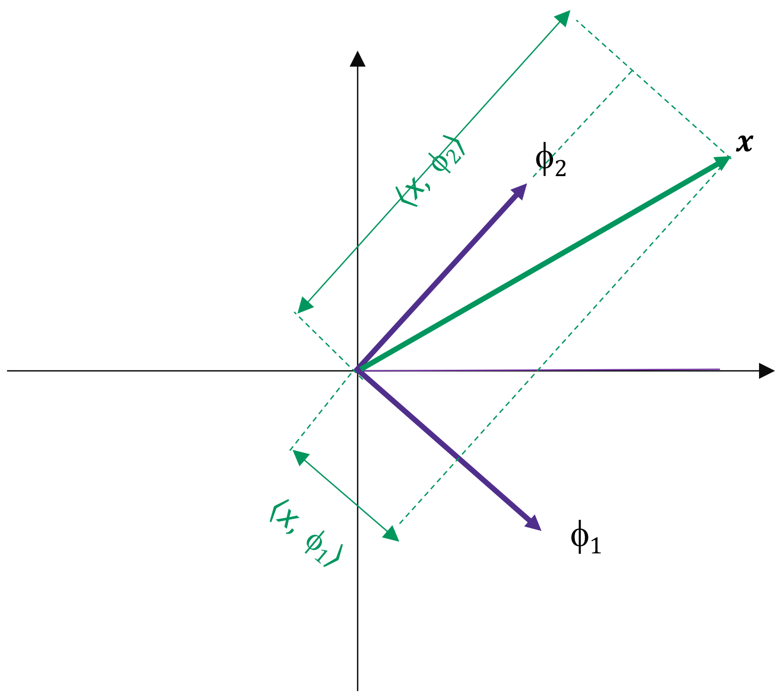
$$\boldsymbol{x} = \begin{bmatrix}1\\1/2\end{bmatrix}$$

Projection on the basis

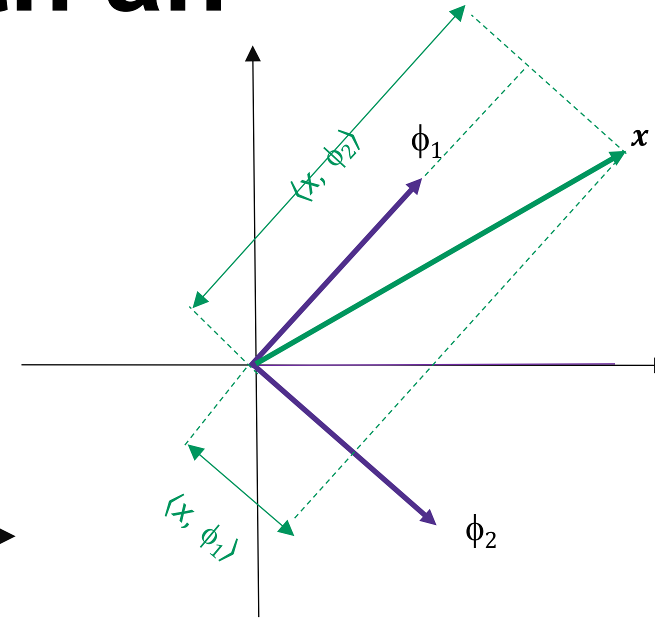
$$\langle x,\ \phi_1\rangle = 1\cdot\frac{1}{\sqrt{2}} + \frac{1}{2}\cdot\frac{-1}{\sqrt{2}} = \frac{1}{2\sqrt{2}}$$

$$\langle x,\ \phi_2\rangle = 1\cdot\frac{1}{\sqrt{2}} + \frac{1}{2}\cdot\frac{1}{\sqrt{2}} = \frac{3}{2\sqrt{2}}$$

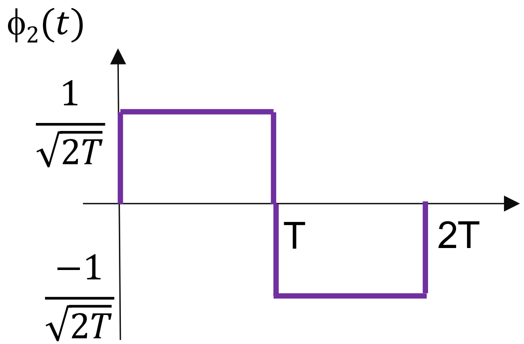
$$\boldsymbol{x} = \frac{1}{2\sqrt{2}}\phi_1 + \frac{3}{2\sqrt{2}}\phi_2$$

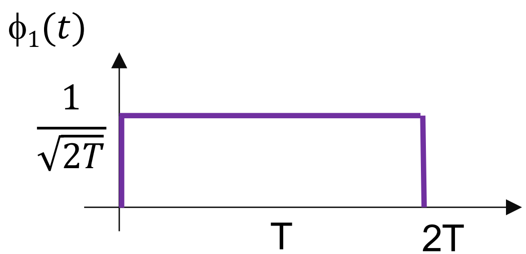


Representing pulse with an orhonormal basis

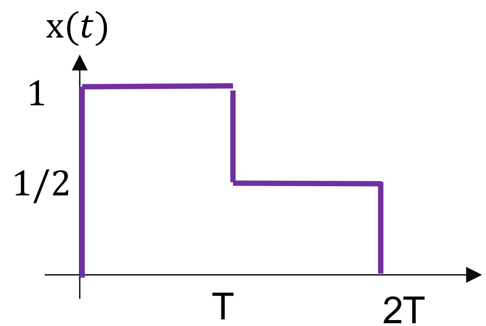


Orhonormal basis (Walsh)





Pulse



Projection on the basis

$$\langle x(t), \phi_2(t)\rangle = \int_0^{T/2} 1 \cdot \frac{1}{\sqrt{2T}}\, dt + \int_{T/2}^{T} \frac{1}{2} \cdot \frac{-1}{\sqrt{2T}}\, dt = \frac{1\sqrt{T}}{2\sqrt{2}}$$

$$\langle x(t),\ \phi_1(t)\rangle = \int_0^{T/2} 1 \cdot \frac{1}{\sqrt{2T}}\, dt + \int_{T/2}^{T} \frac{1}{2} \cdot \frac{1}{\sqrt{2T}}\, dt = \frac{3\sqrt{T}}{2\sqrt{2}}$$

$$x(t) = \frac{\sqrt{T}}{2\sqrt{2}} \phi_2(t) + \frac{3\sqrt{T}}{2\sqrt{2}}\ \phi_1(t)$$

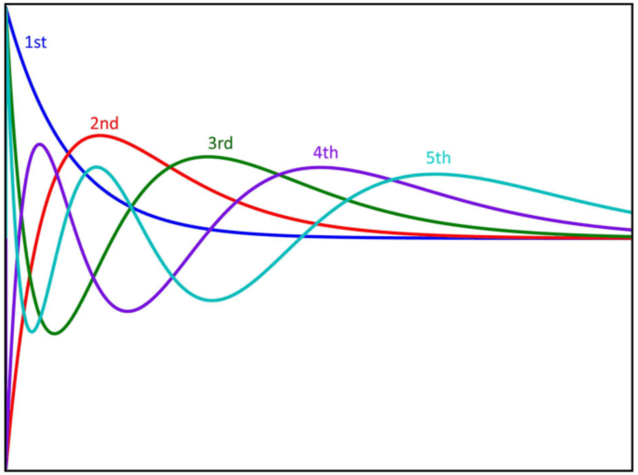
Example: Linear time invariant systems

Laguerre functions form an orthonormal basis to express the responses of stable linear time invariant (LTI) systems

$$\frac{d^n}{dt^n}y(t) = -a_1\frac{d^{n-1}}{dt^{n-1}}y(t) - \cdots - a_n y(t) + b_0\frac{d^m}{dt^m}u(t) + b_1\frac{d^{m-1}}{dt^{m-1}}u(t) + \cdots + b_m u(t)$$

$$y(t) = \sum_{n=0}^{\infty} w_n L_n(t)$$

This can be utilized in system indentification.



Laguerre basis functions

$$L_n(t) = \frac{e^t}{n!}\frac{d}{dt}t^n e^{-t}$$

Gram-Schmidt procedure

Given set of signals $\{s_k(t), k = 1,2, \ldots\}$ **find orthonormal basis** $\{\phi_k(t), k = 1,2, \ldots\}$ **that span the signal set.**

$$\phi_1(t) = \frac{s_1(t)}{\|s_1(t)\|}$$

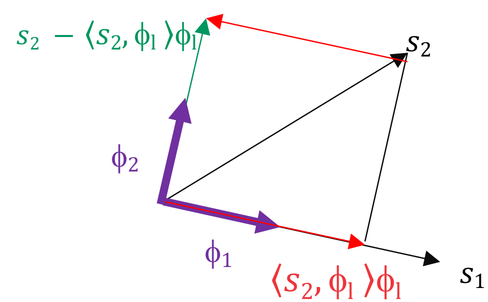
$$\tilde{\phi}_2(t) = s_2(t) - \langle s_2(t), \phi_1(t)\rangle \phi_1(t)$$

$$\phi_2(t) = \frac{\tilde{\phi}_2(t)}{\|\tilde{\phi}_2(t)\|}$$

$$\ldots$$

$$\tilde{\phi}_k(t) = s_k(t) - \sum_{l=1}^{k-1} \langle s_k(t), \phi_l(t)\rangle \phi_k(t)$$

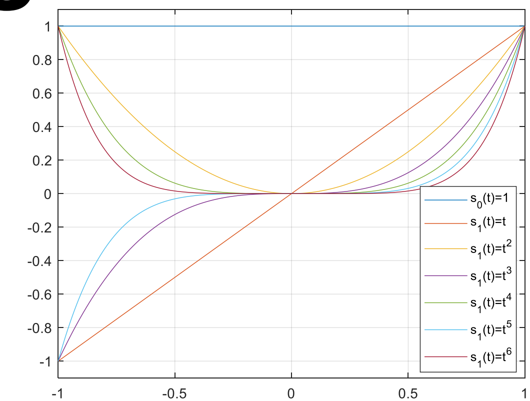
$$\phi_k(t) = \frac{\tilde{\phi}_k(t)}{\|\tilde{\phi}_k(t)\|}$$



Gram-Schmidt procedure: Legendre polynomials

Consider set of signals on $t \in [-1,1]$

$$s_0(t) = 1, s_1(t) = t, s_2(t) = t^2, \dots s_k(t) = t^k$$

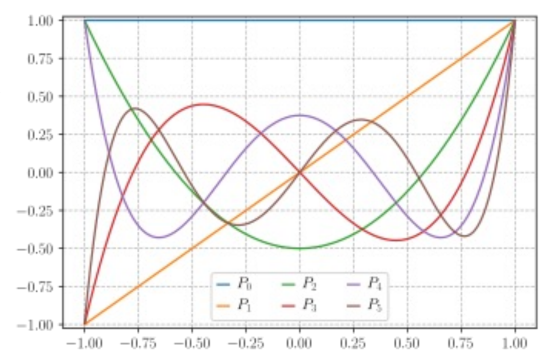


With Gram-Schmidt procedure, we can find the basis

$$\phi_0(t) = \frac{1}{\sqrt{2}}1, \phi_2(t) = \sqrt{\frac{3}{2}}t,$$

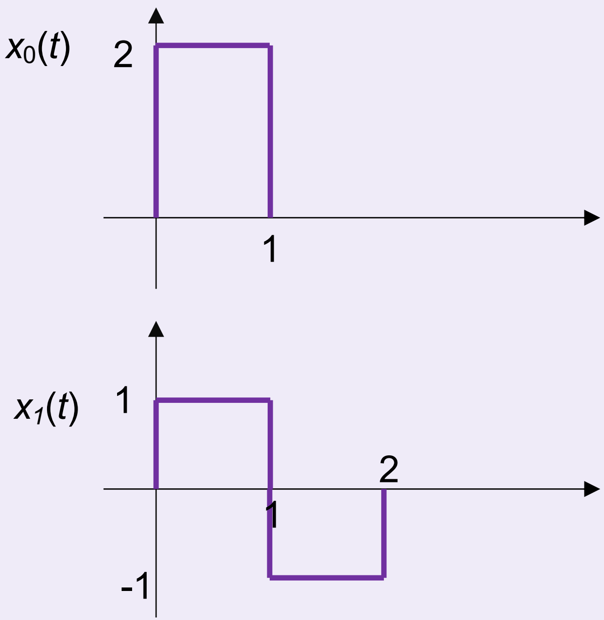
$$\phi_2(t) = \sqrt{\frac{5}{8}}(3t^2 - 1), \dots, \phi_{k(t)} = Pk(t)$$

Legendre polynomial



Problem

Find orhonormal basis for the following two signals



Calculating signal energy / power

- While in vector space one can always find orthonormal basis to represent the vectors, this is not guaranteed in the signal space.
- If a signal can be represented as sum of orthonormal signals

$$x(t) = \sum_{\boldsymbol{k=1}}^{\boldsymbol{n}} x_k \, \phi_k(t)$$

<u>Pulses</u>

Its Energy is given by

$$E = \sum_{k=1}^{n} |x_k|^2$$

Parseval's theorem

<u>Periodic signals</u>

Its power is given by

$$P = \frac{1}{T_0} \sum_{k=1}^{n} |x_k|^2$$

Problem

Consider a periodic signal $x(t) = x(t + T_0)$

that can be expressed in terms of orthonormal basis $\{\phi_k(t), k = 1,2\}$:

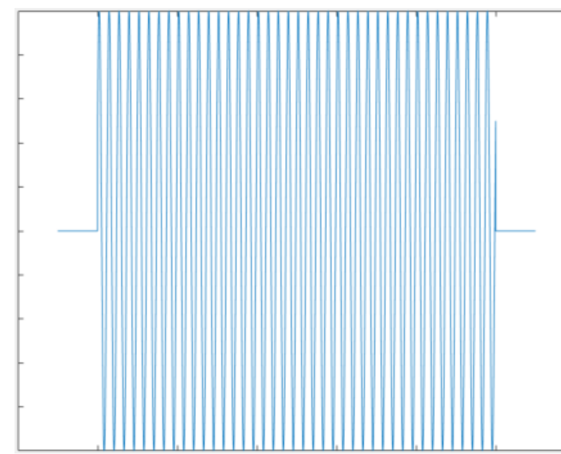
$x(t) = -\phi_1(t) + 2\phi_2(t)$

Determine the average power of the signal

Application of orthonormal basis: Signal constellation

Consider a digital modulation system that transmits *K* bits during one symbol of length *T* by controlling amplitude and phase. There are 2^K different symbols (a_k, θ_k). The transmitter generates waveform (signal) $s_k(t)$ when symbol *k* is transmitted

$$s_k(t) = \sqrt{\frac{2E}{T}} a_k \cos(2\pi f_c t + \theta_k) \qquad 0 \leq t \leq T$$



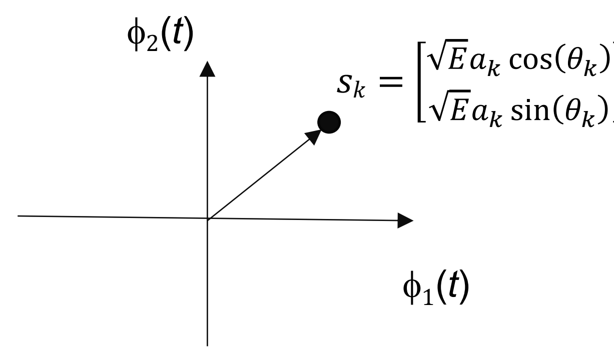
T	*Symbol duration*
E	*Symbol energy*
f_c	*Carrier frequency*
f_c T	Integer

Application of orthonormal basis: Signal constellation

Orthonormal basis

$$\phi_1(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_c t),\ \phi_2(t) = \sqrt{\frac{2}{T}}\sin(2\pi f_c t) \qquad 0 \leq t \leq T$$

Transmitted symbol visualized



Generated waveform

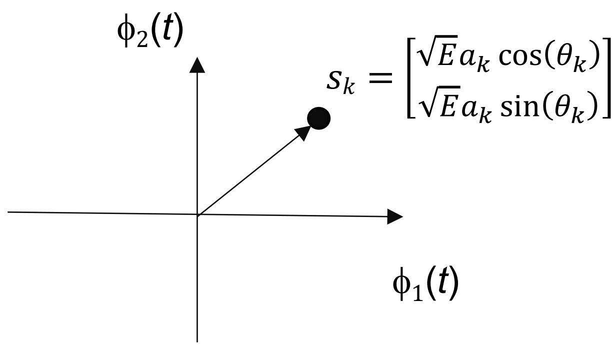
$$s_k(t) = \sqrt{E}a_k\cos(\theta_k)\,\phi_1(t) + \sqrt{E}a_k\sin(\theta_k)\,\phi_2(t)$$

Signal energy= $\left(\sqrt{E}a_k\cos(\theta_k)\right)^2 + (\sqrt{E}a_k\sin(\theta_k))^2 = Ea_k^2$

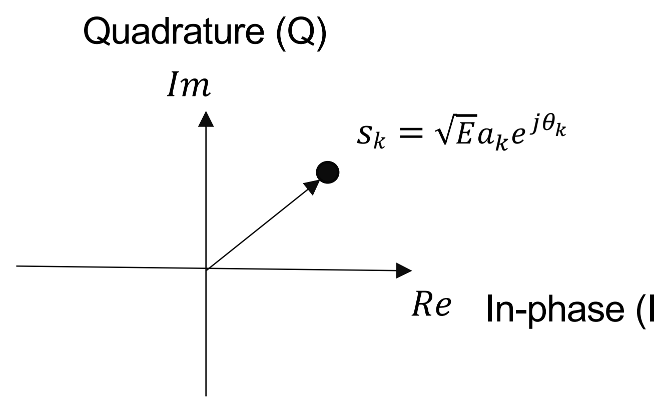
Application of orthonormal basis: Signal constellation

It is customary to represent the modulated signal using complex numbers

Transmitted symbol visualized in signal space

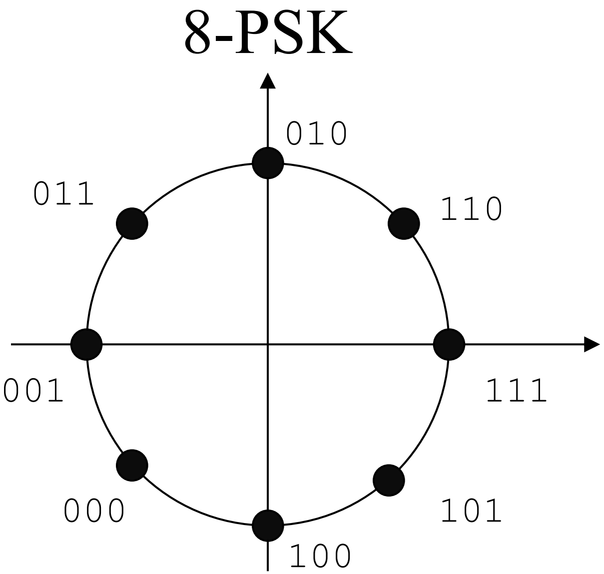


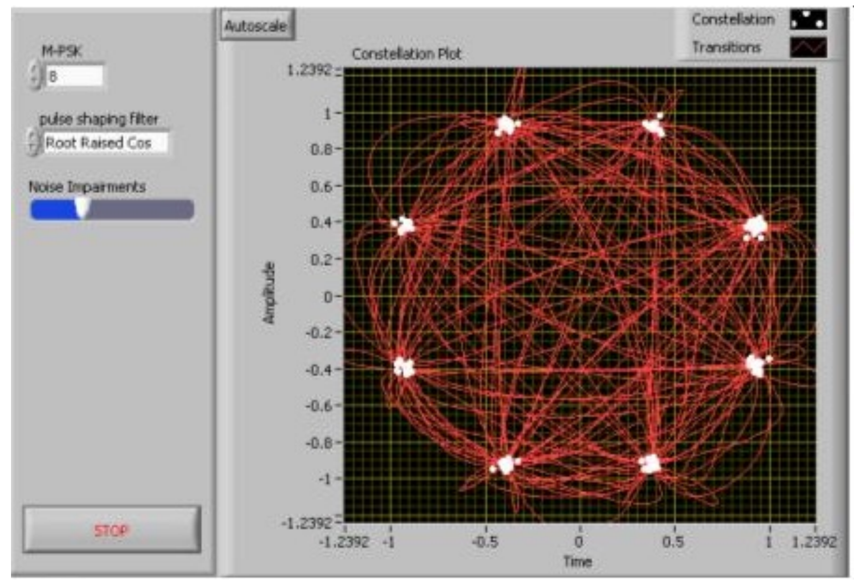
Transmitted symbol visualized in a phasor diagram



Application of orthonormal basis: Signal constellation

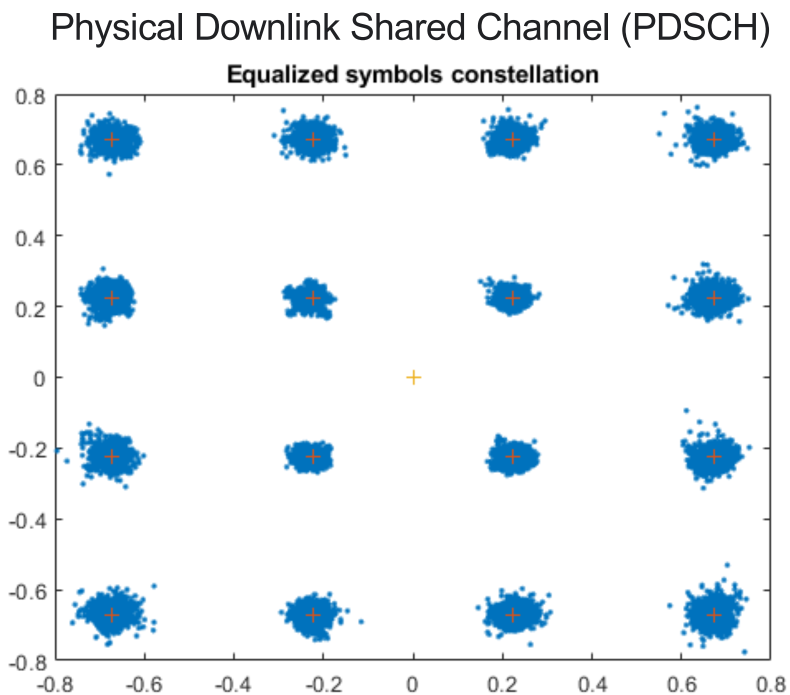
Example: 8-Phase Shift Keying (PSK) modulation signal constillation



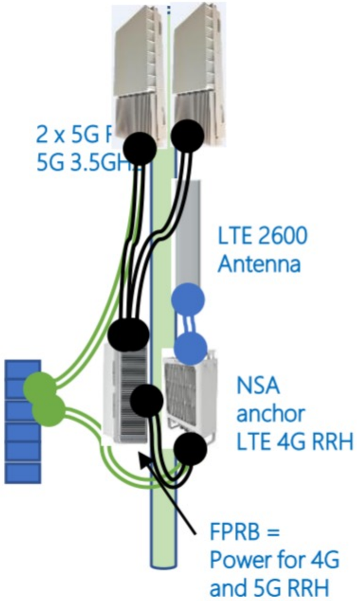


http://zone.ni.com/cms/images/devzone/tut/psk2.JPG

Exaple signal constellation



https://se.mathworks.com/help/5g/ug/evm-measurement-of-5g-nr-pdsch-waveforms.html



Aalto's 5G gNB on the roof of Väre building

Signal representation in orthonormal basis: Fourier-series

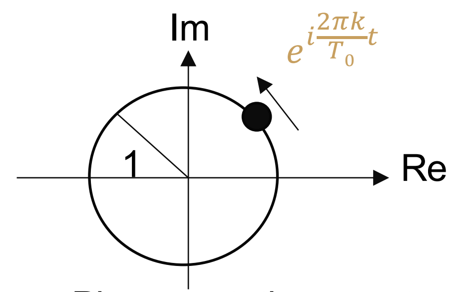
Periodic signal $x(t) = x(t + T_0)$

Orthonormal basis

$$\phi_k(t) = \frac{1}{\sqrt{T_0}} e^{i\frac{2\pi k}{T_0}t}$$

Complex signal

k=…,-2,-1,0,1,2,..



Phasor rotating counter clockwise with frequency $\frac{k}{T_0}$

Signal represented in the orthonormal basis

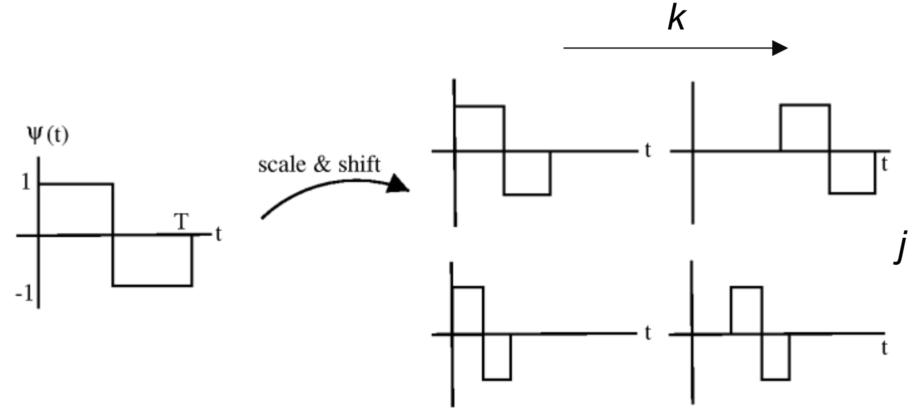
$$x(t) = \sum_{k=-\infty}^{\infty} \langle x(t), \phi_k(t) \rangle \, \phi_k(t) = \sum_{k=-\infty}^{\infty} \int_{T_0} x(t) \frac{1}{\sqrt{T_0}} e^{-i\frac{2\pi k}{T_0}t} dt \, \frac{1}{\sqrt{T_0}} e^{i\frac{2\pi k}{T_0}t} = \sum_{k=-\infty}^{\infty} X(k) \, e^{i\frac{2\pi k}{T_0}t}$$

$$X(k) = \frac{1}{T_0} \int_{T_0} x(t) e^{-i\frac{2\pi k}{T_0}t} dt$$

Coefficients of Exponential Fourier Series

Signal representation in orthonormal basis: Haar wavelets

Define $\psi(t) = \begin{cases} 1 & 0 \le t < \frac{1}{2} \\ -1 & \frac{1}{2} < t \le 1 \\ 0 & otherwise \end{cases}$



and $\psi_{jk}(t)$=$2^{\frac{j}{2}}\psi\left(2^{j}t-k\right)$

Functions are otrhogonal

$\langle\psi_{jk}(t),\psi_{lm}(t)\rangle$=0 if $(j,k)\neq(l,m)$

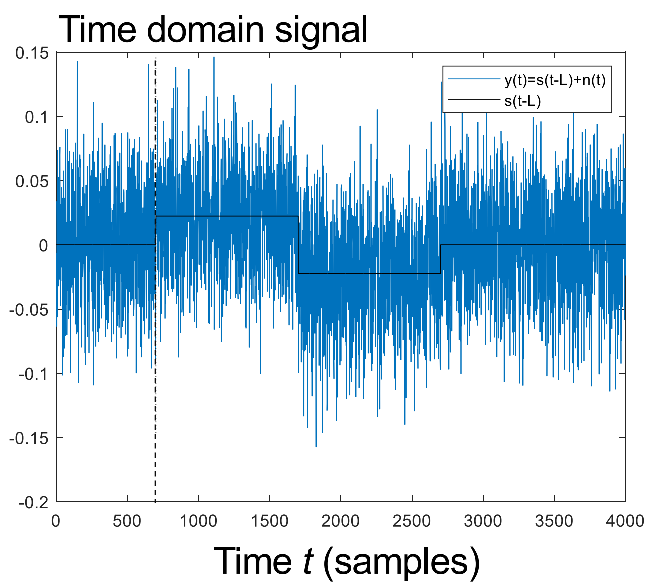
Signal representation $x(t)=c_0+\sum_{j=0}^{\infty}\sum_{k=0}^{2^j-1}c_{jk}\,\psi_{jk}(t)$ c_{jk}= $\langle x(t),\psi_{jk}(t)\rangle$

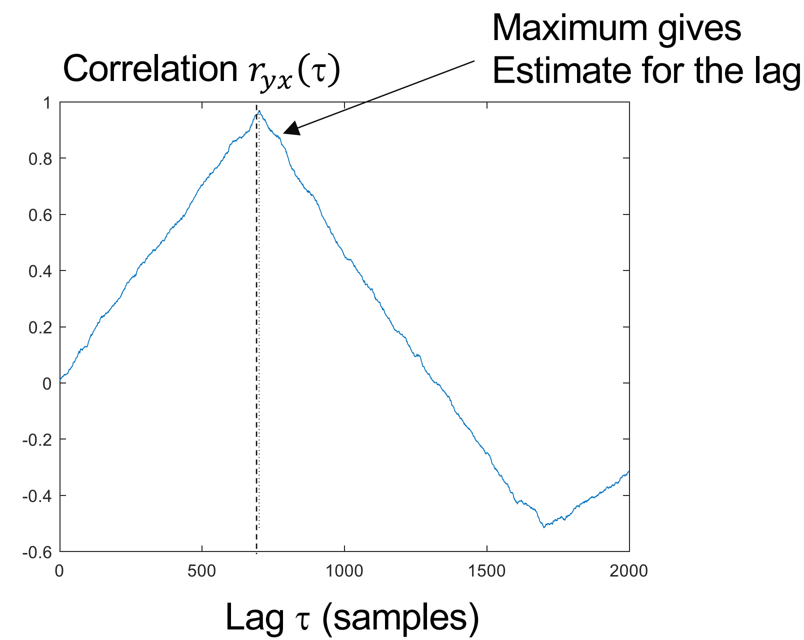
Application of inner product: Correlator

- In signal processing correlation between signals is utilized as a measure of their similarity or to locate a known signal with unknown lag.
- Correlation is calculated as an inner product between two signals.

Application of inner product: Correlator base synchronization

Example: Determine the lag of a known signal buried in noise.





$$r_{yx}(\tau) = \langle y(t), x(t-\tau) \rangle = \int_{t_0}^{t_1} y(t)x^*(t-\tau)dt$$

Application of inner product: Correlator receiver

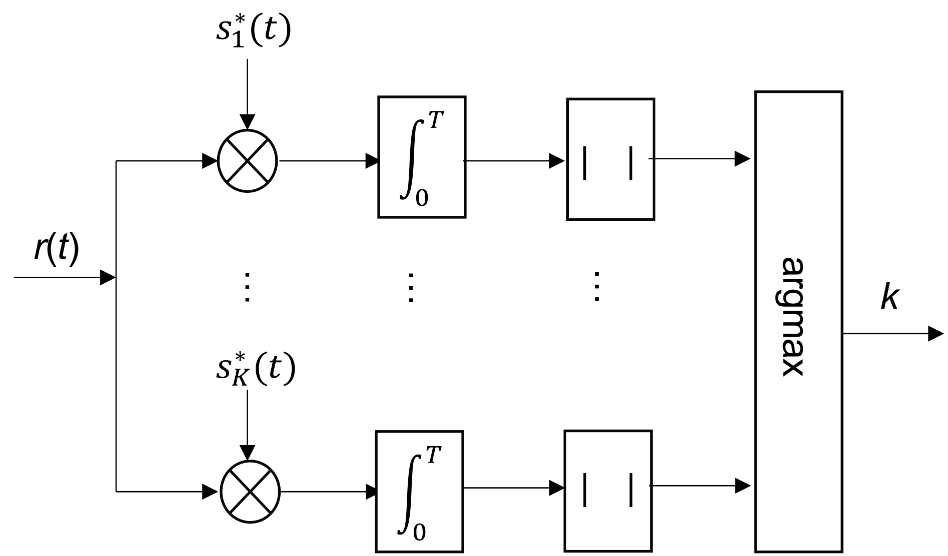
Consider a digital modulation system that transmits *K* bits during one symbol of length *T* by selecting one of the signals $s_k(t)$.

When signal *k* was transmitted, we receive

$r(t) = h\, s_k(t) + n(t)$

where h denotes the channel attenuation factor and n(t) is the noise power.

Incoherent correlator receiver



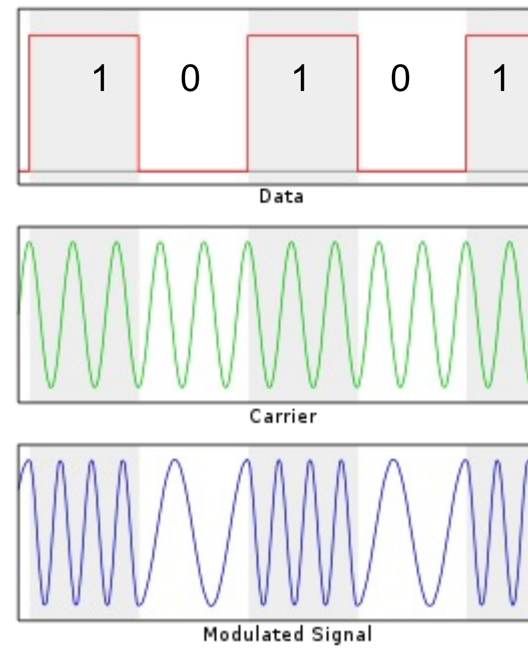
Correlate against all the possible waveforms
Select the one with highest absolute value correlation

Example: Frequency shift keying (FSK)

We wish to transmit either ‘0’ or ‘1’ by selecting one of the waveforms

$$s_0(t) = \sqrt{\frac{2E}{T}} \cos(2\pi f_c t), \qquad 0 \le t \le T \qquad f_c T \in Integer$$

$$s_1(t) = \sqrt{\frac{2E}{T}} \cos(2\pi (f_c + \Delta f) t), \qquad 0 \le t \le T$$



Example: Frequency shift keying (FSK)

Performance of the receiver is maximized when the two signals are taken to be orthogonal

$$\langle s_1(t)|s_0(t)\rangle = \frac{2E}{T}\int_0^T \cos(2\pi(f_c+\Delta f)t)\cos(2\pi f_c t)\,dt$$

$$= E\,\frac{\sin(4\pi f_c T + 2\pi\Delta f T)}{2\pi f_c T + \pi\Delta f T} + E\,\frac{\sin(2\pi\Delta f T)}{\pi\Delta f T} = 0 \quad \Rightarrow \quad \Delta f T \in \left\{\frac{1}{2}, 1, \frac{3}{2}, \ldots\right\}$$

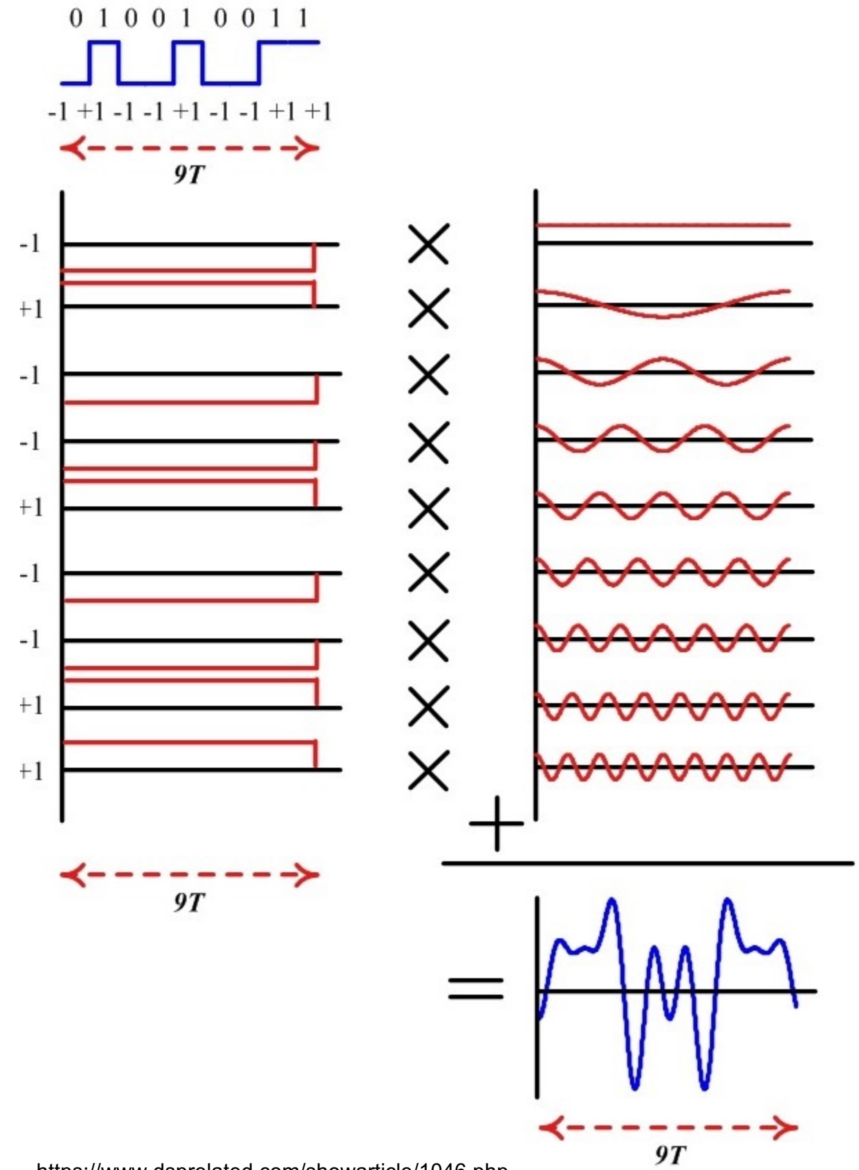
where $f_c T$=integer

Example: Frequency shift keying (FSK)

- Audio frequency FSK were used to transmit digital data over analog circuit switched telephony lines.
- Variants of the FSK include
  - Gaussian FSK where the pulses are first filtered with a Gaussian filter before FSK is applied. GFSK is used e.g. in Bluetooth.
  - Gaussian minimum shift keying is used in GSM (2G) mobile phone systems

Example: OFDM

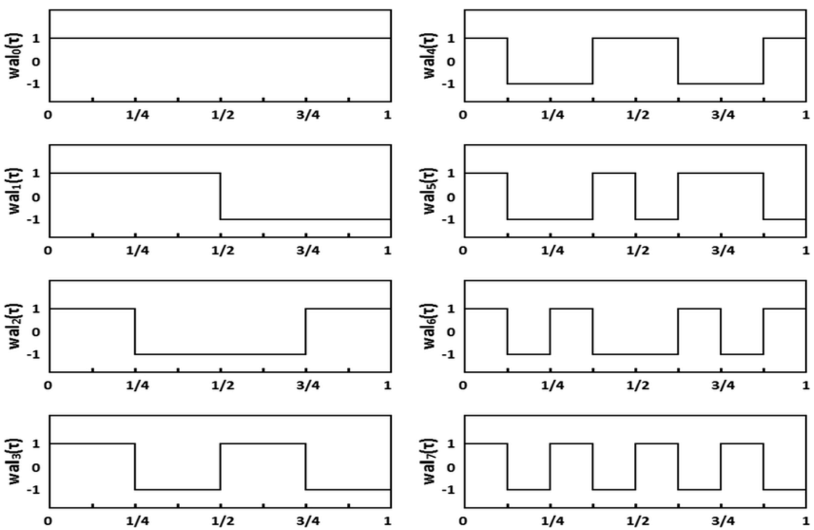
Modern wireless systems (4G LTE, 5G NR & WiFi) use Orthogonal Frequency Division Multiplexing (OFDM) to transmit different data symbols on orthogonal frequencies

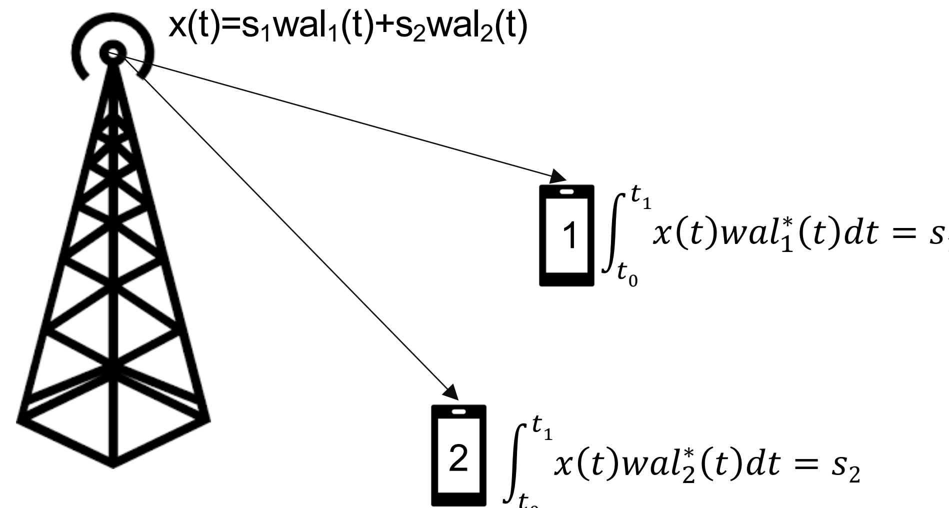


https://www.dsprelated.com/showarticle/1046.php

Application of orthonormal basis and inner product: UMTS

3G UMTS mobile networks used orthogonal Walsh (channelization) codes to distinguish between different users in downlink





Summary

- Inner product
- Orthonormal basis
- Gram-Schidt procedure
- Applications in communications engineering

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