Mathematics for Economists

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Multivariate functions

Functions

- A **function** $f : A \longrightarrow B$ from a set A to a set B is a rule that assigns to each element $a \in A$ one and only one element $b \in B$
- A is the domain of f
- B is the codomain of f
- The *image* (or *range*) of A under f is the set

$$f[A] := \{b \in B : b = f(a) \text{ for some } a \in A\}$$

- In this course (and in much of Economics), $A \subseteq \mathbb{R}^n$ and $B = \mathbb{R}^m$
  - note if $m > 1$, then $f(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \dots, f_m(\mathbf{x}))$, where f_i ,$i = 1, \dots, m$, are the component functions of f

Functions

1. What is the function corresponding to the set $\{(1,2),(2,2),(3,2)\}$ in $X \times Y$? What is the domain, codomain and range of the function?
2. Assume $X = \{-1, 1, 2, 3\}$ and $Y = \mathbb{R}$

   In which of the cases we have a function from X to Y?

a) $f(1) = 2$, $f(2) = 2$, $f(3) = 2$

b) $f(-1) = 0$, $f(1) = 0$, $f(2) = \{1, 2\}$, $f(3) = 1$

c) $f(x) = \sqrt{x}$

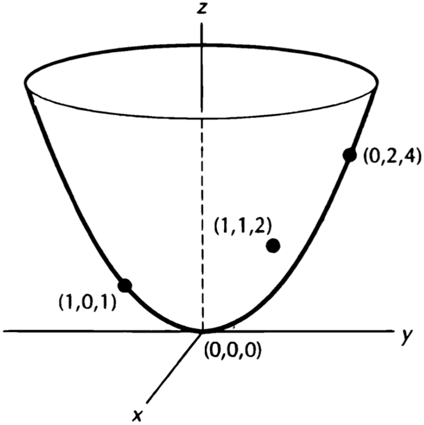
Functions of Several Variables: Examples

- Examples of utility/production functions $f : \mathbb{R}^n \longrightarrow \mathbb{R}$
  - Linear (perfect substitutes):
  $$f(x_1, \ldots, x_n) = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$$
  - Leontief (perfect complements):
  $$f(x_1, \ldots, x_n) = \min \{a_1 x_1, a_2 x_2, \ldots, a_n x_n\}$$
  - Cobb-Douglas:
  $$f(x_1, \ldots, x_n) = C \prod_{i=1}^{n} x_i^{a_i}$$
  - Constant Elasticity of Substitution (CES):
  $$f(x_1, \ldots, x_n) = C \left(\sum_{i=1}^{n} a_i x_i^{\rho} \right)^{\frac{1}{\rho}}, \quad \text{with } \rho \neq 0, \rho < 1$$

Functions of Several Variables: Graph

- The **graph** of a function $f : A \longrightarrow B$ is the set:

$$\{(x, f(x)) : x \in A\}.$$

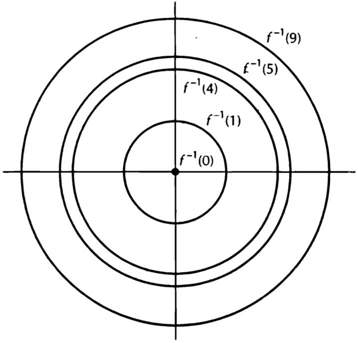


The graph of $f(x, y) = x^2 + y^2$.

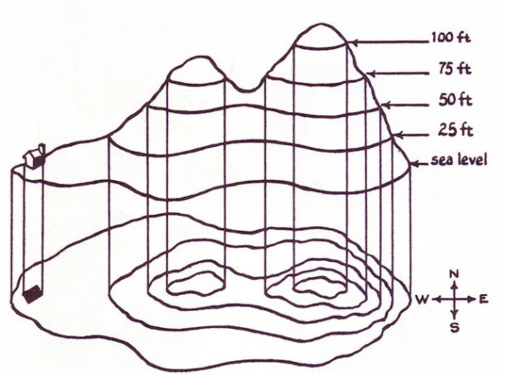
Functions of Several Variables: Level Curves

- It is often easier to represent functions defined over $A \subseteq \mathbb{R}^2$ with level curves (or sets)
- For a fixed value $\bar{b}$, the **level curve** of f is the set:

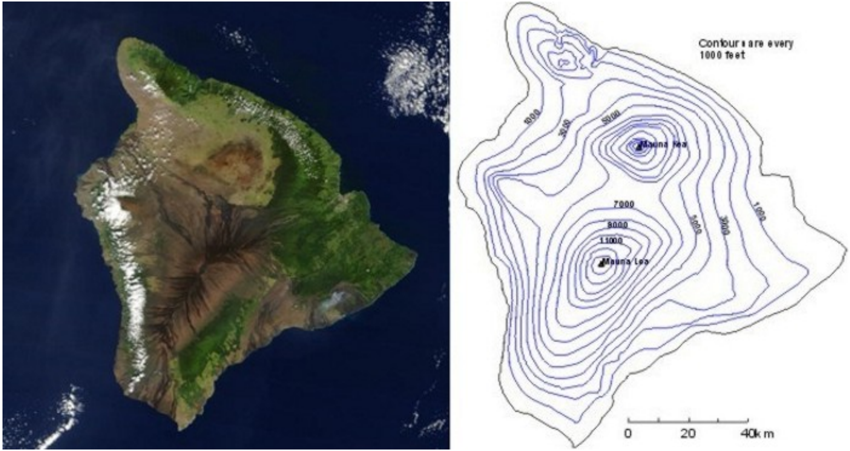
$$\{x \in A : f(x) = \bar{b}\} .$$



Level curves of $z = x^2 + y^2$.



Topographic Maps as Level Curves



Map of Hawaii

Functions of Several Variables: Indifference Curves

- In Economics, level curves of utility and production functions are called **indifference curves** and **isoquants**, respectively

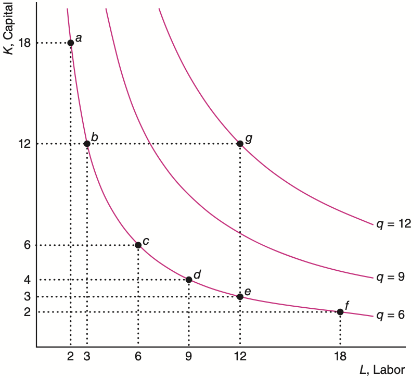


Figure: Three distinct isoquants of a production function $q = f(K, L)$

Injections, Surjections, and Bijections

- A function $f : A \longrightarrow B$ is **one-to-one** or **injective** if, for every $x, y \in A$,

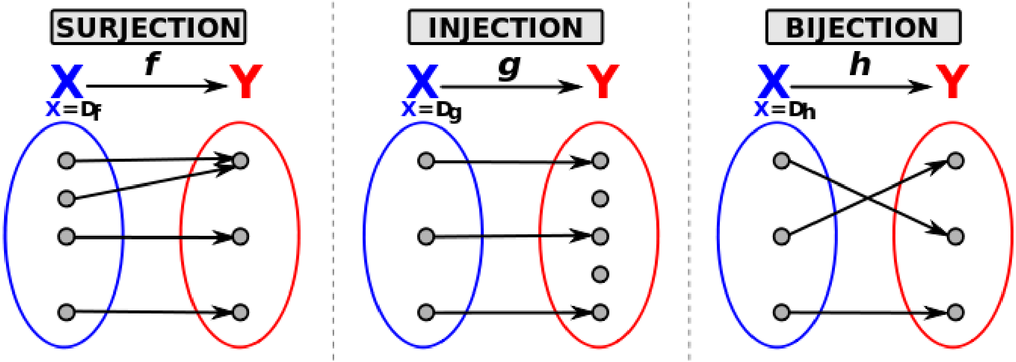
$$x \neq y \implies f(x) \neq f(y).$$

  - Example: $f : \mathbb{R}_+ \longrightarrow \mathbb{R}$ such that $f(x) = x^2$

- A function $f : A \longrightarrow B$ is **onto** or **surjective** if, for every $y \in B$, there exists an element $x \in A$ such that $f(x) = y$.
  - Example: $f : \mathbb{R} \longrightarrow \mathbb{R}_+$ such that $f(x) = x^2$

- A function $f : A \longrightarrow B$ is **bijective** if it is both injective and surjective.
  - Example: $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that $f(x) = x^2$

Injections, Surjections, and Bijections

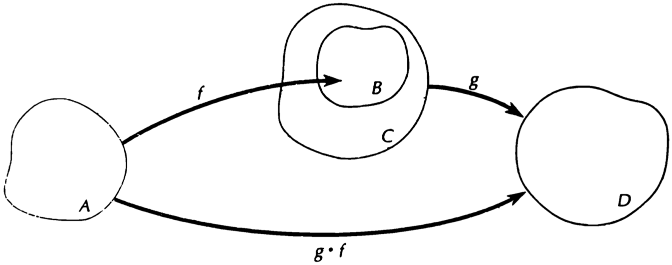


Injections, Surjections, and Bijections

- Which of the following are injections bijections or surjections (and how to define domain in each case)?
- $f(x) = e^x$
- $f(x) = \ln(x)$
- $f(x, y) = xy$
- $f(x, y) = \min\{x, y\}$
- $f(x, y) = (x, x)$
- $f(x_1, \ldots, x_n) = 0$

Composite Functions

- Given two functions $f : A \longrightarrow B$ and $g : C \longrightarrow D$, with $B \subseteq C$, the **composition** of f with g is the function $g \circ f : A \longrightarrow D$ such that
$$(g \circ f)(x) = g(f(x)) \text{ for all } x \in A.$$
- Example:
  - $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ such that $f(x, y) = x + y$
  - $g : \mathbb{R} \longrightarrow \mathbb{R}$ such that $g(x) = x^2$
  - $g \circ f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is such that $(g \circ f)(x, y) = (x + y)^2$



The composition of f with g.

Inverse Function

- For a bijective function $f : A \longrightarrow B$, we can define the **inverse** of f as the function $f^{-1} : B \longrightarrow A$ such that

$$f(x) = y \iff f^{-1}(y) = x.$$

- Example:
  - Take the linear demand function $Q : [0, a/b] \longrightarrow [0, a]$ such that $Q(p) = a - bp$, with $a > b > 0$
  - The so-called inverse demand function $P(q) : [0, a] \longrightarrow [0, a/b]$ such that $P(q) = \frac{1}{b}(a - q)$ is the inverse function of Q

Linear Functions

- Assume that A is an $m \times n$ matrix
- Function $f(\mathbf{x}) = A\mathbf{x}$ is a linear function, $f : \mathbb{R}^n \mapsto \mathbb{R}^m$
- Assume that $m = n$, and A is invertible
  The inverse function of f is $f^{-1}(\mathbf{y}) = A^{-1}\mathbf{y}$
- Assume that B is an $k \times m$ matrix and $g(\mathbf{y}) = B\mathbf{y}$
  Composition of f with g is $(g \circ f)(\mathbf{x}) = BA\mathbf{x}$

Linear Functions: Example

- Assume two firms with quantities produced denoted by q_1 and q_2
- Reaction functions:
  - if firm 1 produces q_1 the other responds by producing $R_2(q_1) = 6 - q_1/2$
  - if firm 2 produces q_2 the other responds by producing $R_1(q_2) = 6 - q_2/2$
- The reactions of firms are characterized by $R : \mathbb{R}^2 \mapsto \mathbb{R}^2$ such that $R(q_1, q_2) = (R_1(q_1), R_2(q_2))$

Sequences

- A **sequence** in $\mathbb{R}$ is a function $s : \mathbb{N} \longrightarrow \mathbb{R}$
- Examples:
  - $s(n) = \frac{1}{n}$, i.e. $\left\{1, \frac{1}{2}, \frac{1}{3}, \dots\right\}$
  - $s(n) = 5$, i.e. $\{5, 5, 5, 5, \dots\}$
  - $s(n) = \frac{1}{n^2}$, i.e. $\left\{1, \frac{1}{4}, \frac{1}{9}, \dots\right\}$
  - $s(n) = (-1)^n$, i.e. $\{-1, 1, -1, 1, \dots\}$
- Oftentimes we write a generic sequence as $\{x_1, x_2, x_3, \dots\}$ or $\{x_n\}_{n=1}^{\infty}$
- A sequence in $\mathbb{R}^n$ is a function $s : \mathbb{N} \longrightarrow \mathbb{R}^n$. That is, a sequence is an assignment of a vector in $\mathbb{R}^n$ to each natural number

Sequences and Limits

- Given a sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathbb{R}$ and a real number L, we say that this sequence **converges** to L if, for every arbitrarily small real number $\epsilon > 0$, there exists a positive integer N such that $|x_n - L| < \epsilon$ for all $n \geq N$.

- When $\{x_n\}_{n=1}^{\infty}$ converges to L, we say that L is the **limit** of this sequence, and we write $\lim_{n \longrightarrow \infty} x_n = L$ or simply $x_n \longrightarrow L$.

Sequences and Limits

- Example: $\lim_{n \longrightarrow \infty} \frac{1}{n^2} = 0$
- How to check that 0 is indeed the limit of this sequence?

1. Fix a small number $\epsilon > 0$
2. Choose any positive integer N such that $N > \frac{1}{\sqrt{\epsilon}}$
3. For any $n \geq N$, we have

$$\left|x_n - L\right| = \left|\frac{1}{n^2} - 0\right| \leq \left|\frac{1}{N^2} - 0\right| < \left|\frac{1}{(1/\sqrt{\epsilon})^2} - 0\right| = \epsilon.$$

Sequences and Limits

- If a sequence converges, its limit is unique
- Not every sequence has a limit. Examples:
  - $\{1, -1, 1, -1, 1, -1, \dots\}$
  - $\left\{1^2, 2^2, 3^2, 4^2, \dots\right\}$
- If $a_n \longrightarrow a$ and $b_n \longrightarrow b$, then $(a_n + b_n) \longrightarrow a + b$
- If $a_n \longrightarrow a$ and $b_n \longrightarrow b$, then $a_n b_n \longrightarrow ab$
- If $a_n \longrightarrow a$ and $b_n \longrightarrow b$, then $\frac{a_n}{b_n} \longrightarrow \frac{a}{b}$ if neither b nor any b_n is equal to zero

Sequences and Limits

- Given a sequence of vectors in $\mathbb{R}^n$, we have that this sequence converges if and only if all n sequences of its components converge in $\mathbb{R}$
- Alternatively, a sequence **converges** to $\mathbf{x}^*$ if, for every arbitrarily small real number $\epsilon > 0$, there exists a positive integer N such that $\|\mathbf{x}_n - \mathbf{x}^*\| < \epsilon$ for all $n \geq N$.
- For example, the sequence of vectors $\left\{\left(1 + \frac{1}{n}, \frac{1}{2n}\right)\right\}_{n=1}^{\infty}$ converges to the vector $(1, 0)$

Continuous Functions

- Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a function and let $x_0 \in \mathbb{R}^n$ be a point in its domain. We say that f is **continuous at $\boldsymbol{x_0}$** if whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence in $\mathbb{R}^n$ that converges to x_0, then the sequence $\{f(x_n)\}_{n=1}^{\infty}$ in $\mathbb{R}$ converges to $f(x_0)$.

- If a function is continuous at every point in its domain, then we say that the function is continuous

- Examples of continuous functions are all the utility/production functions at p. 4

Continuous Functions

- An alternative (and equivalent) definition of continuity (so-called *epsilon-delta* definition) is the following
- A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous at x_0 if, for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x \in \mathbb{R}$ we have

$$|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon.$$

Discontinuity

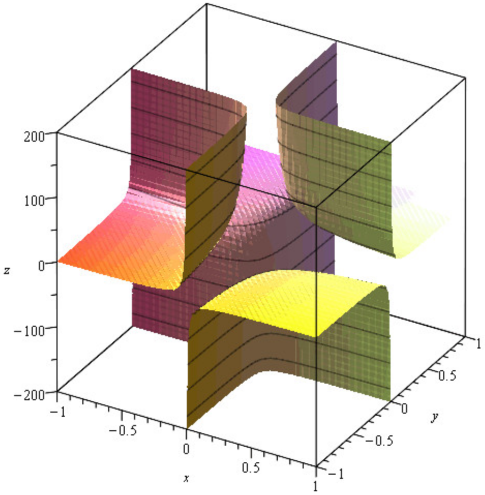
- An example of a discontinuous function is $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

- To see why this function is discontinuous at $x = 0$, take the sequence $\{\frac{1}{n}\}_{n=1}^{\infty}$ in $\mathbb{R}$. This sequence converges to zero, but the sequence $\{f(\frac{1}{n})\}_{n=1}^{\infty}$ converges to 1

Discontinuity

- Another example: $f(x, y) = 1/(xy)$, for $x, y \neq 0$, otherwise $f(x, 1) = 1$



Composites of Continuous Functions

- Let f and g be functions from $\mathbb{R}^n$ to $\mathbb{R}$. Suppose that both f and g are continuous at $x \in \mathbb{R}^n$. Then we have that all the following functions are continuous at x too:
  - $f + g$
  - $f - g$
  - $f \times g$

- Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a continuous function at $x_0 \in \mathbb{R}^n$, and let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function at $f(x_0) \in \mathbb{R}$. Then the composite function $g \circ f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is continuous at x_0.

Derivatives and Partial Derivatives

- For a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ of one variable, the **derivative** of f at x_0 is

$$\frac{df}{dx}(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h},$$

  provided that the limit exists.

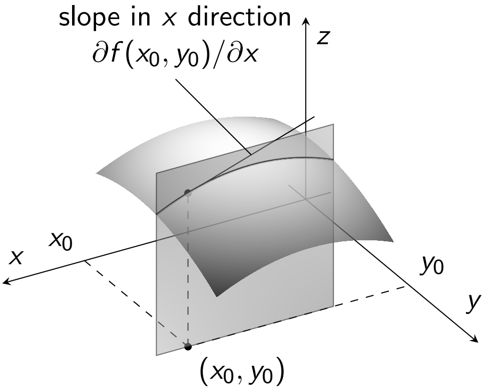
- Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$. The **partial derivative** of f with respect to x_i at $x = (x_1, \ldots, x_n)$ is

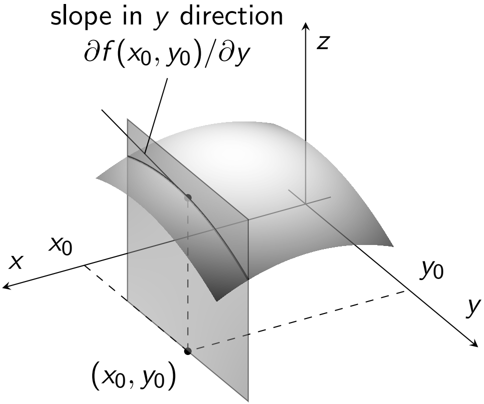
$$\frac{\partial f}{\partial x_i}(x) = \lim_{h \to 0} \frac{f(x_1, ..., x_i + h, ..., x_n) - f(x_1, \ldots, x_i, \ldots, x_n)}{h},$$

  provided that the limit exists.

- NOTE: only x_i changes, all the other variables are treated as constants.
- Intuitively, the partial derivative of f w.r.t. x_i tells you how much the function changes as x_i changes.

Derivatives and Partial Derivatives





Rules of Differentiation

- Linearity: $h(x) = af(x) + bg(x)$, then $h'(x) = af'(x) + bg'(x)$
- Product rule: $h(x) = f(x)g(x)$, then $h'(x) = f'(x)g(x) + f(x)g'(x)$
- The chain rule: $h(x) = f(g(x))$, then $h'(x) = f'(g(x))g'(x)$
- Some elementary derivatives
  - $f(x) = x^r$, $r \neq 0$, $f'(x) = rx^{r-1}$
  - $f(x) = e^{rx}$, $f'(x) = re^{rx}$
  - $f(x) = \ln(x)$, $f'(x) = 1/x$

Derivatives and Partial Derivatives: Examples

- For a production function f, the partial derivative of f w.r.t. x_i is the **marginal product** of input x_i
- For a utility function u, the partial derivative of u w.r.t. x_i is the **marginal utility** of commodity x_i

- Example: Let $f : \mathbb{R}^2_+ \longrightarrow \mathbb{R}$ be the Cobb-Douglas production function

$$f(k, \ell) = Ck^\alpha \ell^\beta,$$

where k is capital and ℓ is labor.

- The marginal products of capital and labor are

$$\frac{\partial f}{\partial k}(k, \ell) = C\alpha k^{\alpha-1}\ell^\beta$$
$$\frac{\partial f}{\partial \ell}(k, \ell) = C\beta k^\alpha \ell^{\beta-1}.$$

Example: Marginal Utility

- Example: Let $u : \mathbb{R}_+^T \longrightarrow \mathbb{R}$ be the CRRA (Constant Relative Risk Aversion) utility function

$$u(c_1, ..., c_T) = \sum_{t=1}^{T} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma},$$

where $\beta \in (0, 1)$ and $\gamma \geq 0$, $\gamma \neq 1$.

- The marginal utility of c_t (consumption in period t) is

$$\frac{\partial u}{\partial c_t} = \beta^t c_t^{-\gamma}.$$