CS–E4500 Advanced Course in Algorithms

Week 02 – Tutorial

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1. For a coin that comes up heads independently with probability p on each flip, what is the expected number of flips until the kth heads?

   Solution. Let X_k denote the number of flips until the kth heads. We employ the law of total expectation, conditioning on the result of the first flip (first).

   $$\begin{aligned}
   \mathrm{E}(X_k) &= \mathrm{E}(\mathrm{E}(X_k \mid \mathsf{first})) \\
   &= p \cdot \mathrm{E}(X_k \mid \mathsf{first} = heads) + (1-p) \cdot \mathrm{E}(X_k \mid \mathsf{first} \neq heads) \\
   &= p \cdot (1 + \mathrm{E}(X_{k-1})) + (1-p) \cdot (1 + \mathrm{E}(X_k)) \\
   p\mathrm{E}(X_k) &= p\mathrm{E}(X_{k-1}) + 1 \\
   \mathrm{E}(X_k) &= \mathrm{E}(X_{k-1}) + 1/p
   \end{aligned}$$

   Solving the recurrence above yields $\mathrm{E}(X_k) = k/p$.

   An alternative approach is to start bottom-up:

   $$\begin{aligned}
   \mathrm{E}(X_1) &= 1/p \\
   \mathrm{E}(X_2) &= \mathrm{E}(X_1) + 1/p = 2/p \\
   &\vdots \\
   \mathrm{E}(X_k) &= k/p
   \end{aligned}$$

   Another approach is to construct a Markov chain and compute the expected passage time (first passage time).