ELEC-E8101 Digital and Optimal Control
Exercise 3
Solutions

1. State-space representation:

$$\frac{d\mathbf{x}(t)}{dt} = \begin{bmatrix} -0.0197 & 0 \\ 0.0178 & -0.0129 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 0.0263 \\ 0 \end{bmatrix}u(t)$$
$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix}\mathbf{x}(t)$$

a) Let's start with $\mathbf{\Phi}$ (with the help of Laplace-transformation):

$$\mathbf{\Phi} = e^{\mathbf{A}h} = \mathrm{L}^{-1}\{(s\mathbf{I} - \mathbf{A})^{-1}\}|_{t=h}$$

$$(s\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} s+0.0197 & 0 \\ -0.0178 & s+0.0129 \end{bmatrix}^{-1} =$$
$$= \frac{1}{(s+0.0197)(s+0.0129)}\begin{bmatrix} s+0.0129 & 0 \\ 0.0178 & s+0.0197 \end{bmatrix} =$$
$$= \begin{bmatrix} \dfrac{1}{s+0.0197} & 0 \\ \dfrac{0.0178}{(s+0.0197)(s+0.0129)} & \dfrac{1}{s+0.0129} \end{bmatrix}$$

The inverse Laplace-transformation from the tables:

$$L^{-1}\{(s\mathbf{I} - \mathbf{A})^{-1}\}|_{t=h} = \begin{bmatrix} e^{-0.0197\,h} & 0 \\ 2.62(e^{-0.0129\,h} - e^{-0.0197\,h}) & e^{-0.0129\,h} \end{bmatrix}$$

By writing h=12, we'll get:

$$\mathbf{\Phi} = \begin{bmatrix} 0.79 & 0 \\ 0.18 & 0.86 \end{bmatrix}$$

Then $\mathbf{\Gamma}$:

$$\mathbf{\Gamma} = \int_0^h e^{\mathbf{A}s}ds\mathbf{B} = \int_0^{12}\begin{bmatrix} 0.0263e^{-0.0197s} \\ +0.069e^{-0.0129s} - 0.069e^{-0.0197s} \end{bmatrix}ds = \begin{bmatrix} 0.282 \\ 0.0297 \end{bmatrix}$$

The state-space representation is:

$$\mathbf{x}(12(k+1)) = \begin{bmatrix} 0.79 & 0 \\ 0.18 & 0.86 \end{bmatrix}\mathbf{x}(12k) + \begin{bmatrix} 0.282 \\ 0.0297 \end{bmatrix}u(12k)$$
$$y(12k) = \begin{bmatrix} 0 & 1 \end{bmatrix}\mathbf{x}(12k)$$

The same with Matlab®:

```
» cont=ss([-0.0197 0;0.0178 -0.0129],[0.0263;0],[0 1],0);
» contd=c2d(cont,12)

a =
                        x1            x2
           x1      0.78946             0
           x2       0.1757       0.85659

b =
                        u1
           x1      0.28107
           x2     0.029621

c =
                        x1            x2
           y1            0             1

d =
                        u1
           y1            0
Sampling time: 12
Discrete-time model.
```

b) Determination of the pulse transfer function :

$$H(q) = \mathbf{C}\,(q\mathbf{I} - \mathbf{\Phi})^{-1}\mathbf{\Gamma} + \mathbf{D} = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix} q-0.79 & 0 \\ -0.18 & q-0.86 \end{bmatrix}^{-1}\begin{bmatrix}0.282 \\ 0.0297\end{bmatrix} =$$

$$= \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix} \dfrac{1}{q-0.79} & 0 \\ \dfrac{0.18}{(q-0.79)(q-0.86)} & \dfrac{1}{q-0.86} \end{bmatrix}^{-1}\begin{bmatrix}0.282 \\ 0.0297\end{bmatrix}$$

$$H(q) = \frac{0.282\cdot 0.18}{(q-0.79)(q-0.86)} + \frac{0.0297}{q-0.86} = \frac{0.03q+0.026}{q^2-1.65q+0.68}\,.$$

(**C** and **D** are the same as in the continuous case.)

Matlab: contdtf=tf(contd);

2.

The system

$y(k) + 0.5y(k-1) = u(k-1)$.

z-transformed version (initial values assumed to be zero):

$Y(z) + 0.5z^{-1}Y(z) = z^{-1}U(z) \;\Rightarrow\; Y(z)[1 + 0.5z^{-1}] = z^{-1}U(z)$

Pulse transfer function (respect to z^{-1}):

$$H^*(z^{-1}) = \frac{Y(z)}{U(z)} = \frac{z^{-1}}{1+0.5z^{-1}}\,.$$

Pulse transfer function (with respect to *z*):

$$H(z) = \frac{1}{z+0.5}$$

Note that we could have obtained *H* directly by shifting the time index $k \to k+1$ in the original difference equation.

Let’s write the pulse transfer function with q^{-1} (backward-shift operator):

$[1 + 0.5q^{-1}]\, y(k) = q^{-1}u(k)$ (generally: $A^*(q^{-1})y(k) = q^{-d}B^*(q^{-1})u(k)$)

where d = 1-0=1 is the pole excess i.e. the number of poles minus the number of zeros).

The pulse transfer operator is obtained:

$$H^*(q^{-1}) = \frac{q^{-d}B^*(q^{-1})}{A^*(q^{-1})} = \frac{q^{-1}}{1+0.5q^{-1}}\,.$$

Of course the shift operator *q* (forward-shift operator) can be used:

$y(k + 1) + 0.5y(k) = u(k)$ (generally: $A(q)y(k) = B(q)u(k)$)

The pulse transfer operator is obtained:

$$H(q) = \frac{B(q)}{A(q)} = \frac{1}{q+0.5}$$

A linear system with a pulse transfer function *H*(*z*) has an impulse response satisfying

$Y(z) = H(z)U(z) = H(z)$,   (since $U(z) = \mathrm{Z}\{\delta(k)\} = 1$)

z-transformation:

$$H(z) = \sum_{k=0}^{\infty} h(k) z^{-k}$$

or in operator format if you wish

$$H(q) = H(q) = \sum_{k=0}^{\infty} h(k) q^{-k} \ .$$

Now

$$H(q) = H^*(q^{-1}) = \frac{q^{-1}}{1+0.5q^{-1}} \ .$$

Anyway the following must be inverse-transformed:

$$H(z) = \frac{1}{z+0.5} = z^{-1} \frac{z}{z+0.5}$$

$$\Rightarrow \begin{cases} h(k) = (-0.5)^{k-1}, k \geq 1 \\ h(k) = 0, k = 0 \end{cases}$$

3. The discrete time state-space representation is:

$$\mathbf{x}(kh+h) = \begin{bmatrix} e^{-h} & 0 \\ 1-e^{-h} & 1 \end{bmatrix} \mathbf{x}(kh) + \begin{bmatrix} 1-e^{-h} \\ h-1+e^{-h} \end{bmatrix} u(kh)$$

$$y(kh) = \begin{bmatrix} 0 & 1 \end{bmatrix} \mathbf{x}(kh)$$

a) The corresponding pulse transfer function:

$$H(z) = C(z\mathbf{I} - \mathbf{\Phi})^{-1}\mathbf{\Gamma} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} z-e^{-h} & 0 \\ e^{-h}-1 & z-1 \end{bmatrix}^{-1} \begin{bmatrix} 1-e^{-h} \\ h+e^{-h}-1 \end{bmatrix} =$$

$$= \frac{\begin{bmatrix} 0 & 1 \end{bmatrix}}{\left(z-e^{-h}\right)\left(z-1\right)} \begin{bmatrix} z-1 & 0 \\ 1-e^{-h} & z-e^{-h} \end{bmatrix} \begin{bmatrix} 1-e^{-h} \\ h+e^{-h}-1 \end{bmatrix} = \frac{\left(h+e^{-h}-1\right)z + \left(1-e^{-h}-he^{-h}\right)}{\left(z-e^{-h}\right)\left(z-1\right)} =$$

$$= \frac{\left(h+e^{-h}-1\right)z + \left(1-e^{-h}-he^{-h}\right)}{z^2 - \left(1+e^{-h}\right)z + e^{-h}}$$

b) Pulse response (from the state space model):

$$h(k) = \begin{cases} 0, & k = 0 \\ \mathbf{C\Phi}^{k-1}\mathbf{\Gamma}, & k \geq 1 \end{cases}$$

Since $\mathbf{\Phi}^k = \left(e^{\mathbf{A}h}\right)^k = e^{\mathbf{A}kh}$, we'll get

$$\mathbf{C}\mathbf{\Phi}^{k-1}\mathbf{\Gamma} = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix} e^{-(k-1)h} & 0 \\ 1-e^{-(k-1)h} & 1 \end{bmatrix}\begin{bmatrix} 1-e^{-h} \\ h+e^{-h}-1 \end{bmatrix} =$$

$$= \left(1-e^{-h}\right)\left(1-e^{-(k-1)h}\right) + h + e^{-h} - 1 = h - e^{-(k-1)h} + e^{-kh}$$

The pulse response can be acquired also by the inverse z-transform. The poles are:

$$z_{1,2} = \frac{1+e^{-h} \pm \sqrt{\left(1+e^{-h}\right)^2 - 4e^{-h}}}{2} = \frac{1+e^{-h} \pm \left(1-e^{-h}\right)}{2} \Rightarrow \begin{cases} z_1 = 1 \\ z_2 = e^{-h} \end{cases}$$

Partial fractions:

$$\frac{\left(h+e^{-h}-1\right)z + \left(1-e^{-h}-he^{-h}\right)}{z^2 - \left(1+e^{-h}\right)z + e^{-h}} \equiv \frac{A}{z-1} + \frac{B}{z-e^{-h}}$$

$$\Rightarrow \begin{cases} A + B = h + e^{-h} - 1 \\ Ae^{-h} + B = he^{-h} + e^{-h} - 1 \end{cases} \Rightarrow \begin{cases} A = h \\ B = e^{-h} - 1 \end{cases}$$

The inverse z-transformation:

$$Z^{-1}\left\{\frac{h}{z-1} + \frac{e^{-h}-1}{z-e^{-h}}\right\} = Z^{-1}\left\{z^{-1}h\frac{z}{z-1} + \left(e^{-h}-1\right)z^{-1}\frac{z}{z-e^{-h}}\right\} =$$

$$= h \cdot H(h(k-1)) + \left(e^{-h}-1\right)e^{-(k-1)h} = h \cdot H(h(k-1)) + e^{-kh} - e^{-(k-1)h}$$

($H(kh)$ is the discrete unit step in $kh = 0$.)

Answer is the same as above, just as it should be.

c) Difference equation corresponding to $H(q)$:

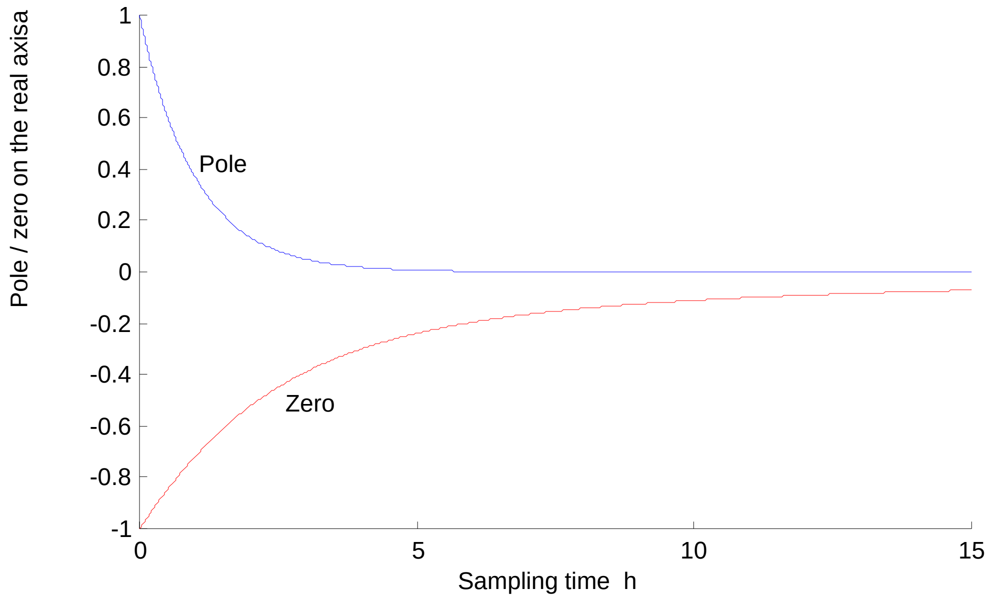
$$y(kh+2h) - \left(1+e^{-h}\right)y(kh+h) + e^{-h}y(kh) = \left(h+e^{-h}-1\right)u(kh+h) + \left(1-e^{-h}-he^{-h}\right)u(kh)$$

d) Poles: $p_1 = 1$ and $p_2 = e^{-h}$.

Zero: $z_0 = \dfrac{he^{-h} + e^{-h} - 1}{h + e^{-h} - 1}$

Let's examine the movement of the pole and zero, when h goes from 0 to infinity (one of the poles doesn't move at all):

```
» h=0:0.01:15;plot(h,exp(-h));
» hold on;plot(h,(h.*exp(-h)+exp(-h)-1)./(h+exp(-h)-1),'r');
```



Vocabulary: "Napa" = pole, "Nolla" = zero, "Diskretointiaika" = sampling time.

4. The transfer function under consideration: $G(s) = \dfrac{6}{(s+2)(s+3)}$

Result:

$$H(z) = \frac{z-1}{z} \cdot Z\left\{ L^{-1}\left\{ G(s)\frac{1}{s} \right\}\Big|_{t=kh} \right\}$$

So let us calculate:

Step response:

$$Y(s) = G(s)\frac{1}{s} = \frac{6}{s(s+2)(s+3)} = \frac{1}{s} - \frac{3}{s+2} + \frac{2}{s+3}$$

Inverse transformation:

$$y(t) = 1 - 3e^{-2t} + 2e^{-3t}$$

With sampling (period h = 1) the following sequence can be acquired:

$$y(k) = 1 - 3e^{-2k} + 2e^{-3k}$$

Z-transform:

$$Y(z) = \frac{z}{z-1} - \frac{3z}{z-e^{-2}} + \frac{2z}{z-e^{-3}}$$

$$= \frac{z\left(z-e^{-2}\right)\left(z-e^{-3}\right) - 3z(z-1)\left(z-e^{-3}\right) + 2z(z-1)\left(z-e^{-2}\right)}{(z-1)\left(z-e^{-2}\right)\left(z-e^{-3}\right)}$$

$$= \frac{\left(1+2e^{-3}-3e^{-2}\right)z^2 + \left(e^{-5}-3e^{-3}+2e^{-2}\right)z}{(z-1)\left(z-e^{-2}\right)\left(z-e^{-3}\right)}$$

$$= \frac{z\left[\left(1+2e^{-3}-3e^{-2}\right)z + \left(e^{-5}-3e^{-3}+2e^{-2}\right)\right]}{(z-1)\left(z-e^{-2}\right)\left(z-e^{-3}\right)}$$

Divide by the Z-transform of a unit step $U(z) = \dfrac{z}{z-1}$:

$$G(z) = \frac{Y(z)}{U(z)} = \frac{\left(1+2e^{-3}-3e^{-2}\right)z + \left(e^{-5}-3e^{-3}+2e^{-2}\right)}{\left(z-e^{-2}\right)\left(z-e^{-3}\right)}$$

$$= \frac{\left(1+2e^{-3}-3e^{-2}\right)z + \left(e^{-5}-3e^{-3}+2e^{-2}\right)}{z^2 - \left(e^{-2}+e^{-3}\right)z + e^{-5}}$$

$$= \frac{0.6936z + 0.128}{z^2 - 0.1851z + 0.0067}$$

Note: The problem sheet contains a table ($G(s)$ and ZOH-equivalent $H(z)$). Using that you can determine the pulse transfer function directly. In the correct formula take $a = 2$, $b = 3$.

The final value theorem in continuous time:

$$\bar{k} = \lim_{s\to 0}\{sY(s)\} = \lim_{s\to 0}\left\{\frac{6}{(s+2)(s+3)}\right\} = 1$$

The final value theorem in discrete time:

$$\bar{k} = \lim_{z\to 1}\left\{\frac{z-1}{z}Y(z)\right\} = \lim_{z\to 1}\left\{\frac{0.6936z + 0.128}{z^2 - 0.1851z + 0.0067}\right\} = 1$$

Let's simulate with Matlab:

```
» step(tf([0.6936 0.128],[1 -0.1851 0.0067],1),5)
» hold on;step(tf(6,[1 5 6]),5)
```