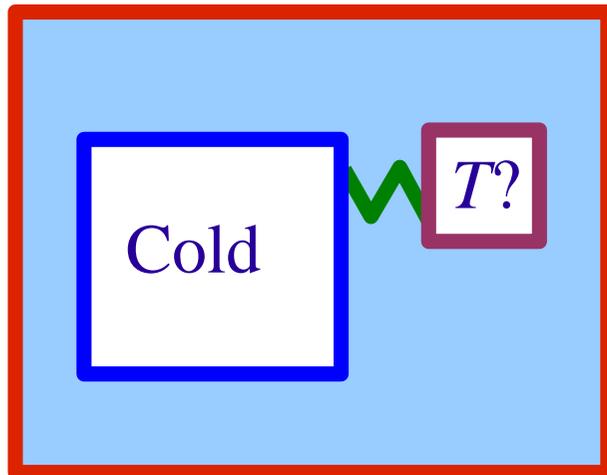


Heat Transfer

- What does it take to cool down specimens; “ R of $\tau = RC$ ”?
- Protect the low temperature systems from external heat
- Reliability of thermometer readings
- Relationship between heat capacity and thermal conductivity
- Relationship between electrical and thermal conductivity
- Poor heat transfer between subsystems can result in several differing temperatures: T_e , T_{ph} , T_n , ...



Hot

Sources:

P.V.E. McClintock, D.J. Meredith, and J.K. Wigmore,
Low-Temperature Physics: an introduction
for scientists and engineers (1992)

F. Pobell,
Matter and Methods at Low Temperatures (2007)

Mechanisms

- Thermal conduction
 - lattice vibrations (phonons)
 - conduction electrons (Fermi gas)
 - gas particles (“vacuum”)
- Thermal radiation (vacuum)
- Thermal convection (flow *i.e.* mass transfer)

Thermal conductivity κ relates temperature gradients ∇T to heat currents dQ/dt :

$$dQ/dt = -A\kappa\nabla T \quad (\kappa = Cv^2\tau /3)$$

Simple sum rules: (transfer/scattering mechanism n/i)

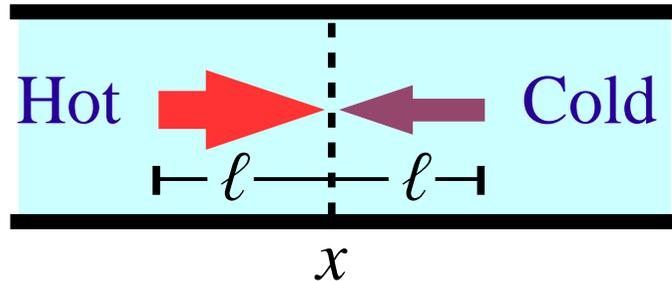
$$1/\tau_n = 1/\tau_{n1} + 1/\tau_{n2} + \dots \quad (\text{Mathiessen rule, sum over } i)$$

$$\rho_n = \rho_{n1} + \rho_{n2} + \dots \quad (\kappa_n = 1/\rho_n)$$

$$\kappa = \kappa_1 + \kappa_2 + \dots \quad (\text{sum over } n)$$

Heat capacity and thermal conductivity

Kinetic theory relates the two (diffusion problem):



$$\frac{\dot{Q}}{A} = \frac{1}{2} nv \left\{ E[T(x-\ell)] - E[T(x+\ell)] \right\}$$

Series expansion

$$\frac{\dot{Q}}{A} = \frac{1}{2} nv \left\{ E[T(x)] + \frac{dE}{dT} \frac{dT}{dx} (-\ell) - E[T(x)] - \frac{dE}{dT} \frac{dT}{dx} (\ell) \right\}$$

$$= -nv\ell \frac{dE}{dT} \frac{dT}{dx} = -nv^2 \tau \frac{dE}{dT} \frac{dT}{dx}$$

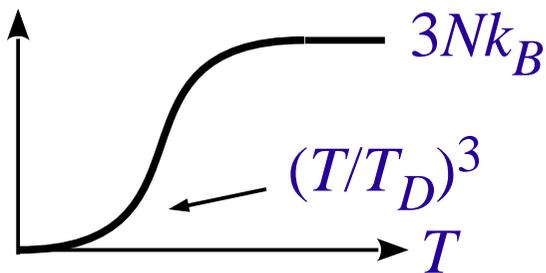
Since in 3D: $\langle v_x^2 \rangle = \langle v_y^2 \rangle = \langle v_z^2 \rangle = \frac{1}{3} \langle v^2 \rangle$ and $C = n \frac{dE}{dT}$

$$\Rightarrow \kappa = \frac{1}{3} C v^2 \tau = \frac{1}{3} C v \ell$$

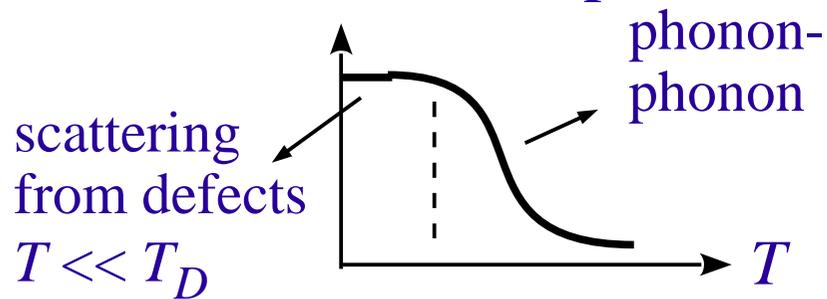
Phonons

Continuum approximation $\times C \times \ell$

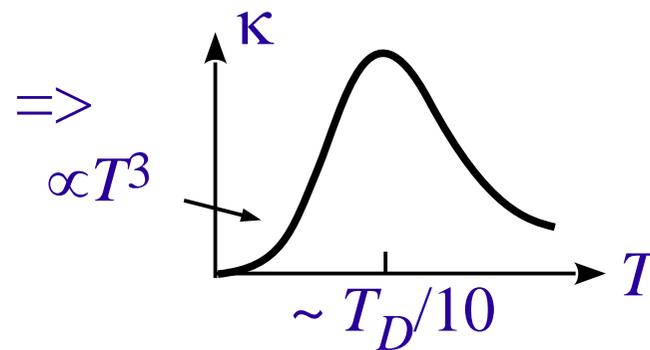
- v is constant
typically $\sim 3 - 5$ km/s
- heat capacity:



- mean free path:



Thermal conductivity has a maximum



Scattering from dislocations:

$\ell(T)$ at low T

Examples

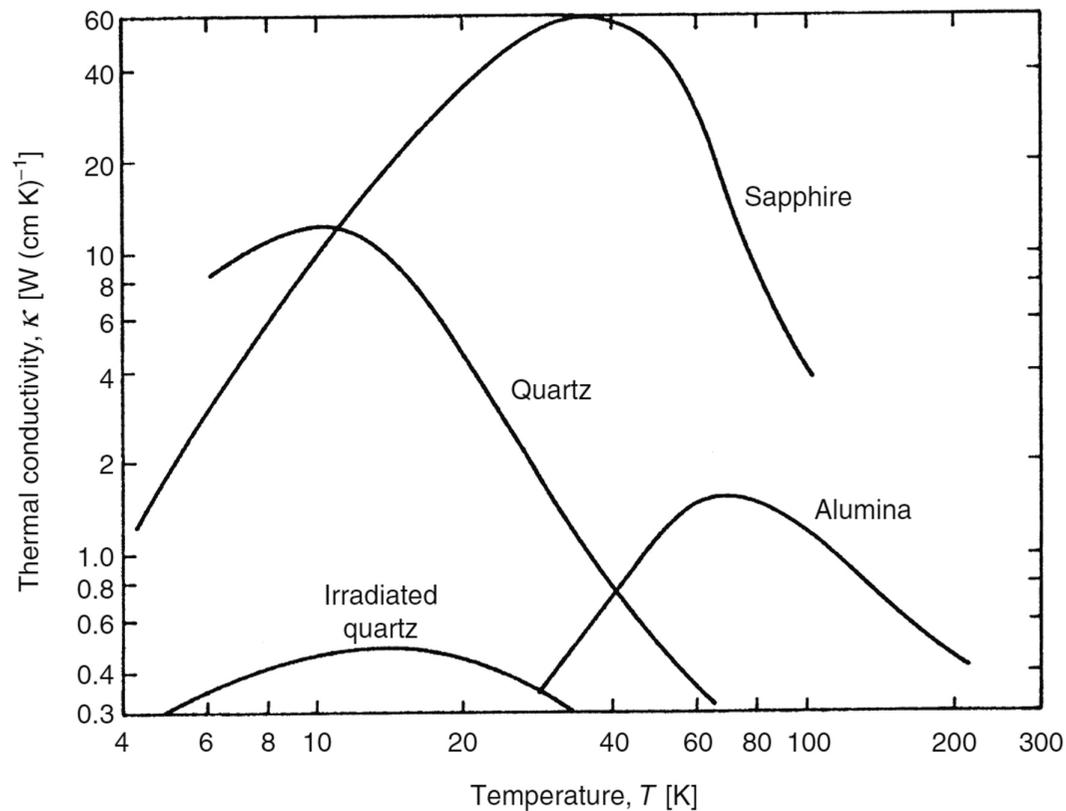
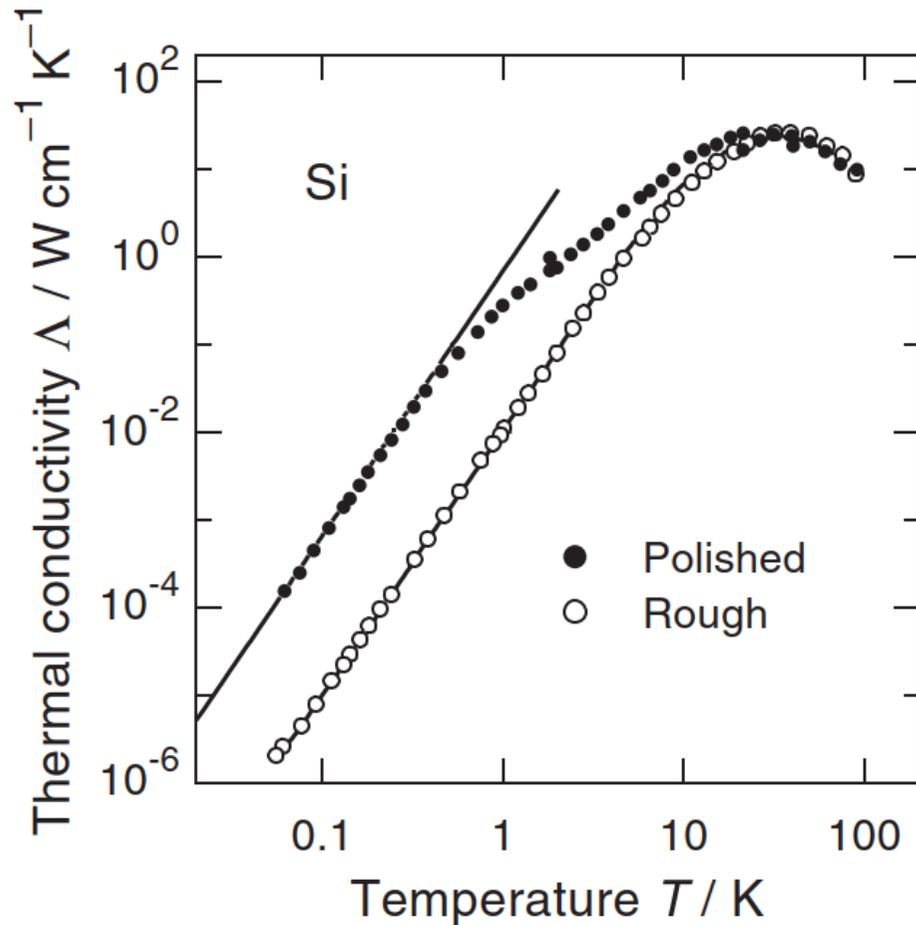
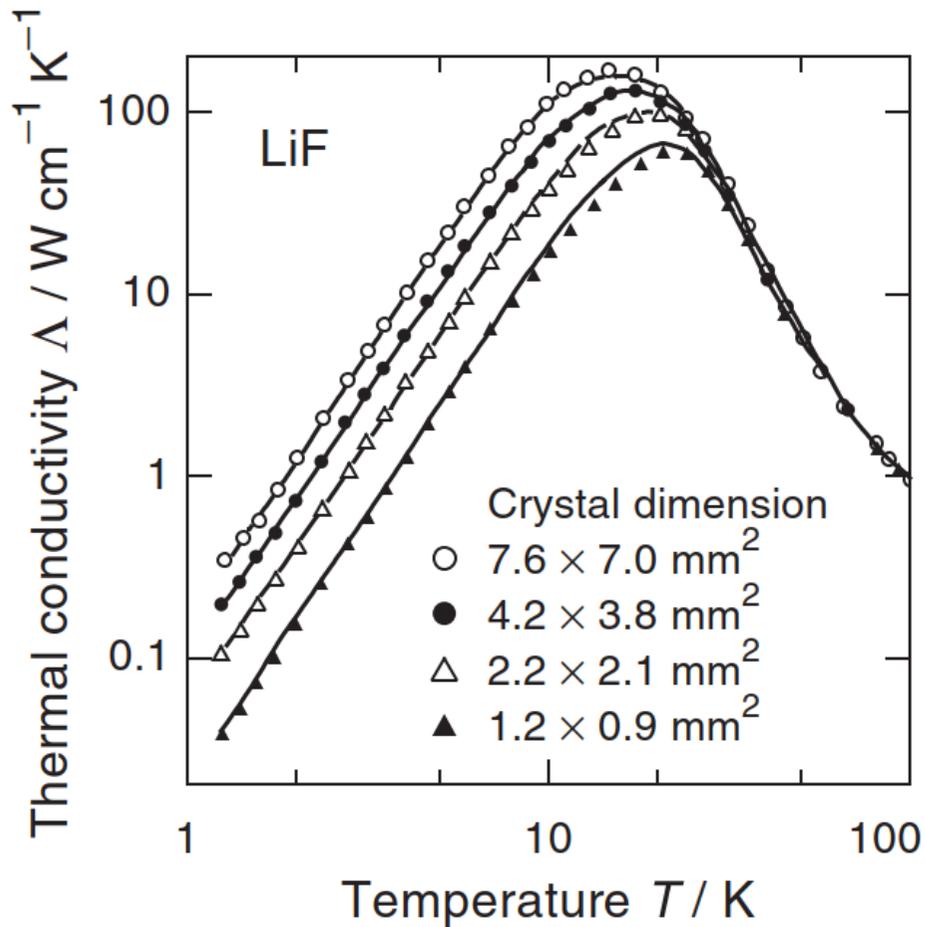


Fig. 3.19. Temperature dependence of the thermal conductivities of some dielectric solids

- Thermal conductivity of insulators depends crucially on the crystal quality
- Big flawless single crystals may conduct as well as metals
- Amorphous solids have poor thermal conductivity

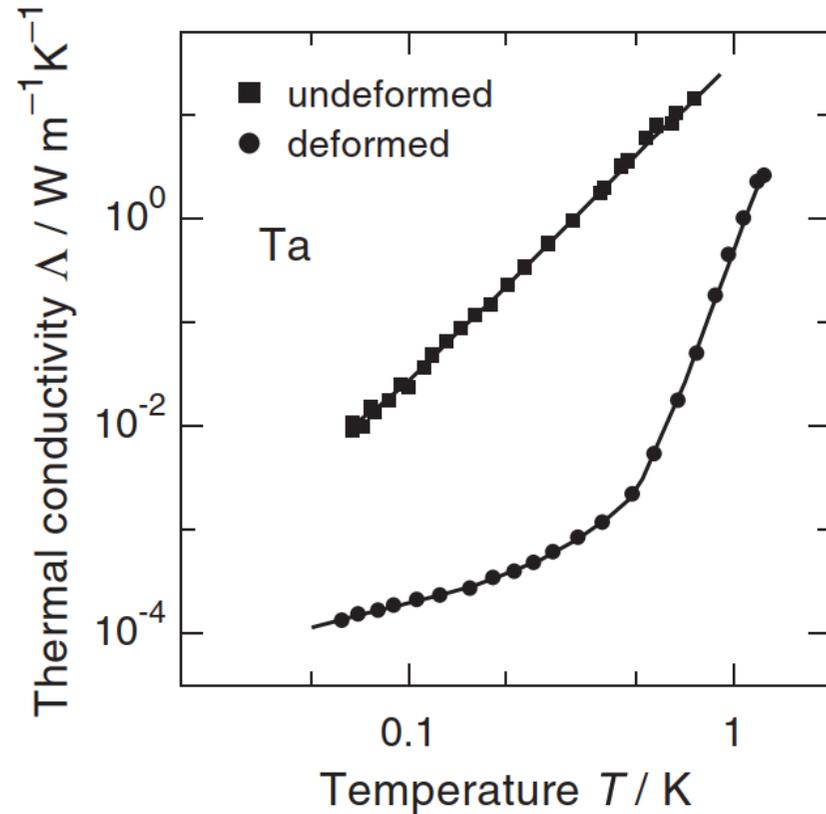
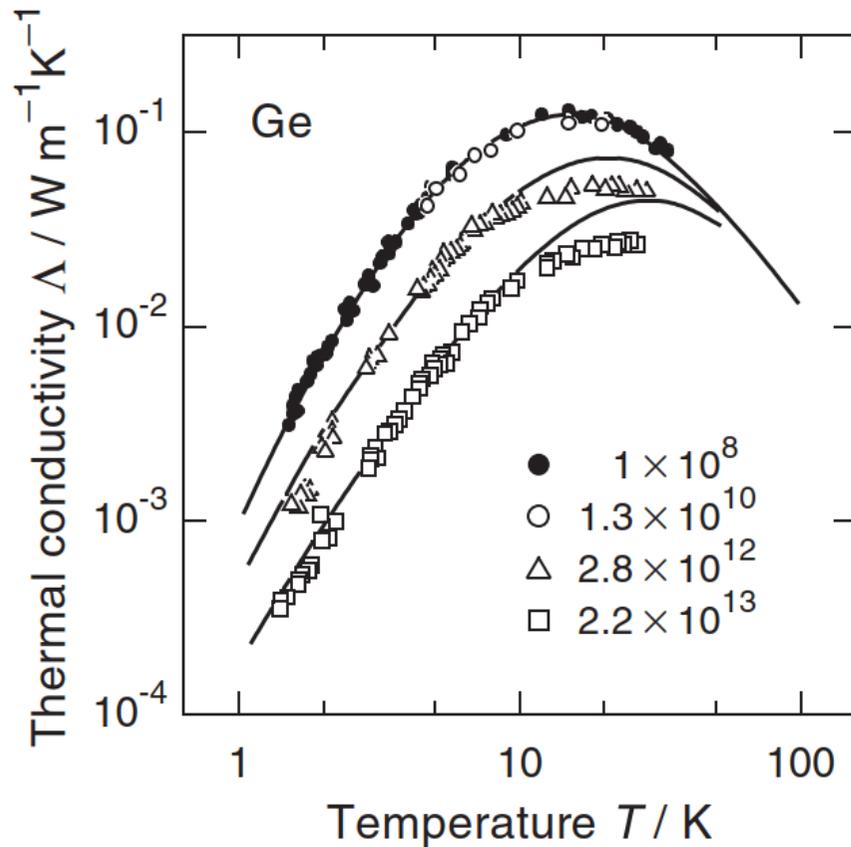
Surface quality matters

- Specular reflection at the boundary enhances the mean free path beyond D
- Even a factor of 50 increase in κ can be seen in Si crystals



Effect of dislocations

- Dislocations absorb and re-emit phonons, interaction depends on frequency



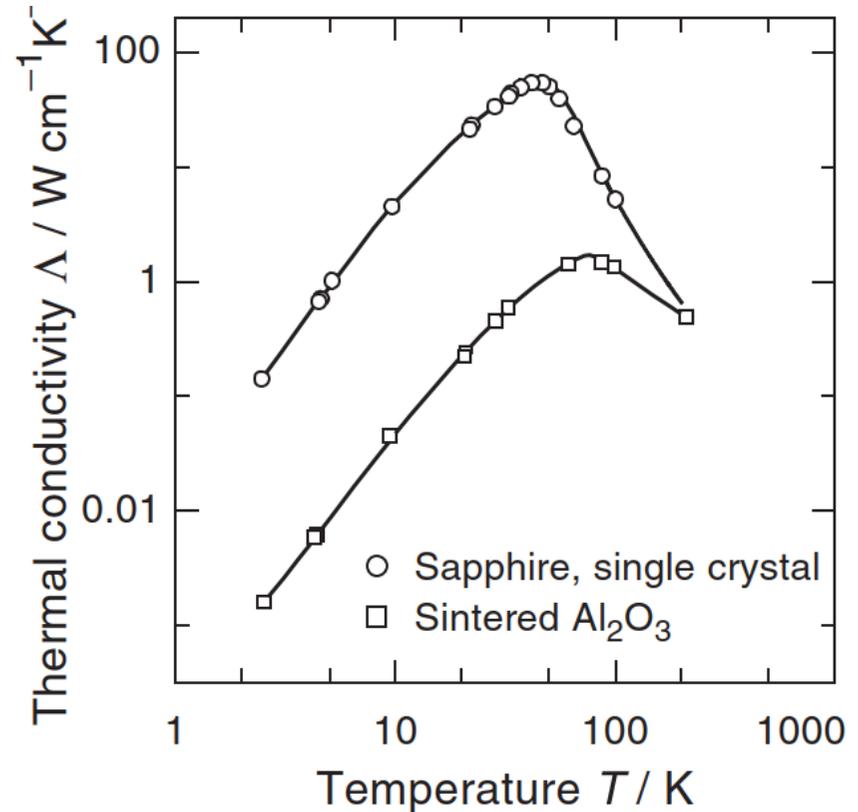
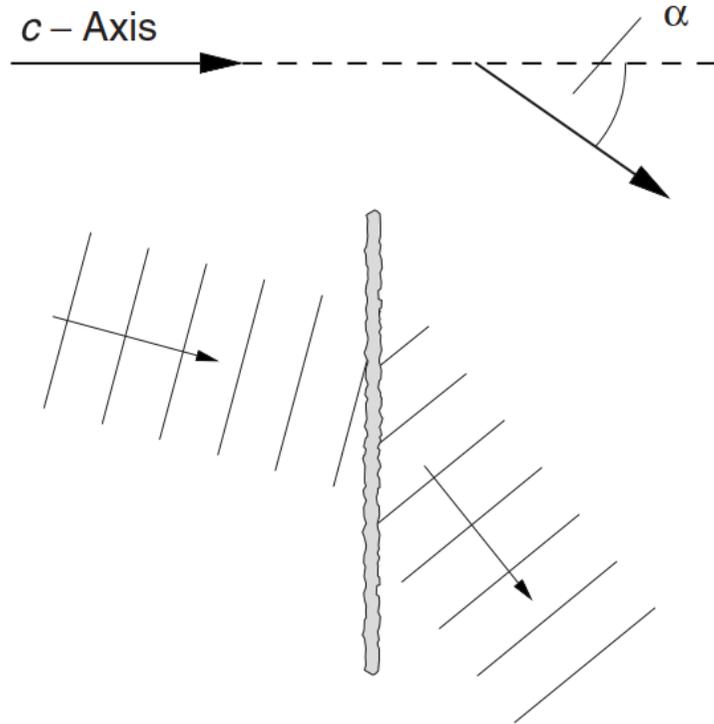
Vibrating string model:

$$\ell^{-1} = A n_d \frac{\omega^3}{(\omega_0^2 - \omega^2)^2 + \beta^2 \omega^4}$$

- High purity Tantalum
- Copper behaves the same way

Grain boundaries

- No proper theory exists as real polycrystalline materials are complex



Refraction at grain boundary ($\lambda \gg d_{\text{boundary}}$):

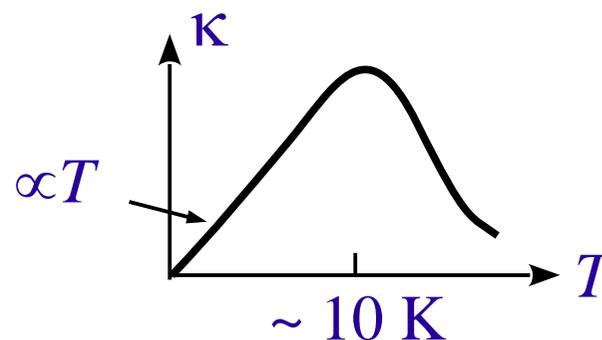
$$\ell \approx L / \langle \sin^2 \alpha \rangle$$

- Strong reduction in thermal conductance of Al_2O_3

Conduction electrons

- $v = v_F \sim 100 \dots 1000 \text{ km/s}$
- heat capacity $C \propto T$
- mean free path:
 - high T : electron-phonon scattering;
 T increase \Rightarrow more phonons $\Rightarrow \kappa$ decreases
 - low T : scattering from defects and impurities
 ℓ is constant, $\kappa \propto Cv\ell \propto T$

$\Rightarrow \kappa_{el}$ displays
a maximum
at around 10 K



Superconductors

Cooper pairs do not contribute to thermal conductivity ($C_{CP} = 0$)

Only normal electrons do and their number decreases exponentially far below T_c

$$n \propto e^{-\Delta/k_B T}$$

$$\Rightarrow \kappa \propto T e^{-\Delta/k_B T}$$

Magnetic field
can be used to
destroy the super-
conducting state

Used as
heat switches!

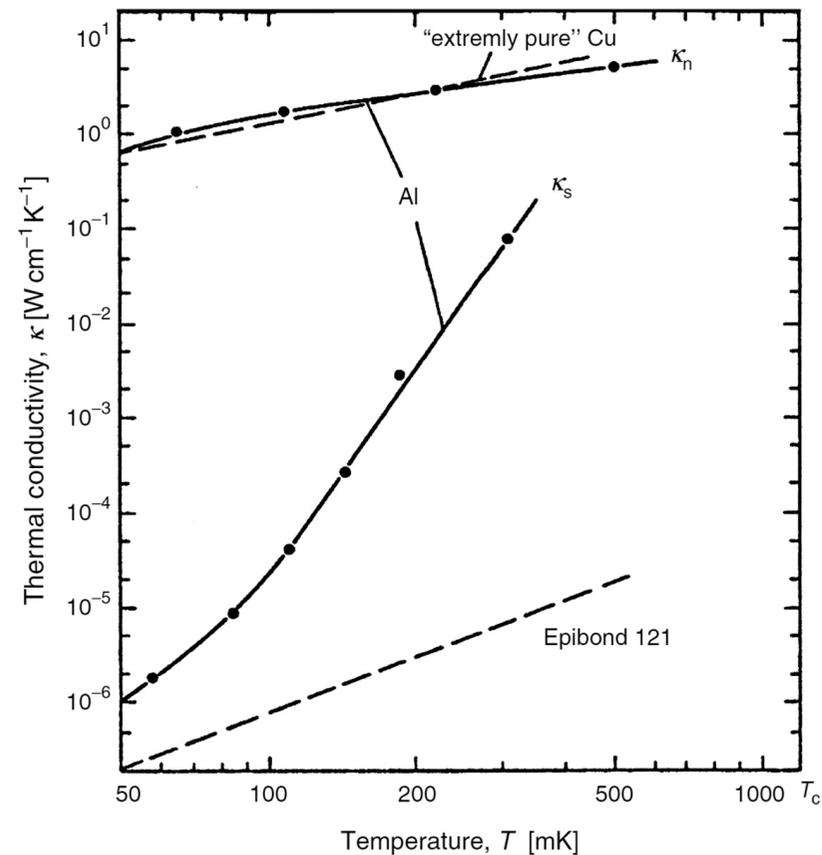
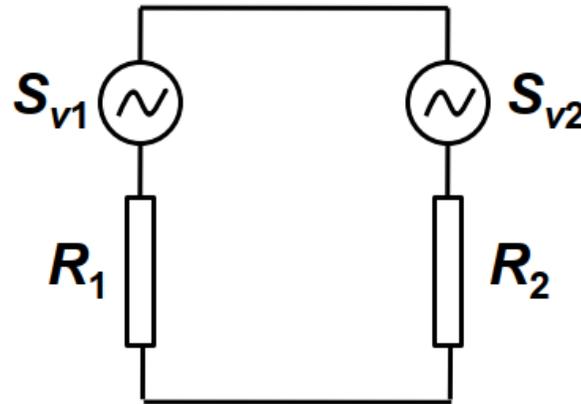
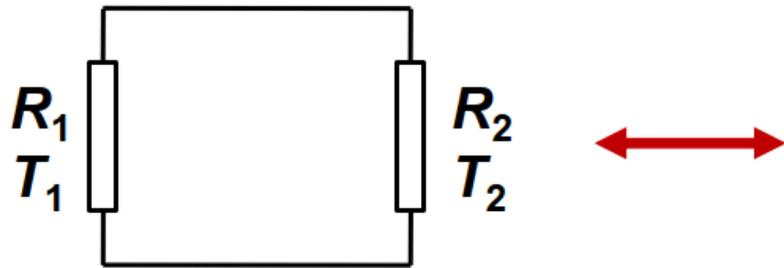


Fig. 4.1. Thermal conductivity κ of Al in the normal-conducting state (compared to κ of Cu) and in the superconducting state (compared to κ of the dielectric Epibond 121) at $T < 50$ mK [4.19]

Heat transported between two resistors



$$P = r \frac{\pi k_B^2}{12\hbar} (T_1^2 - T_2^2)$$

$$r \equiv \frac{4R_1 R_2}{(R_1 + R_2)^2}$$

For small temperature difference $\Delta T = T_1 - T_2$:

$$P = r G_Q \Delta T$$

$$G_Q = \frac{\pi k_B^2}{6\hbar} T$$

Johnson, Nyquist 1928

Photons

Schmidt et al., PRL 93, 045901 (2004)

Meschke et al., Nature 444, 187 (2006)

Timofeev et al., PRL 102, 200801 (2009)

Partanen et al., Nature Physics 12, 460 (2016)

Phonons

K. Schwab et al., Nature 404, 974 (2000)

Electrons

Jezouin et al., Science 342, 601 (2013)

Banerjee et al., Nature 545, 75 (2017)

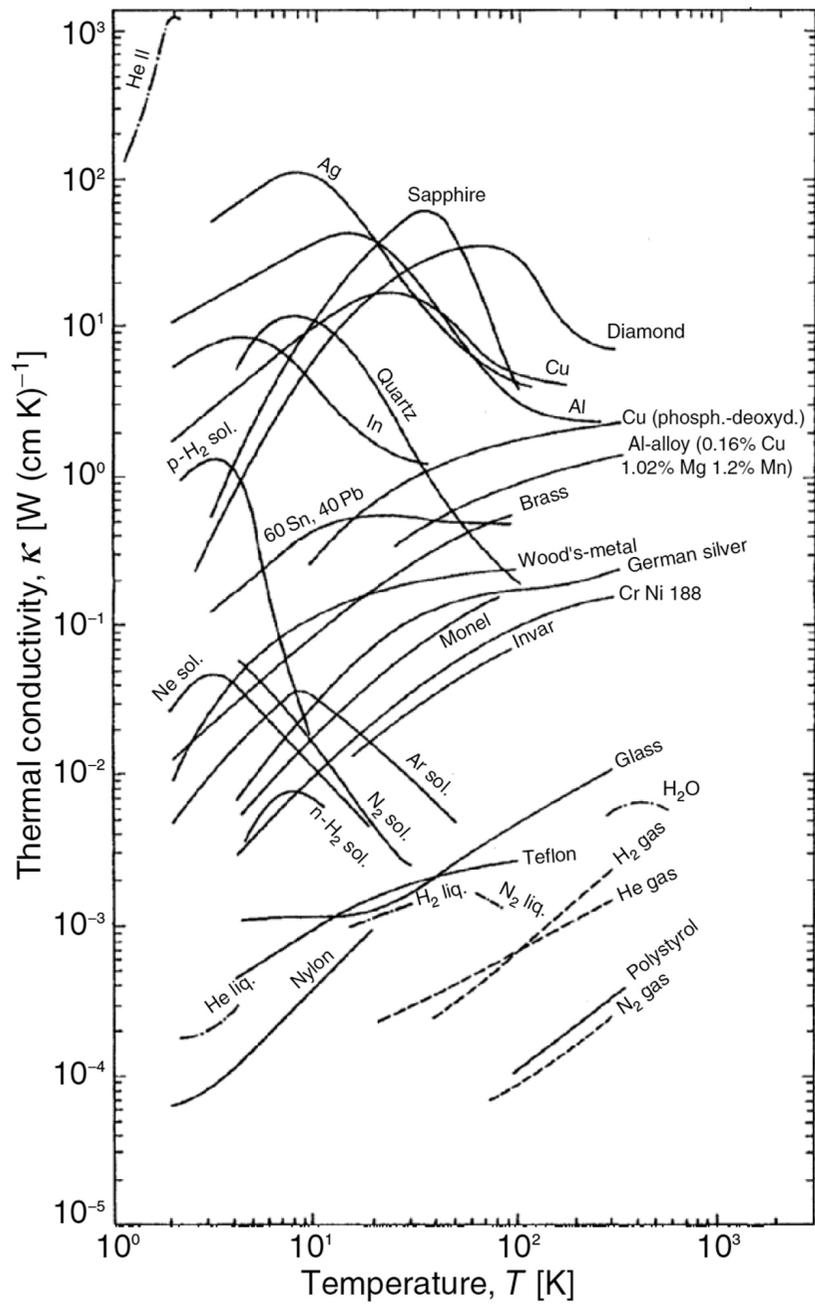


Fig. 3.20. Typical thermal conductivities κ of various materials at $T > 2$ K. Remember that κ depends on the purity and crystalline perfection of a material

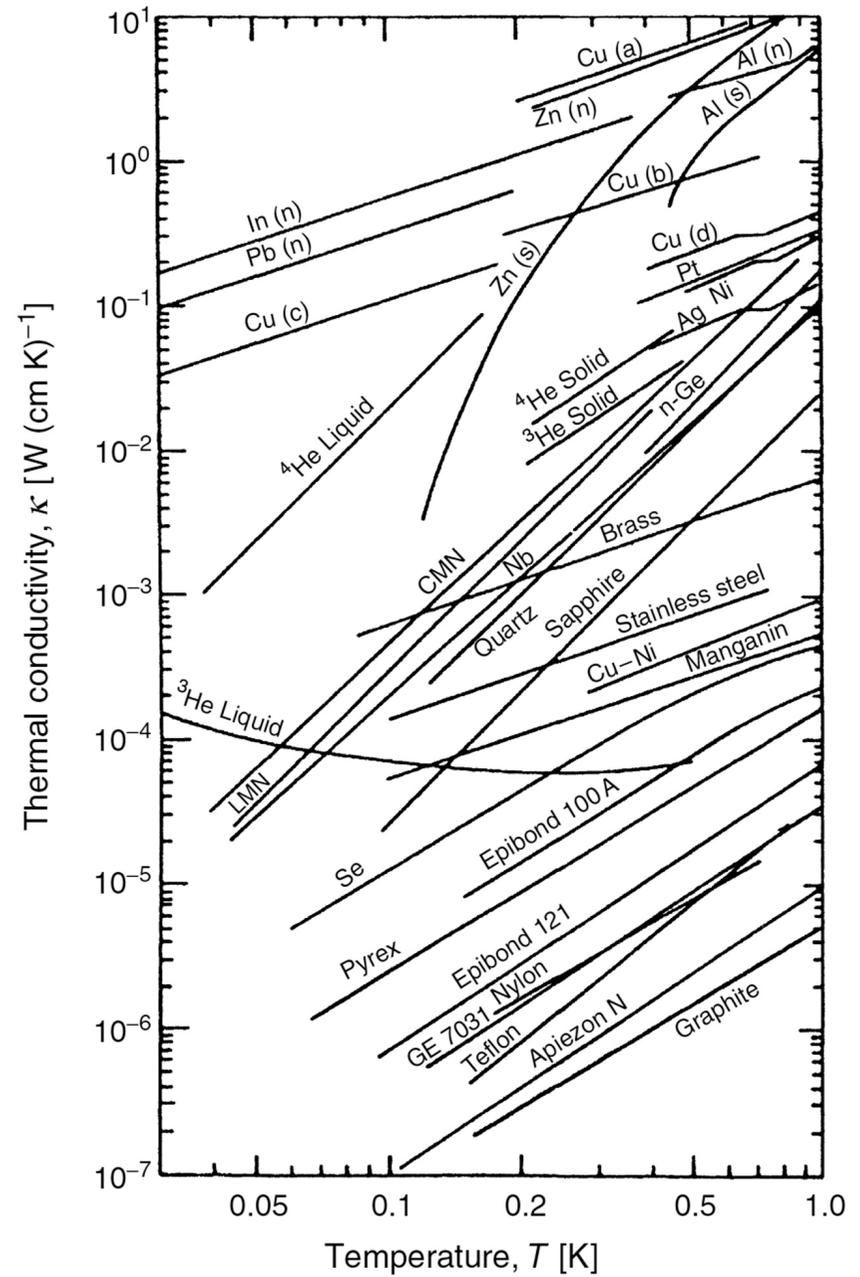
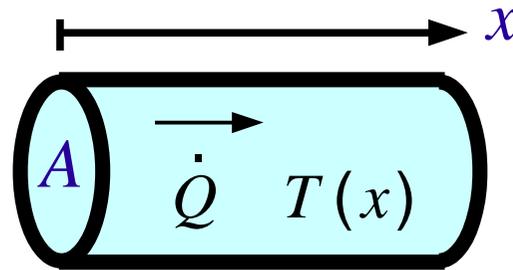


Fig. 3.21. Typical thermal conductivities κ of various materials at $T < 1$ K.

Practical formulae for heat flow

$$\dot{Q} = -A\kappa \frac{dT}{dx}$$



$$T(0) = T_2 \quad T(L) = T_1$$

$$\int_0^L \dot{Q} dx = -A \int_{T_2}^{T_1} \kappa dT$$

Metals: $\kappa = k_0 T$ for $T < 10$ K

$$\dot{Q} = -\frac{Ak_0}{2L} (T_2^2 - T_1^2)$$

Insulators: $\kappa = bT^3$ for $T < T_D/10$

$$\dot{Q} = -\frac{Ab}{4L} (T_2^4 - T_1^4)$$

Note !!

do not use the linear approximation

$$\dot{Q} = -\frac{A\kappa}{L} \Delta T$$

unless $\Delta T \ll T$

Higher temperatures

Use tabulated conductivity integrals, when the range of temperatures is large (e.g. from RT to 4K)

$$\bar{\kappa} = \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \kappa(T) dT$$

$$\Rightarrow \dot{Q} = \frac{A}{L} \bar{\kappa} (T_2 - T_1)$$

TABLE 5.1

Mean values of thermal conductivity expressed in W/m · K; values in brackets are extrapolated

$\bar{\kappa}$	$\bar{\lambda}$				
	$T_2 = 300 \text{ K}$ $T_1 = 77 \text{ K}$	$T_2 = 300 \text{ K}$ $T_1 = 4 \text{ K}$	$T_2 = 77 \text{ K}$ $T_1 = 4 \text{ K}$	$T_2 = 4 \text{ K}$ $T_1 = 1 \text{ K}$	$T_2 = 1 \text{ K}$ $T_1 = 0.1 \text{ K}$
Nylon	0.31	0.27	0.17	0.006	0.001
Pyrex glass	0.82	0.68	0.25	0.06	0.006
Machineable glass-ceramic	2	1.6	1.3	0.03	0.004
Graphite (AGOT)				0.0025	0.0002
18/8 stainless steel	12.3	10.3	4.5	0.2	0.06
Constantan (60 Cu, 40 Ni)	20	18	14	0.4	0.05
Brass (70 Cu, 30 Zn)	81	67	26	1.7	0.35
Copper (phosphorus deoxidized)	190	160	80	5	(1)
Copper (electrolytic)	410	570	980	200	(40)

Thermal (κ) and electric (σ) conductivities

Wiedemann-Franz law:

$$\kappa = \frac{1}{3} C v_F^2 \tau = \frac{\pi^2}{3} \frac{n k_B \tau}{m} T$$

$$\sigma = \frac{n e^2 \tau}{m}$$

$$\Rightarrow \frac{\kappa}{\sigma} = \frac{\pi^2}{3} \frac{k_B^2}{e^2} T = \underline{L_0 T}$$

with

Lorenz number $L_0 = 24.5 \frac{\text{nW}\Omega}{\text{K}^2}$

Measure σ (easier) and use W-F to find κ

Wiedemann-Franz law

$$\frac{\kappa}{\sigma} = L_0 T = 24.5 T \frac{\text{nW}\Omega}{\text{K}^2}$$

Condition for validity:

Both electric and thermal conductivities must be limited by the same scattering process

- works at room temperature (lots of phonons)
- works at very low temperatures (defect scattering)
- fails in between !

W-F failure at intermediate T

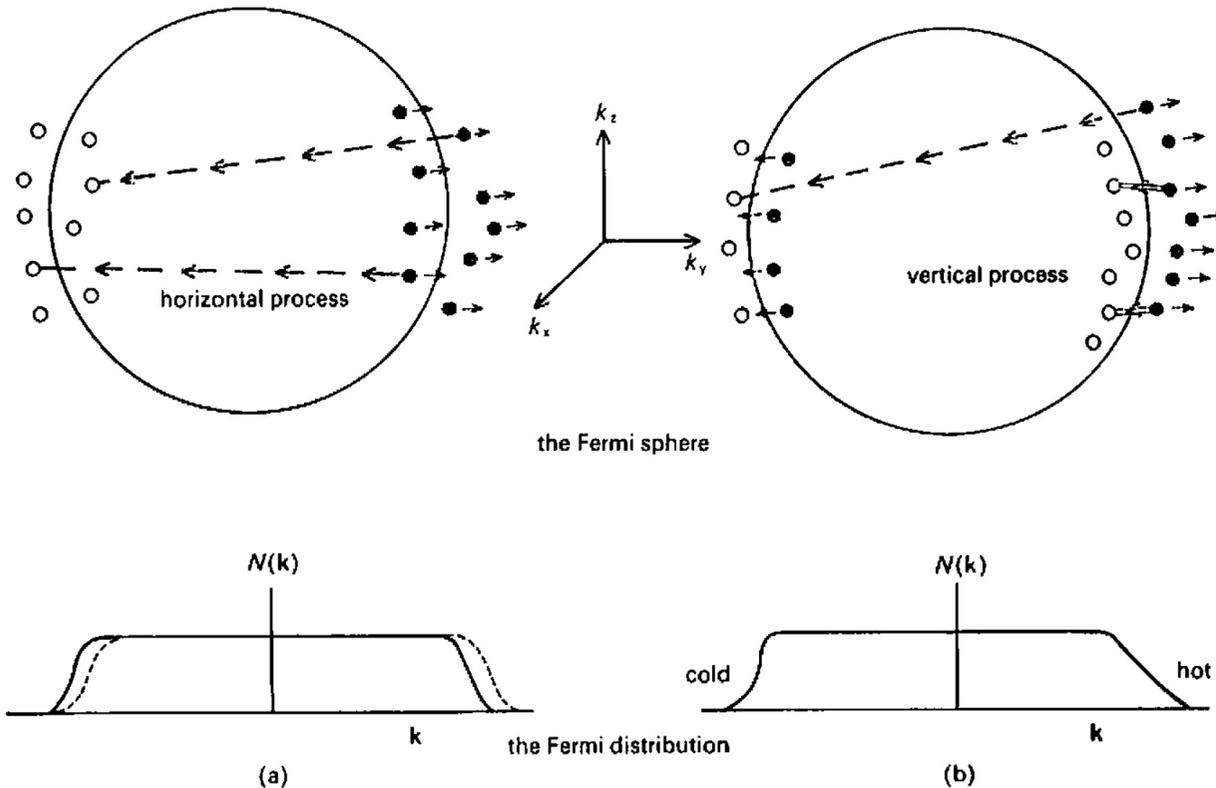


Figure 3.12 The disturbed Fermi distributions produced by the application of (a) an electric field to measure J and (b) a temperature gradient to determine U .

- excess of an electron relative to equilibrium distribution,
- deficiency of an electron relative to equilibrium.

It is seen that electric field shifts the whole distribution in k -space, whereas temperature difference merely makes the distribution asymmetric.

Schematic of the idea (after McClintock et al.)

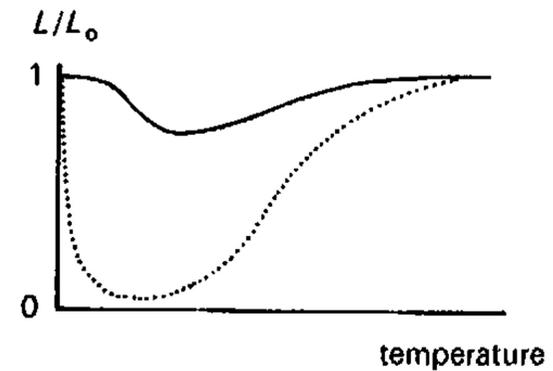


Figure 3.10 Dependence of the Lorenz number on temperature and sample purity.

- impure sample;
 - highest purity crystal attainable.
- (schematic)

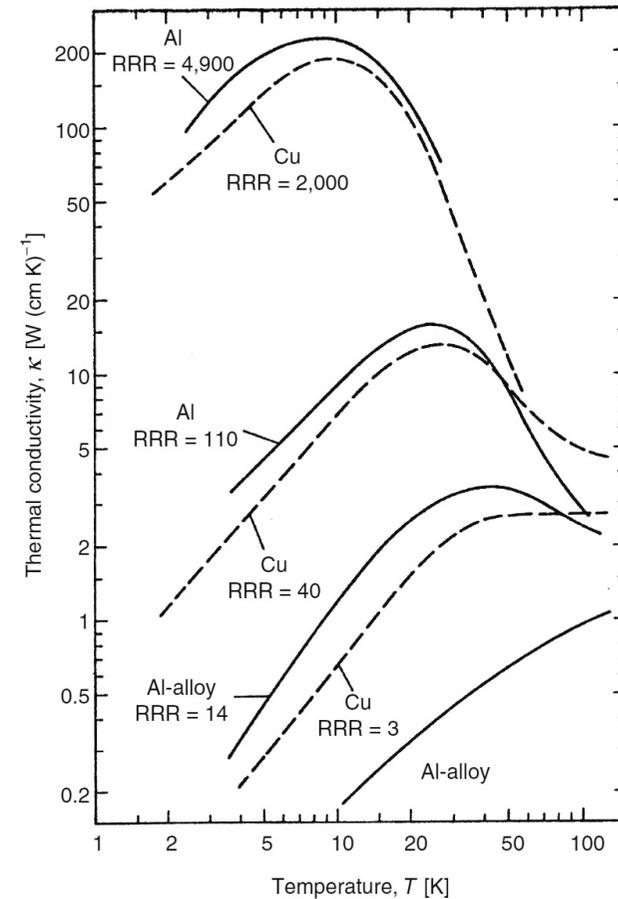
Ways to improve κ

- Lattice imperfections can be reduced by simple annealing
- Impurities deteriorate thermal conductivity at low temperatures

Order of magnitude (for Cu) typically

- nonmagnetic: $\sim 0.1 \text{ n}\Omega\text{cm/ppm}$
- magnetic (Cr, Fe, Co, ...)
 $\sim 1 \text{ n}\Omega\text{cm/ppm}$

(remember $\sigma \rightarrow \kappa$ by W-F)



As a measure of purity one often uses the residual resistivity ratio: $RRR = R_{RT}/R_{4K}$

Oxygen annealing

In some cases (like Cu, Ag) magnetic impurities can be largely neutralized by annealing close to the melting temperature in low oxygen atmosphere

- paramagnetic impurities oxidize to ferromagnetic compounds
- oxide molecules cluster into small crystals ($r \sim 0.1 \mu\text{m}$)

\Rightarrow less scatterers & smaller scattering cross section

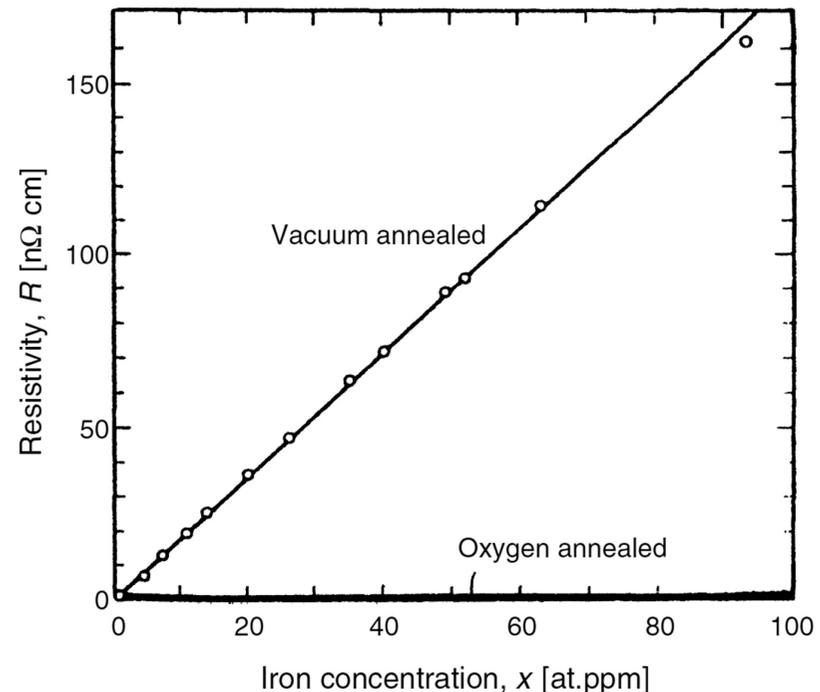


Fig. 3.26. Resistivity of Cu-Fe at 4 K, as a function of Fe impurity concentration, annealed in vacuum and in an O_2 atmosphere (92 h; about 10^{-6} bar air) [3.130]

Residual gas

1) dense gas; $\ell \ll D$ (size of the container)

molecules collide with each other

$$\ell \propto 1/p \quad (\text{indep. of } T)$$

$$C \propto p \quad (\text{indep. of } T)$$

$$v \propto T^{1/2} \quad (\text{indep. of } p)$$

$$\Rightarrow \kappa \text{ is independent of } p \text{ (but } \propto T^{1/2}\text{)}$$

$$\kappa = \frac{1}{3} C \bar{v} \ell$$

$$\ell = \frac{1}{3\rho_N a_0^2}$$

 Scattering cross section

2) rarefied gas; $\ell > D$ (in fact effectively $\ell \rightarrow D$)

Knudsen limit or molecular transport regime
no collisions between the molecules

$$\Rightarrow \kappa \propto p$$

Helium fluids

^4He above $T_\lambda \sim 2\text{ K}$ and ^3He above $\sim 0.1\text{ K}$ behave more or less like classical gases with $\kappa \propto T^{1/2}$

^4He at $\sim 1 \dots 2\text{ K}$ has huge thermal conductivity because of normal-superfluid counterflow

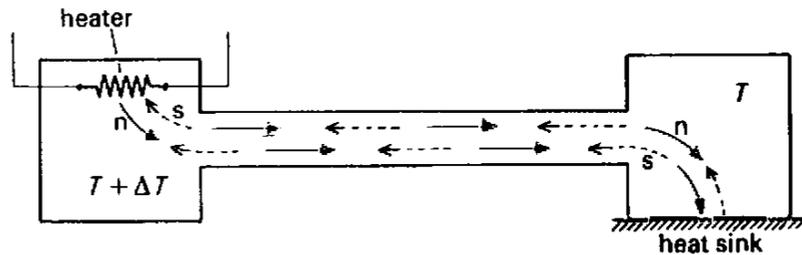
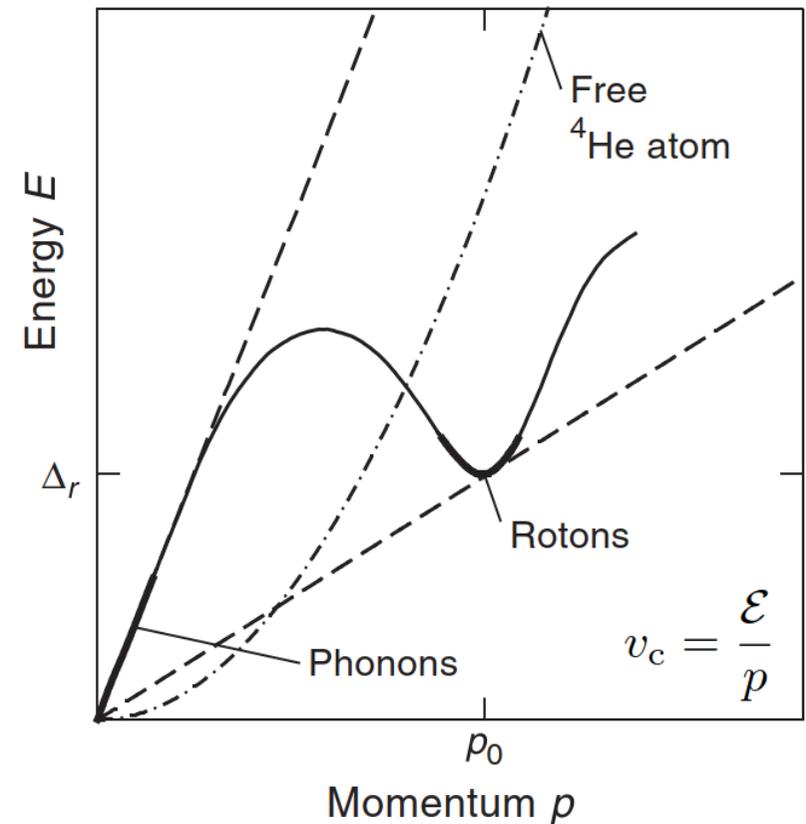


Figure 5.8 Sketch to indicate the peculiarly effective mode of thermal conduction in HeII. Heat generated at the heater converts superfluid into normal fluid, which flows (full arrows) to the heat sink. Here, it gives up all of its entropy in becoming re-converted to superfluid, which returns (dashed arrows) once more to the heater.

Below $\sim 0.5\text{ K}$ ballistic phonons result in $\kappa \propto DT^3$



Helium fluids

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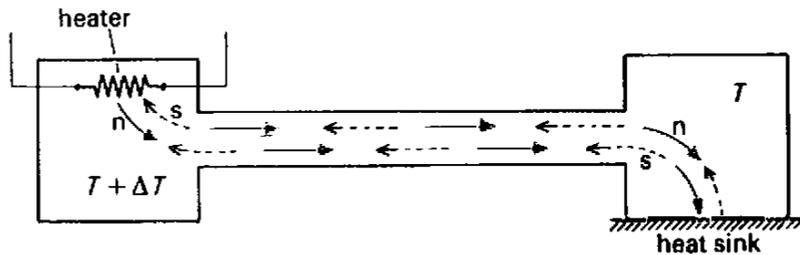


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Below $\sim 0.5 \text{ K}$ ballistic phonons result in $\kappa \propto DT^3$

$$\rho = \rho_n + \rho_s$$

$$\mathbf{j} = \rho_n \mathbf{v}_n + \rho_s \mathbf{v}_s$$

$$\nabla \cdot \mathbf{j} = -\frac{\partial \rho}{\partial t}$$

$$\nabla \cdot (\rho S \mathbf{v}_n) = -\frac{\partial (\rho S)}{\partial t}$$

$$\frac{\partial \mathbf{j}}{\partial t} = -\nabla P$$

$$\frac{\partial \mathbf{v}_s}{\partial t} = S \nabla T - \frac{1}{\rho} \nabla P$$

^3He

^3He and helium mixtures at low temperatures ($T \ll T_F$) are fermi systems with

$$\left. \begin{aligned} \ell &\propto 1/T^2 \\ v &= v_F \\ C &\propto T \end{aligned} \right\} \Rightarrow \kappa \propto 1/T$$

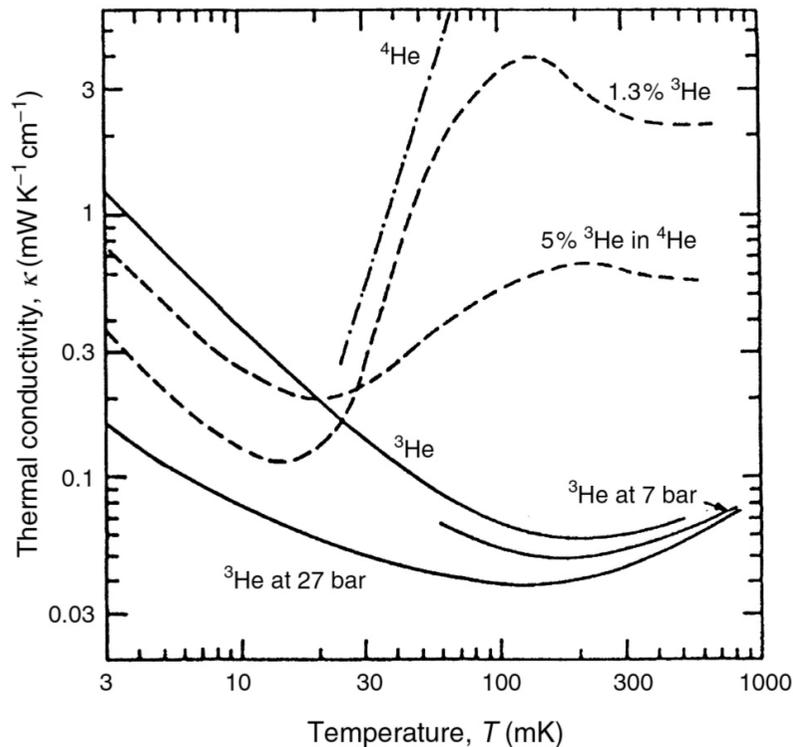


Fig. 2.18. Thermal conductivities of liquid ^3He , liquid ^4He , and dilute ^3He - ^4He mixtures

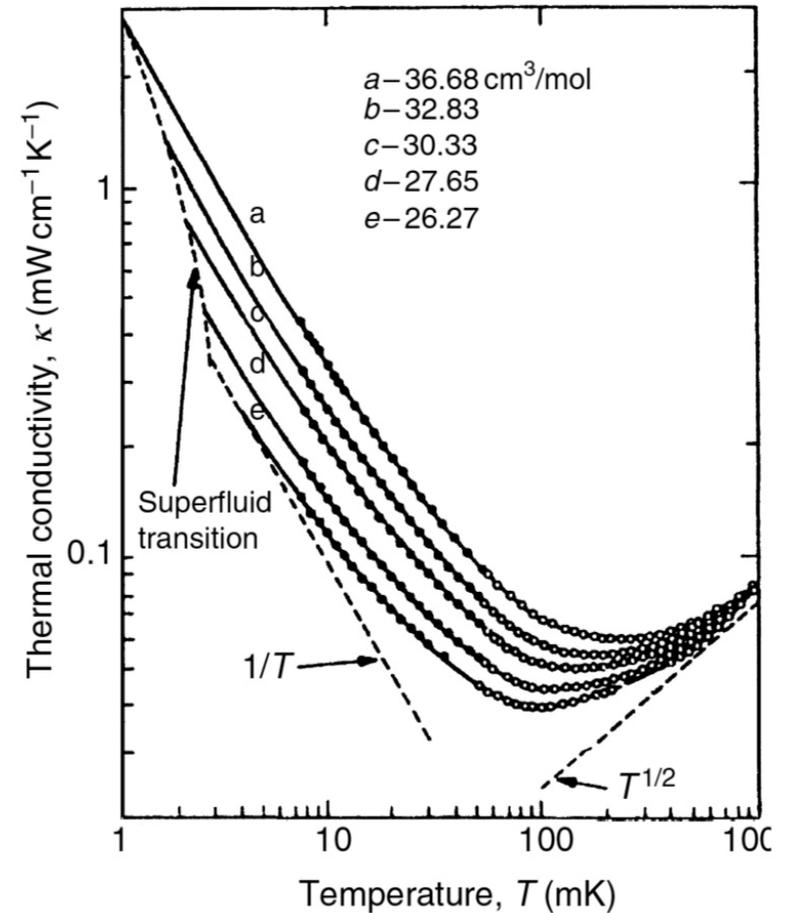


Fig. 2.17. Thermal conductivity of liquid ^3He at the given molar volumes. At low temperatures, one finds the Fermi-liquid behavior $\kappa \propto T^{-1}$

Thermal radiation (black body radiation)

Stefan-Boltzmann law $\dot{Q} = A \sigma T^4$; $\sigma = 56.7 \frac{\text{nW}}{\text{m}^2 \text{K}^4}$

Real surfaces are not “black” emissivity ϵ (= absorptivity) is obtained from the index of reflectivity R : $\epsilon = 1 - R$

$$\dot{Q} = A\sigma \left(T_1^4 - T_2^4 \right) \frac{\epsilon_1 \epsilon_2}{\epsilon_1 + \epsilon_2 - \epsilon_1 \epsilon_2}$$

→ $A\sigma \left(T_1^4 - T_2^4 \right) \quad [\epsilon_1 = \epsilon_2 = 1]$

→ $A\sigma \left(T_1^4 - T_2^4 \right) \frac{\epsilon}{2} \quad [\epsilon_1 \approx \epsilon_2 \ll 1]$

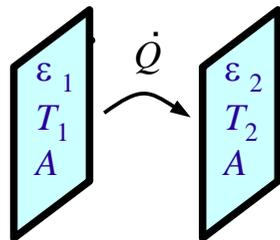


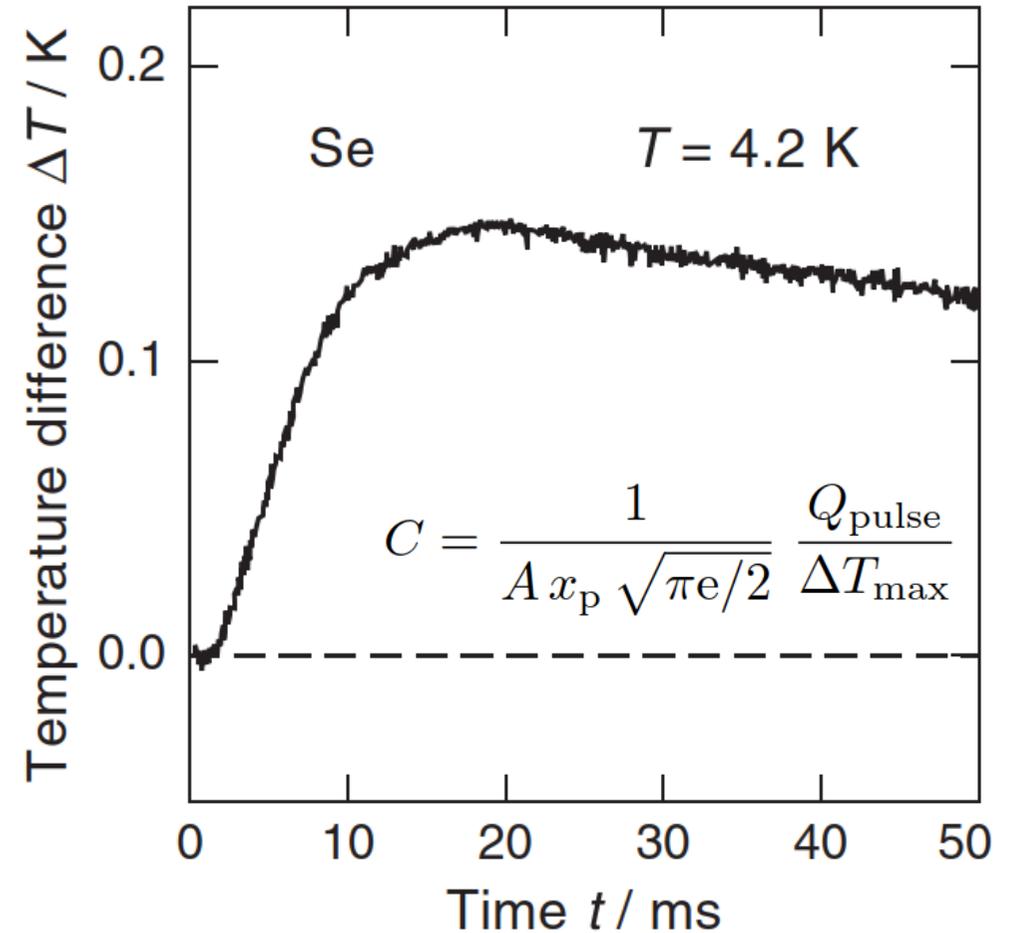
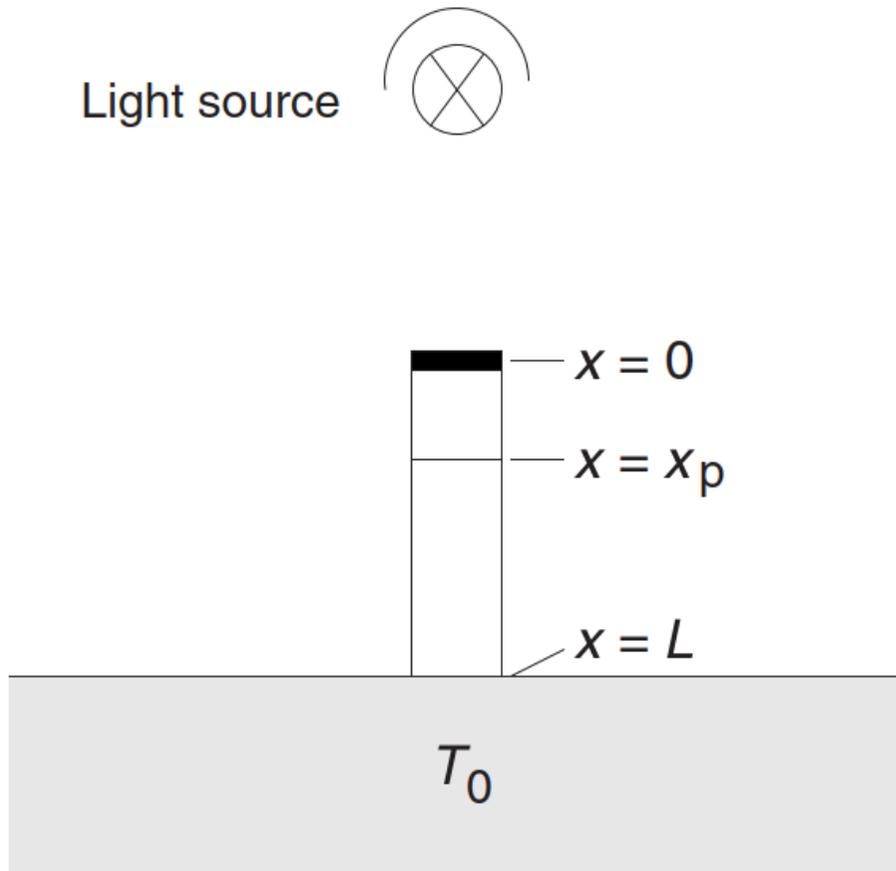
TABLE 5.2
Experimental values of emissivity

Material	Fulk, Reynolds, and Park (1955). 300 K radiation on 78 K surface	McAdams (1954). room temperature	Ramanathan (1952), 14 μm radiation on 2 K surface	Ziegler and Cheung (1957). 273 K radiation on 77 K surface
Al, clean polished foil	0.02	0.04	0.011†	0.043‡
Al, highly oxidized	—	0.31	—	—
Brass, clean polished	0.029	0.03	0.018†	0.10‡
Brass, highly oxidized	—	0.6	—	—
Cu, clean polished	0.015–0.019	0.02	0.0062–0.015†	—
Cu, highly oxidized	—	0.6	—	—
Cr, plate	0.08	0.08	—	0.084‡
Au, foil	0.010–0.023	0.02–0.03	—	—
Ni, polished	—	0.045	—	—
Ag, plate	0.008	0.02–0.03	—	—
Stainless steel	0.048	0.074	—	—
Sn, clean foil	0.013	0.06	0.013†	—
Soft solder	0.03	—	—	0.047‡
Glass	—	0.9	—	—

† These surfaces were electro-polished (Ramanathan).

‡ These surfaces were neither highly polished nor heavily oxidized, but as encountered in normal practice (Ziegler). Ziegler observed that a thin layer of oil or Apiezon grease on a low emissivity surface raised the emissivity to 0.2 or 0.3. He also found that varnishes such as GEC adhesive No. 7031 and bakelite lacquer gave an emissivity $\epsilon \approx 0.87$; similarly, Scotch tape (Sellotape) had an emissivity of about 0.88.

Measurement of κ and C simultaneously



$$\frac{\partial^2 T(x, t)}{\partial x^2} = \frac{1}{\kappa} \frac{\partial T(x, t)}{\partial t}$$

$$\Delta T(x, t) \approx \frac{Q_{\text{pulse}}}{A \sqrt{\pi \Lambda C t}} \exp\left(-\frac{x^2}{4 \kappa t}\right)$$

Table of reference for thermal conductivities

Table 3.2. Thermal conductivity of solids frequently used in low temperature apparatus

material	κ [mW (cm K) ⁻¹]	T range [K]	ref.
Manganin	$0.94 T^{1.2}$	1–4	[3.98]
Nb–Ti	$0.075 T^{1.85}$	4–9	[3.99]
Nb–Ti	$0.15\text{--}0.27 T^{2.0}$	0.1–1	[3.100]
Cu _{0.70} Ni _{0.30}	$0.93 T^{1.23}$	0.3–4	[3.101]
Cu _{0.70} Ni _{0.30}	$0.64 T$	0.05–3.0	[3.100, 3.102]
Pyrex	$0.15 T^{1.75}$	0.18–0.8	[3.103]
Al ₂ O ₃	$2.7 T^{2.5}$	2–8	[3.104]
Al ₂ O ₃	$0.29 T^{2.7}$	0.1–2	[3.94]
Stycast 1266	$0.49 T^{1.98}$	0.05–0.5	[3.105]
Stycast 1266	$0.39 T^{1.9}$	0.06–1	[3.100]
Stycast 2850 GT	$78 \times 10^{-3} T^{1.8}$	1–4	[3.98]
Stycast 2850 FT	$53 \times 10^{-3} T^{1.8}$	2–10	[3.106]
Stycast 2850 FT	$92 \times 10^{-3} T^{2.65}$	0.06–1	[3.100]
Vespel SP 1	$18 \times 10^{-3} T^{1.2}$	0.1–1	[3.94]
Vespel SP 22	$17 \times 10^{-3} T^2$	0.1–2	[3.94, 3.100]
Teflon	$30 \times 10^{-3} T^2$	0.2–1	[3.107]
Teflon	$38 \times 10^{-3} T^{2.4}$	0.3–0.7	[3.103]
Nylon	$26 \times 10^{-3} T^{1.75}$	0.2–1	[3.94, 3.103]
Macor	$58 \times 10^{-3} T^{2.24}$	0.4–1.1	[3.108]
Nuclear graphite	$15 \times 10^{-3} T^{1.13}$	0.1–2	[3.94]
AGOT graphite	$5.1 \times 10^{-3} T^{1.76}$	0.1–2	[3.94]
	$4.9 \times 10^{-3} T^{1.86}$	0.3–3	[3.109]
a-SiO ₂	$0.248 T^{1.91}$	0.06–1	[3.110]
Wood	$9.3 \times 10^{-3} T^{2.7}$	0.04–1	[3.111]
Kevlar	$3.9 \times 10^{-5} T^{1.17}$	0.1–2.5	[3.112]
Polypropylene	$27.4 \times 10^{-3} T^{1.28}$	0.1–1	[3.56]
PVC	$1.8 \times 10^{-4} T^{2.05}$	0.05–0.12	[3.113]
Torlon	$6.13 \times 10^{-2} T^{2.18}$	0.1–0.8	[3.114]

The given temperature ranges are the ranges where the given equations describe the data; in the cited literature, often data for a much wider temperature range are given.

(m) Allen-Bradley carbon resistors:

Alterovitz and Gershenon (1974), 0.3–4 K, measured a 116 Ω bare carbon resistor and concluded that its heat capacity was similar to graphite in this range, viz. $C \sim 1.1T + 2.3T^3 \mu\text{J/g K}$.

Other measurements on Allen-Bradley resistors (Keesom and Seidel 1959; see also Stephens 1973) have given values of C an order of magnitude higher than this.

It is important to realize that the heat capacities per gram of the resins, cements, and plastics mentioned above are far greater than that of copper at low temperatures. For example, at 2 K, $C \approx 28 \mu\text{J/g K}$ for copper compared with a $300 \mu\text{J/g K}$ for GE7031. Thus cements used in the assembly of a calorimeter should be of minimum quantity and carefully weighed. Also the large hyperfine contribution below 1 K in manganin, nichrome, and constantan resistance wires should be recognized.

Frequently in cryogenic design we also need to know the 'integrated' heat capacity or enthalpy of a component in order to calculate the quantity of refrigerant needed to cool it. The enthalpy is given by

$$H(T) = U + PV = \int_0^T C_p dT.$$

Since $C_p - C_v$ is not a unique function of T/θ but depends on the expansion coefficient and compressibility, no general table of H as a function of T/θ can be compiled. However at temperatures below θ , C_v is a fair approximation to C_p and therefore $U = \int C_v dT$ will approximate the heat content for most practical purposes. In Table 11.1 are given values of $(U - U_0)/T$ as a function of θ/T based on the Debye function and taken from Landolt-Börnstein tables. Apart from inadequacies of the Debye model, errors will also occur if there are significant contributions to C from magnetic or other interactions.

As an example consider cooling from 90 to 4 K one mole of copper (63.6 g) for which $\theta_D \approx 310$ K.

At 90 K,

$$\theta/T = 3.44,$$

therefore

$$U - U_0 = 90 \times 5.5 = 495 \text{ J/mol.}$$

TABLE 11.1

Values of $\frac{U - U_0}{T} = \frac{1}{T} \int_0^T C_v dT$ in J/mol K, from the Debye model

θ/T	$\frac{U - U_0}{T}$	θ/T	$\frac{U - U_0}{T}$	θ/T	$\frac{U - U_0}{T}$
0.2	23.10	2.2	10.08	6.0	1.93
0.4	21.38	2.4	9.23	6.5	1.59
0.6	19.76	2.6	8.44	7.0	1.31
0.8	18.24	2.8	7.73	7.5	1.09
1.0	16.81	3.0	7.07	8.0	0.912
1.2	15.47	3.5	5.66	9.0	0.653
1.4	14.23	4.0	4.53	10.0	0.481
1.6	13.07	4.5	3.64	11.0	0.364
1.8	11.99	5.0	2.93	12.0	0.281
2.0	11.00	5.5	2.37	15.0	0.144

At 4 K,

$$\theta/T = 78,$$

therefore

$$U - U_0 \approx 0.$$

Hence

$$U_{90} - U_4 = 495 \text{ J/mol.}$$

About 190 ml of liquid helium would be needed to cool 63.6 g of copper from 90 to 4 K, using only the latent heat of vaporization.

2. Thermal expansion

Introduction

For most solids the volume coefficient of thermal expansion $\beta = 3\alpha$ varies with temperature in much the same way as does the heat capacity. At temperatures $T \geq \theta_D$, β is fairly constant and as $T \rightarrow 0$, $\beta \rightarrow 0$ with similar temperature dependence to C_v . The thermal vibrations usually contribute the major part of β (and C_v) at all but the lowest temperatures.

Grüneisen first observed that β/C_v was essentially constant at

Real example:

Cooling capacity: 1st 30 W @ 65 K
2nd 0.5 W @ 4.2 K

Lowest temperature: 2.5 K

Weight: 25 kg

Cool down time: 100 min

